Dependencies in risk models and their dual queueing models

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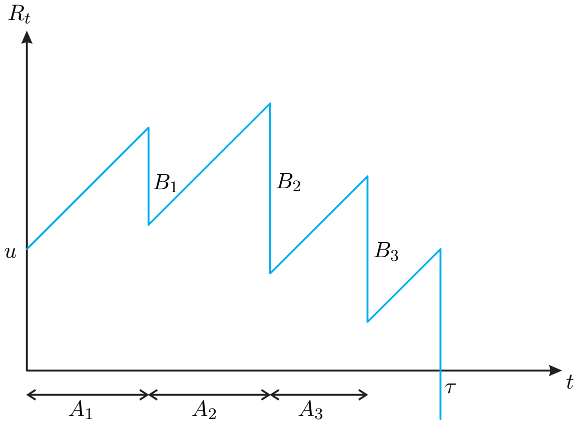
Outline:
Introduction
Model description
Analysis of the maximum aggregate loss

Introduction: The risk reserve process of an insurance company, starting with initial capital u and premium rate 1:

$$R_t(u) = u + \sum_{i:\sigma_i < t} (A_i - B_i) + (t - \sigma_{N(t)}).$$



Assumptions:

- the inter-claim times $(A_i)_i$ are i.i.d. (σ_i is a renewal process)
  the claim sizes $(B_i)_i$ are also i.i.d.
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  but within a pair, (A_i, B_i) *are allowed to depend*
  hence a generalized Sparre-Andersen model
- main quantity of interest: the time to ruin for initial capital u:

$$\tau(u) = \inf\{t > 0 \ : \ R_t(u) < 0\}$$

  $\tau(u)$ related to the maximum aggregate loss up to the n-th arrival epoch:

$$L_n = \max_{1 \leq k \leq n} \left\{0, \sum_{i=1}^{k} (B_i - A_i)\right\},$$

  via $\{\tau(u) \leq \sigma_n\} = \{L_n > u\}$.

- Under the stability condition $\mathbb{E}(B_i - A_i) < 0$, $L_n \stackrel{\mathcal{D}}{\to} L$ as $n \to \infty$, and we can write: $L = \sup_{k \geq 0} \left\{ \sum_{i=1}^{k} (B_i - A_i) \right\}$. The relation with time to ruin becomes:

$$\tau(u) < \infty \Leftrightarrow L > u.$$

Thus we can focus on the distribution of L. This is also the steady state workload at arrival epochs in a GI/G/1 queue with *reverted dependence* between the service requirement of a customer and the time elapsed since the arrival of the previous customer.

Modeling the dependence structure between A and B:

- fairly general class called bivariate matrix exponential (BME) Bladt&Nielsen 2010
  (A,B) are BME if if its joint Laplace-Stieltjes transform is a rational function:

  $$\mathbb{E}e^{-s_1 A - s_2 B} = \frac{F(s_1, s_2)}{G(s_1, s_2)}, \text{ where } F(s_1, s_2) \text{ and } G(s_1, s_2)$$

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  are polynomial functions.
- The dependence structure is not obvious from this definition. Moreover, it may be the case that A and B are independent.

Some examples of explicit dependence structures:

- Bivariate phase-type (Kulkarni 1989) Consider an absorbing CTMC $X(t)$ with finite state space $\mathcal{S}$, together with a reward matrix $(r_x^{(1)}, r_x^{(2)})_x$, $r_x^{(j)} \geq 0$ for $x \in \mathcal{S}\backslash\{\Delta\}$ (Δ-absorbing state). Assume that as long as we stay in state x, we 'earn' at rate vector $\mathbf{r}_x = (r_x^{(1)}, r_x^{(2)})$. Then the total accumulated 'rewards' until absorbtion are

$$A = \int_0^{\zeta} r_{X(t)}^{(1)} dt, \quad B = \int_0^{\zeta} r_{X(t)}^{(2)} dt$$

with ζ the time to absorbtion.

- These can be rewritten as:

$$A = \sum_{i=1}^{N} r_{X_i}^{(1)} H_i, \quad B = \sum_{i=1}^{N} r_{X_i}^{(2)} H_i,$$

where N is the number of jumps until absorbtion of the underlying DTMC X_n and H_i is the holding time in state X_i.
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- Cheriyan and Ramabhadran bivariate gamma distribution can be realized as above. For nonnegative integers m_0, m_1, m_2, consider the state space $\mathcal{S} = \{1, ..., m_0 + m_1 + m_2, \Delta\}$, with the set of transient states partitioned as $\mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$ with $\mathcal{S}_0 = \{1, ..., m_0\}$, $\mathcal{S}_1 = \{m_0 + 1, ..., m_1\}$, $\mathcal{S}_2 = \{m_1 + 1, ..., m_2\}$. The transition rule is $p_{i,i+1} = 1$ and the jump rates are β_k while in state $x \in \mathcal{S}_k$. The reward rates in state x are $r_x^{(1)} = r_x^{(2)} = 1$ for $x \in \mathcal{S}_0$; $r_x^{(1)} = 1, r_x^{(2)} = 0$, for $x \in \mathcal{S}_1$, and $r_x^{(1)} = 0$, $r_x^{(2)} = 1$ for $x \in \mathcal{S}_2$.

- Then the bivariate total accumulated reward has distribution of the form

$$(A, B) \triangleq (Z_0 + Z_1, Z_0 + Z_2)$$

where Z_k are mutually independent $\sim$Erlang(m_k, β_k), $k \in \{0, 1, 2\}$.

Analysis of the maximum aggregate loss:

- Denote by $Y_i = B_i - A_i$ the claim surplus. Since Y_i are i.i.d., the random vector $(Y_1, ..., Y_n)$ has the same distribution as $(Y_n, ..., Y_1)$. Hence we have the identity in distribution

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- L_n satisfies the recursion: $L_n \triangleq \max\{0, Y_n + L_{n-1}\}$; and in the limit, L satisfies the identity in distribution

$$L \triangleq \max\{L + Y, 0\}$$

with Y having the same distribution as $B - A$.

- rewriting this identity in terms of Laplace-Stieltjes transforms gives:

$$\mathbb{P}(L+Y \leq 0) - \mathbb{E}e^{-s(L+Y)}1_{\{L+Y<0\}} = \mathbb{E}e^{-sL}\left[1 - \Phi_{(A,B)}(-s,s)\right]$$

with $\Phi_{(A,B)}(-s,s)$ the Laplace-Stieltjes transform of Y. This is of the form $\frac{f(s)}{g(s)}$, with f and g polynomial functions.

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- **How to find the transform of L?**
- The above is of the form:

$$\Phi_{-}(s) = \mathbb{E}e^{-sL}\frac{g(s) - f(s)}{g(s)}$$

  $\Phi_{-}(s)$ is analytic for $\mathcal{R}e\ s < 0$. $\mathbb{E}e^{-sL}$ is analytic for $\mathcal{R}e\ s > 0$.

- The idea is to separate the above identity by taking the zeros of $g(s)$ from $\mathcal{R}e\ s \geq 0$ on the other side:

$$g_+(s)\Phi_-(s) = \mathbb{E}e^{-sL}\frac{g(s) - f(s)}{g_-(s)}$$

  Now both sides are analytic in their respective regions, and they coincide on $\mathcal{R}e\ s = 0$. This implies they are analytic continuations of each other. In particular, $\mathbb{E}e^{-sL}\frac{g(s)-f(s)}{g_-(s)}$ is *everywhere* analytic. Also remark that $\mathbb{E}e^{-sL}\frac{g(s)-f(s)}{g_-(s)}$ is $O(s^{m_+})$, with $m_+ = deg[g/(g_-)]$. We use a version of Liouville's theorem:

Theorem (*Liouville*)

If $P(s)$ is analytic for all finite values of s, and as $|s| \to \infty$

$$P(s) = O(s^{m_+}),$$

then $P(s)$ is a polynomial of order $\leq m_+$.

- On the other hand, the identity

$$\mathbb{E}e^{-sL} = \frac{g_{-}(s)}{g(s) - f(s)} P(s)$$

implies that $P(s)$ must have all the zeros of $g(s) - f(s)$ from $\mathcal{R}e\ s \geq 0$.

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- An application of Rouché's theorem shows that $g(s) - f(s)$ has exactly m_{+} zeros in $\mathcal{R}e\ s \geq 0$, which means $P(s)$ is determined up to a constant:

$$c \prod_{s_k^+} (s - s_k^+),$$

  s_k^+ are the zeros of $g(s) - f(s)$ from $\mathcal{R}e\ s \geq 0$.

- Setting $s = 0$ determines the constant and

$$Ee^{-sL} = \frac{\prod_{\tilde{s}_j^-}(1 - \frac{s}{\tilde{s}_j^-})}{\prod_{s_k^-}(1 - \frac{s}{s_k^-})}.$$

From this follows the distribution of L by inversion.