Algorithmic methods for branching processes

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Outline

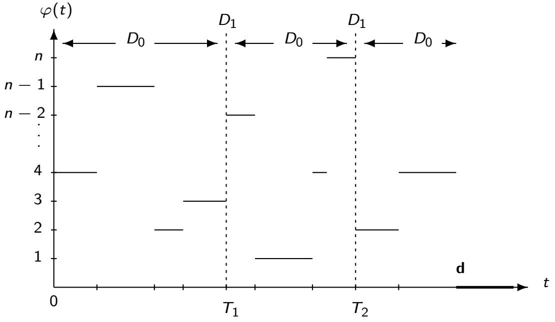
1. The Markovian binary tree (MBT)
2. Algorithms to compute the extinction probability
3. Catastrophes

Markovian binary trees (MBTs)

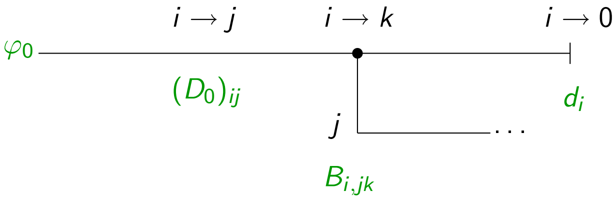
Markovian binary trees are mathematical objects at the intersection of branching processes and matrix analytic methods.

- The lifetime of individuals is controlled by a transient Markovian arrival process ;
- We use techniques inspired from the matrix analytic methods to compute the extinction probability of the process ;
- We give a probabilistic interpretation to all of our algorithms.

The transient Markovian arrival process $(\alpha, D_0, D_1, \mathbf{d})$

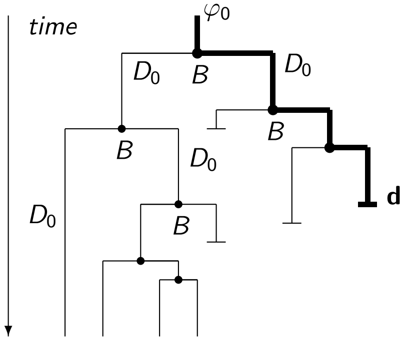


The individual's lifetime in an Markovian binary tree



- n transient phases, 1 absorbing phase 0 ;
- φ_0 : the initial phase ;
- D_0 : the matrix of phase transition rates between two events ;
- B : the Birth rate matrix ; $D_1 = B(\mathbf{1} \otimes I)$;
- $\mathbf{d}$: the death rate vector.

The MBT representation

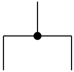


- An MBT models the evolution of a family or population over time.
- We assume that the individuals behave independently of each other.

Extinction probability of an MBT

Let $\mathbf{q} = \mathrm{P}[\text{the MBT becomes extinct} \,|\, \varphi_0]$.

We define

- $\boldsymbol{\theta} = (-D_0)^{-1}\,\mathbf{d}$: the death probability of a branch, 
- $\Psi = (-D_0)^{-1}\,B$: the birth probability of a branch. 

$\rightarrow$ **q** is the minimal nonnegative solution of the matrix extinction equation

$$\boxed{\mathbf{s} = \boldsymbol{\theta} + \Psi\,(\mathbf{s} \otimes \mathbf{s}).}$$

Linear algorithms to compute the extinction probability

$$\mathbf{s} = \boldsymbol{\theta} + \Psi\,(\mathbf{s} \otimes \mathbf{s})$$

$$\equiv$$

$$\mathbf{s} = [I - \Psi\,(I \otimes \mathbf{s})]^{-1}\,\boldsymbol{\theta}$$

$$\equiv$$

$$\mathbf{s} = [I - \Psi\,(\mathbf{s} \otimes I)]^{-1}\,\boldsymbol{\theta}$$

1. The Depth and the Order algorithms
   (Bean, Kontoleon and Taylor, 2008)
2. The Thicknesses algorithm
   (Hautphenne, Latouche and Remiche, 2011).

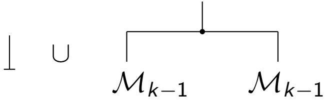
The Depth algorithm

$$\mathbf{s}_0 = \boldsymbol{\theta}$$
$$\mathbf{s}_k = \boldsymbol{\theta} + \Psi\left(\mathbf{s}_{k-1} \otimes \mathbf{s}_{k-1}\right), \qquad k \geq 1$$

For $k \geq 0$,

- $\mathcal{M}_k$ = the set of MBTs considered at stage k
- $\mathbf{s}_k = \mathbb{P}[\mathcal{M}_k \,|\, \varphi_0]$ = the kth approximation of $\mathbf{q}$.

$\mathcal{M}_0 =$ 

$\mathcal{M}_k =$ 

for $k \geq 1$.

Probabilistic interpretation of the Depth algorithm

$$\begin{aligned} \mathbf{s}_0 &= \boldsymbol{\theta} \\ \mathbf{s}_k &= \boldsymbol{\theta} + \Psi\left(\mathbf{s}_{k-1} \otimes \mathbf{s}_{k-1}\right), \qquad k \geq 1 \end{aligned}$$

Depth of an MBT = number of branching points along the longest branch

For $k \geq 0$,

- $\mathcal{M}_k$ = the set of extinct MBTs with a depth $\leq k$ (constraint on the shape of the tree)
- $\mathcal{M}_k \subseteq \mathcal{M}_{k+1} \subseteq \cdots \subseteq \mathcal{M}$ = the set of all extinct MBTs.
- $\mathbf{s}_k = \mathbb{P}[\mathcal{M}_k \,|\, \varphi_0] \xrightarrow{k\to\infty} \mathbf{q}$.

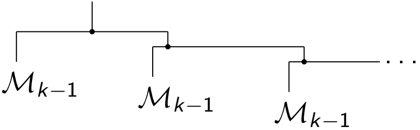
The Order algorithm

$$
\begin{aligned}
\mathbf{s}_0 &= \boldsymbol{\theta} \\
\mathbf{s}_k &= [I - \Psi(\mathbf{s}_{k-1} \otimes I)]^{-1} \boldsymbol{\theta}, \qquad k \geq 1
\end{aligned}
$$

For $k \geq 0$,

- $\mathcal{M}_k =$ the set of MBTs considered at stage k
- $\mathbf{s}_k = \mathbb{P}[\mathcal{M}_k \mid \varphi_0] =$ the kth approximation of $\mathbf{q}$.

$\mathcal{M}_0 =$ 

$\mathcal{M}_k =$ 

for $k \geq 1$.

Probabilistic interpretation of the Order algorithm

$$\begin{aligned} \mathbf{s}_0 &= \boldsymbol{\theta} \\ \mathbf{s}_k &= [I - \Psi(\mathbf{s}_{k-1} \otimes I)]^{-1} \boldsymbol{\theta}, \qquad k \geq 1 \end{aligned}$$

Order of an MBT = total number of children generations

For $k \geq 0$,

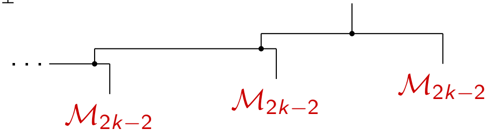
- $\mathcal{M}_k$ = the set of extinct MBTs with an order $\leq k$ (constraint on the shape of the tree)
- $\mathcal{M}_k \subseteq \mathcal{M}_{k+1} \subseteq \cdots \subseteq \mathcal{M}$ = the set of all extinct MBTs.
- $\mathbf{s}_k = \mathbb{P}[\mathcal{M}_k \,|\, \varphi_0] \overset{k\to\infty}{\longrightarrow} \mathbf{q}$.

The Thicknesses algorithm

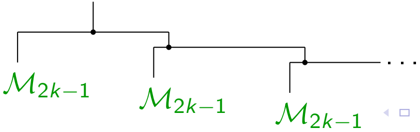
$$\begin{aligned} \mathbf{s}_0 &= \boldsymbol{\theta} \\ \mathbf{s}_{2k-1} &= [I - \Psi\,(I \otimes \mathbf{s}_{2k-2})]^{-1}\,\boldsymbol{\theta}, \qquad k \geq 1 \\ \mathbf{s}_{2k} &= [I - \Psi\,(\mathbf{s}_{2k-1} \otimes I)]^{-1}\,\boldsymbol{\theta}, \qquad k \geq 1 \end{aligned}$$

$\mathcal{M}_0 =$ 

$\mathcal{M}_{2k-1} =$



$\mathcal{M}_{2k} =$

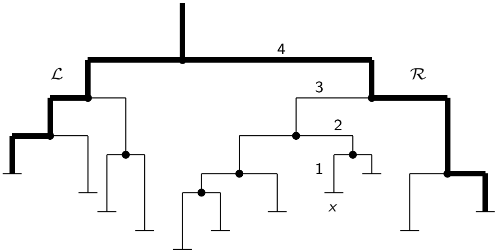


for $k \geq 1$.

Left and right thicknesses of a tree

We define the left thickness $LT(\mathcal{T})$ and the right thickness $RT(\mathcal{T})$ of a tree $\mathcal{T}$.

Example where $LT(\mathcal{T}) = 4$ and $RT(\mathcal{T}) = 3$:



Probabilistic interpretation of the Thicknesses algorithm

$$
\begin{aligned}
\mathbf{s}_0 &= \boldsymbol{\theta} \\
\mathbf{s}_{2k-1} &= [I - \Psi(I \otimes \mathbf{s}_{2k-2})]^{-1}\boldsymbol{\theta}, \qquad k \geq 1 \\
\mathbf{s}_{2k} &= [I - \Psi(\mathbf{s}_{2k-1} \otimes I)]^{-1}\boldsymbol{\theta}, \qquad k \geq 1
\end{aligned}
$$

For $k \geq 0$,

- $\mathcal{M}_{2k-1}$ the set of extinct MBTs with $LT \leq 2k-1$
- $\mathcal{M}_{2k}$ the set of extinct MBTs with $RT \leq 2k$
- $\mathcal{M}_k \subseteq \mathcal{M}_{k+1} \subseteq \cdots \subseteq \mathcal{M}$ = the set of all extinct MBTs.
- $\mathbf{s}_k = \mathbb{P}[\mathcal{M}_k \,|\, \varphi_0] \xrightarrow{k\to\infty} \mathbf{q}$.

Comparison of the linear algorithms

- The Depth algorithm is slower than the Order algorithm and the Thicknesses algorithm ;
- The performance of the Thicknesses algorithm compared to the Order algorithm depends on the example considered.

Quadratically convergent algorithm : Newton

$$\mathcal{F}(\mathbf{s}) = \mathbf{s} - \boldsymbol{\theta} - \Psi\,(\mathbf{s} \otimes \mathbf{s}) = \mathbf{0}$$

$\Rightarrow$ Newton's iteration method :

$$\mathbf{x}_k = \mathbf{x}_{k-1} - (\mathcal{F}'_{\mathbf{x}_{k-1}})^{-1}\,\mathcal{F}(\mathbf{x}_{k-1}), \quad k \geq 0,$$

which leads to the Newton algorithm :

$$\begin{aligned} \mathbf{x}_0 &= \boldsymbol{\theta}, \\ \mathbf{x}_k &= [I - \Psi\,(\mathbf{x}_{k-1} \oplus \mathbf{x}_{k-1})]^{-1}\,[\boldsymbol{\theta} - \Psi\,(\mathbf{x}_{k-1} \otimes \mathbf{x}_{k-1})], \quad k \geq 1 \\ &= x_{k-1} + \Delta_k \end{aligned}$$

where

$$\Delta_k = [I - \Psi\,(\mathbf{x}_{k-1} \oplus \mathbf{x}_{k-1})]^{-1}\,\Psi\,(\Delta_{k-1} \otimes \Delta_{k-1})$$

Probabilistic interpretation of the Newton algorithm

$$\mathbf{x}_0 = \boldsymbol{\theta},$$
$$\mathbf{x}_k = x_{k-1} + \Delta_k$$

where

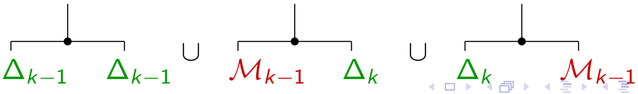
$$\Delta_k = [I - \Psi(\mathbf{x}_{k-1} \oplus \mathbf{x}_{k-1})]^{-1}\,\Psi(\Delta_{k-1} \otimes \Delta_{k-1})$$

$\mathcal{M}_0 = \Delta_0 =$ 

For $k \geq 1$:

$$\mathcal{M}_k = \mathcal{M}_{k-1} \cup \Delta_k,$$

$\Delta_k =$



Link between MBTs and QBDs

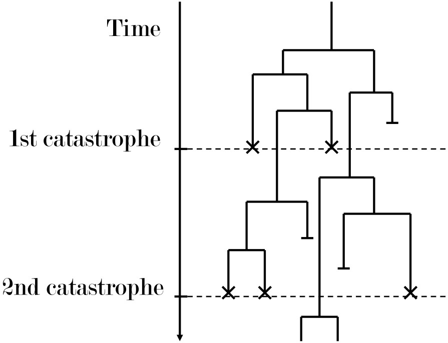
Markovian binary trees can be represented as level-dependent quasi-birth-and-death processes $(X(t), \varphi(t))$ with

- $X(t)$ = the total population size at time t = the level,
- $\varphi(t) = (Z_1(t), Z_2(t), \ldots, Z_n(t))$ = the phase.

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ Q_{10} & Q_{11} & Q_{12} & 0 & 0 & \cdots \\ 0 & Q_{21} & Q_{22} & Q_{23} & 0 & \cdots \\ 0 & 0 & Q_{32} & Q_{33} & Q_{34} & \cdots \\ & & \vdots & & & \ddots \end{bmatrix}$$

Extinction probability $\mathbf{q}$ ≡ Probability to go from level 1 to level 0, given by the matrix $G^{(1)}$.

MBT with catastrophes



MBT with catastrophes

Assume that

- the catastrophes occur following a Poisson process with parameter β (or more generally following a MAP),
- they arrive independently of the evolution of the MBT,
- an individual in phase i is killed with probability ε_i.

$\hat{\mathbf{q}}$ = the extinction probability of the MBT with catastrophes, given the initial phase.

Loss of independence $\Rightarrow \hat{\mathbf{q}} \neq \boldsymbol{\theta} + \Psi\left(\hat{\mathbf{q}} \otimes \hat{\mathbf{q}}\right)$.

Structured Markov chain approach

Two-dimensional $G/M/1$-type Markov process $(X(t), \varphi(t))$ with

- $X(t) =$ the total population size at time $t =$ the level,
- $\varphi(t) = (Z_1(t), Z_2(t), \ldots, Z_n(t)) =$ the phase.

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ Q_{10} & Q_{11} & Q_{12} & 0 & 0 & \cdots \\ Q_{20} & Q_{21} & Q_{22} & Q_{23} & 0 & \cdots \\ Q_{30} & Q_{31} & Q_{32} & Q_{33} & Q_{34} & \cdots \\ & & \vdots & & & \ddots \end{bmatrix}$$

Extinction probability $\hat{\mathbf{q}} \equiv$ Probability to go from level 1 to level 0.

Structured Markov chain approach

γ_i = first passage time to level i.

$$\hat{\mathbf{q}} = \mathbf{G}^{(1)} = P[\gamma_0 < \infty\,, \varphi(\gamma_0)\,|\,X(0) = 1, \varphi(0)].$$

$\mathbf{G}^{(1)} = \mathbf{1} - \lim_{M\to\infty}(L_1\,L_2\cdots L_M)\,\mathbf{1}$ with

$$L_i \;=\; \mathrm{P}[\gamma_{i+1} < \gamma_0\,, \varphi(\gamma_{i+1})\,|\,X(0) = i, \varphi(0)],$$

$$L_1 \;=\; (-Q_{11})^{-1}\,Q_{12},$$

$$L_i \;=\; \left[I - (-Q_{ii})^{-1}\sum_{j=1}^{i-1} Q_{i\,(i-j)} \prod_{i-j\le k\le i-1} L_k\right]^{-1} (-Q_{ii})^{-1}\,Q_{i\,(i+1)},$$

$i \geq 2$.

References

1. N. Bean, N. Kontoleon, and P. Taylor. Markovian trees : properties and algorithms. *Annals of Operations Research*, 160 :31–50, 2008.
2. S. Hautphenne, G. Latouche, and M.-A. Remiche. Newton's iteration for the extinction probability of a Markovian Binary Tree. *Linear Algebra and its Applications*, 428(11-12) :2791–2804, 2008.
3. S. Hautphenne. An algorithmic look at phase-controlled branching processes, PhD Thesis, 2009.
4. S. Hautphenne, G. Latouche, and M.-A. Remiche. Algorithmic approach to the extinction probability of branching processes. *Methodology and Computing in Applied Probability*, 13(1) :171–192, 2011.
5. S. Hautphenne and G. Latouche. Markovian trees subject to catastrophes : transient features and extinction probability. *Stochastic Models*, 27 :569–590, 2011.