Perturbation Analysis for Multiserver Retrial Queues

Tuan Phung-Duc
Tokyo Institute of Technology

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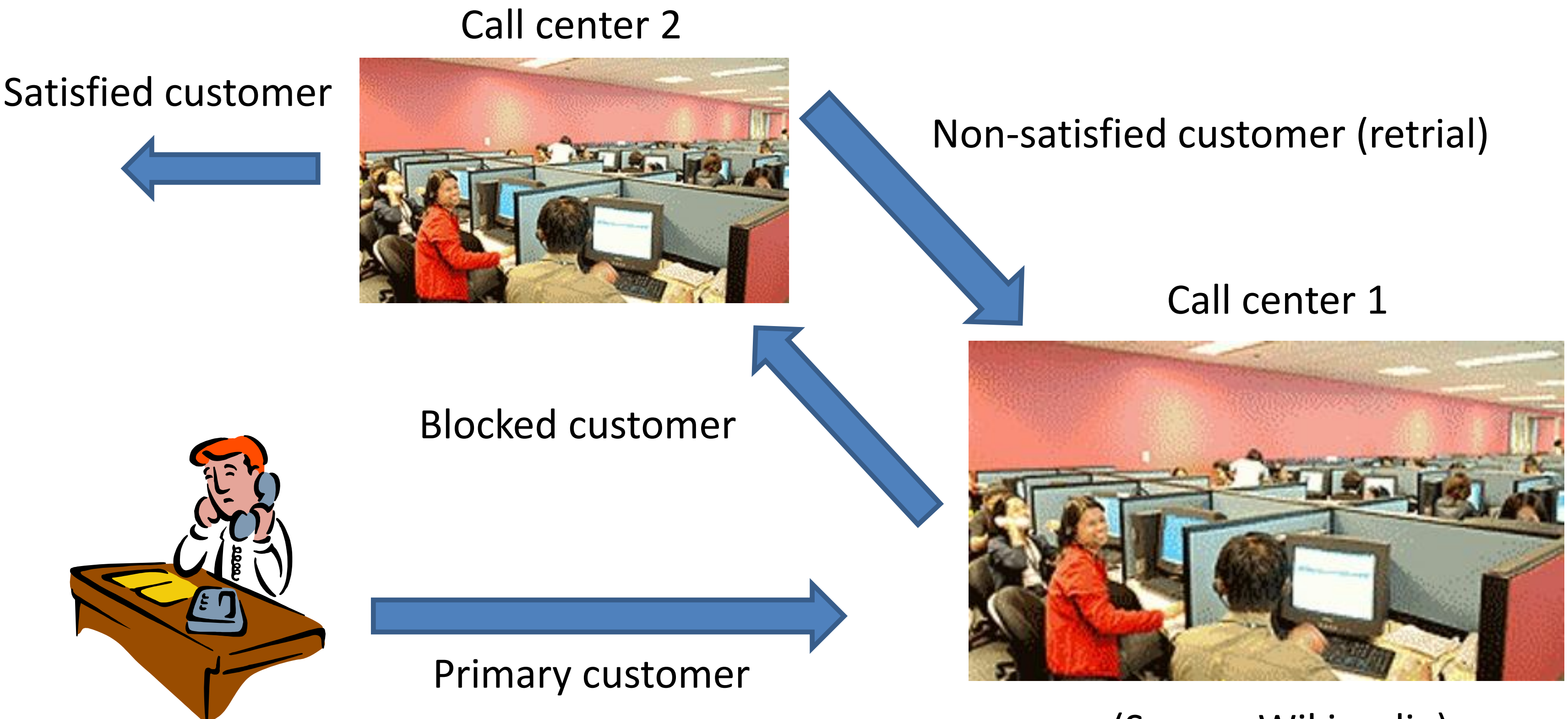
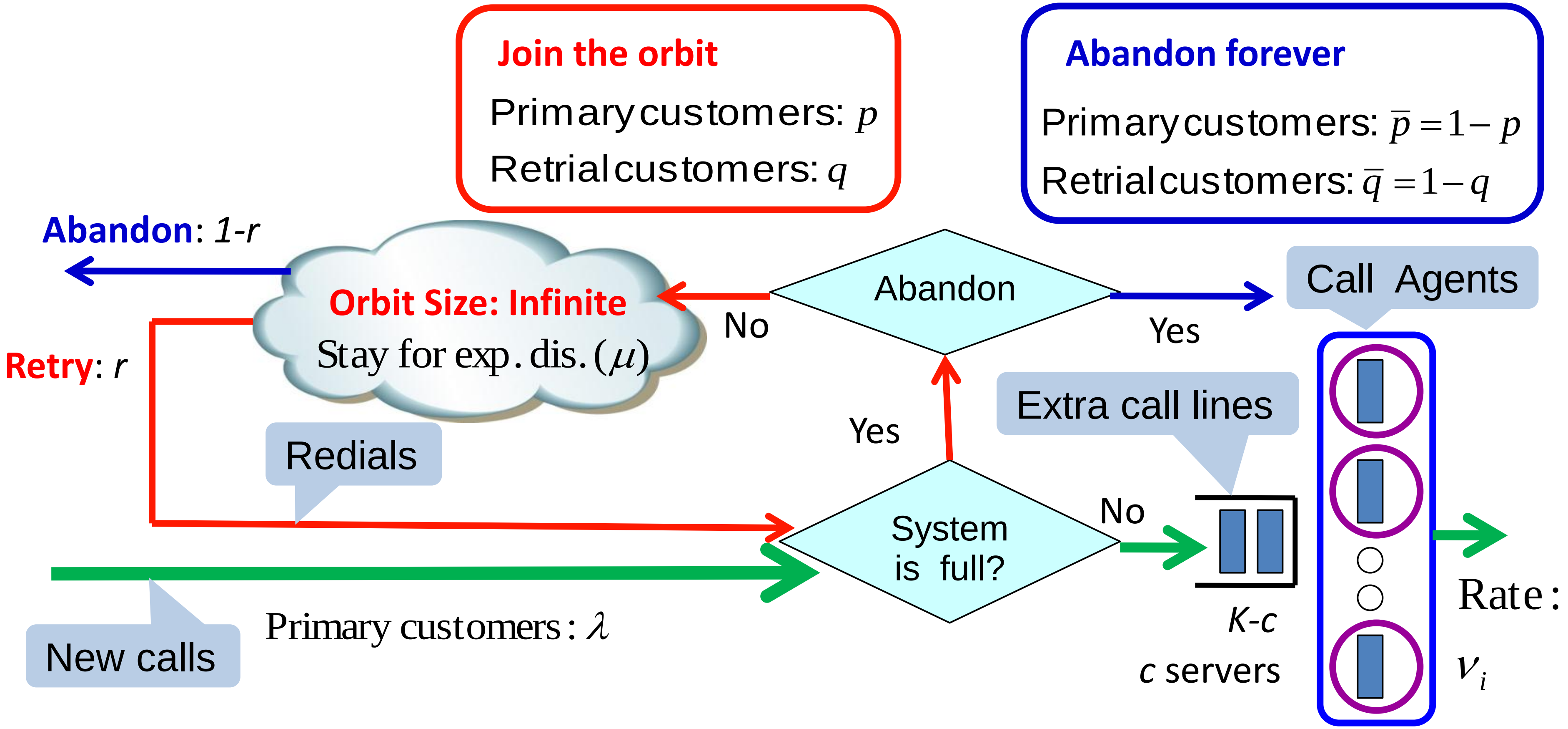
Motivation: Collaborative call centers



(Source:Wikipedia)

- Blocked calls → Forwarded to another call center
  - Satisfied customers → departure
  - Non-satisfied customers → reattempt to the original call center

Model



Related work: Choi et al. (1999), Gomez-corral et al. (1999)

Phung-Duc et al. (JIMO 2010, Annals of OR 2011)

- Level-dependent **quasi-birth-and-death** process (LDQBD)

LDQBD of the M/M/c/K retrial queue

- $C(t)$: # of customers in the system at time t (phase)
- $N(t)$: # of customers in the orbit at time t (level)
- $X(t) = (C(t), N(t))$ forms a **LDQBD** on the state space

$$S = \{0,1,\ldots,K\} \times \{0,1,2,\ldots\}.$$

$$\pi_{k,n} = \lim_{t\to\infty} \Pr(C(t) = k, N(t) = n), \quad (k,n) \in S$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_1^{(0)} & \mathbf{Q}_0^{(0)} & \mathbf{O} & \mathbf{O} & \cdots \\ \mathbf{Q}_2^{(1)} & \mathbf{Q}_1^{(1)} & \mathbf{Q}_0^{(1)} & \mathbf{O} & \cdots \\ \mathbf{O} & \mathbf{Q}_2^{(2)} & \mathbf{Q}_1^{(2)} & \mathbf{Q}_0^{(2)} & \cdots \\ \mathbf{O} & \mathbf{O} & \mathbf{Q}_2^{(3)} & \mathbf{Q}_1^{(3)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$$\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_n, \ldots),$$

$$\boldsymbol{\pi}_n = (\pi_{0,n}, \pi_{1,n}, \ldots, \pi_{K,n})$$

$$\mathbf{e} = (1,1,\ldots,1,\ldots)^T,$$

$$\mathbf{0} = (0,0,\ldots,0,\ldots)^T.$$

Block matrices

Increases by 1

$$\boldsymbol{Q}_0^{(n)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \boxed{\lambda p} \end{pmatrix}$$

Decreases by 1

$$\boldsymbol{Q}_2^{(n)} = \begin{pmatrix} n\mu\bar{r} & n\mu r & 0 & \cdots & 0 \\ 0 & n\mu\bar{r} & n\mu r & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & n\mu\bar{r} & n\mu r \\ 0 & \cdots & \cdots & 0 & n\mu(\bar{r}+r\bar{q}) \end{pmatrix}$$

Unchanged

$$\boldsymbol{Q}_1^{(n)} = \begin{pmatrix} b_0^{(n)} & \lambda & 0 & \cdots & \cdots & 0 \\ \nu_1 & b_1^{(n)} & \lambda & \ddots & & \vdots \\ 0 & \nu_2 & b_2^{(n)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & b_{K-1}^{(n)} & \lambda \\ 0 & \cdots & \cdots & 0 & \nu_K & b_K^{(n)} \end{pmatrix}$$

$$b_i^{(n)} = -(\lambda + n\mu + \nu_i),$$

$$i = 0, 1, 2, \ldots, K-1,$$

$$b_K^{(n)} = -(p\lambda + n\mu(\bar{r}+r\bar{q}) + \nu_K).$$

Existent result on LDQBD

Ramaswami and Taylor (1996)

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_{n-1}\mathbf{R}^{(n)} \quad (n = 1,2,\ldots).$$

$\{\mathbf{R}^{(n)}; n = 1,2,\ldots\}$ is the minimal nonnegative solution of

$$\mathbf{Q}_0^{(n-1)} + \mathbf{R}^{(n)}\mathbf{Q}_1^{(n)} + \mathbf{R}^{(n)}\mathbf{R}^{(n+1)}\mathbf{Q}_2^{(n+1)} = \mathbf{O} \quad (n = 1,2,\ldots).$$

Bright and Taylor (95): expression of $R^{(n)}$ in terms of infinite sum

Matrix continued fraction (Phung-Duc et al. 2010)

$$\mathbf{R}^{(n)} = \mathbf{Q}_0^{(n-1)}(-\mathbf{Q}_1^{(n)} - \mathbf{Q}_0^{(n)}(-\mathbf{Q}_1^{(n+1)} - \mathbf{R}^{(n+2)}\mathbf{Q}_2^{(n+2)})^{-1}\mathbf{Q}_2^{(n+1)})^{-1} = \cdots$$

$\mathbf{R}_k^{(n)} = R_n \circ R_{n+1} \circ \circ \circ R_{n+k-1}(\mathbf{O})$. We have $\lim_{k\to\infty} \mathbf{R}_k^{(n)} = \mathbf{R}^{(n)}$.

$f^\circ g(x) = f(g(x))$

→ Numerical algorithms for $R^{(n)}$

Today's talk → Taylor series expansion for $R^{(n)}$

Special sparse structure

$$\mathbf{R}^{(n)} = \mathbf{Q}_0^{(n-1)}(-\mathbf{Q}_1^{(n)} - \mathbf{R}^{(n+1)}\mathbf{Q}_2^{(n+1)})^{-1}$$

$$\mathbf{Q}_0^{(n-1)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \boxed{\lambda p} \end{pmatrix} \Rightarrow \mathbf{R}^{(n)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ r_0^{(n)} & r_1^{(n)} & r_2^{(n)} & \cdots & r_K^{(n)} \end{pmatrix}$$

Instead of $\mathbf{R}^{(n)}$, we investigate $\mathbf{r}^{(n)} = (r_0^{(n)}, r_1^{(n)}, \ldots, r_K^{(n)})$.

Liu and Zhao (QUESTA 2010): $p=q=r=1$

Theorem 3.1 (First-order formula) *For* $k = 0, 1, 2, \ldots, c$,

$$r_{n,c-k} = \rho\left(\frac{\mu}{n\theta}\right)^k \frac{c!}{(c-k)!} + o\left(\frac{1}{n^k}\right).$$

Corollary 3.2 (Higher-order formulas)

$$r_{n,c-2} = \rho\left(\frac{\mu}{n\theta}\right)^2 \frac{c!}{(c-2)!} - \rho\left(\frac{\mu}{n\theta}\right)^3 \frac{c!}{(c-2)!}(2\rho + 2c - 3) + O\left(\frac{1}{n^4}\right),$$

$$r_{n,c-1} = \rho\left(\frac{\mu}{n\theta}\right) \frac{c!}{(c-1)!} - \rho\left(\frac{\mu}{n\theta}\right)^2 \frac{c!}{(c-2)!} + \rho\left(\frac{\mu}{n\theta}\right)^3 \frac{c!}{(c-2)!}(2\rho + c - 1) + O\left(\frac{1}{n^4}\right),$$

"kth-order for $r_i^{(n)}$ => $(k+1)$th-order for $r_i^{(n)}$. No single expression for a general k. The proof will soon become too cumbersome!!"

and

$$r_{n,c} = \rho + \rho^2 \frac{\mu}{n\theta} + \rho^2(c-1)\frac{\mu}{n^2\theta^2}(\theta + \mu\rho - \mu) + O\left(\frac{1}{n^3}\right).$$

Asymptotic result for $\pi_{c,n}$

- Our contribution
  - Explicit Taylor series expansion for $r_i^{(n)}$ $(i = 0, 1, \ldots, K)$

Our results

Today's talk

- Retrial customers may give up: $q < 1$ or $r < 1$

$$r_{K-k}^{(n)} = \sum_{i=1}^{m} \gamma_i^{(k)} (-1)^{i+1} \frac{1}{n^{k+i}} + O(\frac{1}{n^{k+m+1}}) \qquad m = 1,2,\ldots$$

- Retrial customers never give up: $q = r = 1$

$$r_{K-k}^{(n)} = \sum_{i=0}^{m} \theta_i^{(k)} (-1)^{i} \frac{1}{n^{k+i}} + O(\frac{1}{n^{k+m+1}}) \qquad m = 0,1,\ldots$$

Different order!!

$$\lim_{x \to 0} |O(x)/x| = C \geq 0$$

Equation for rate matrices

$$\mathbf{Q}_0^{(n-1)} + \mathbf{R}^{(n)}\mathbf{Q}_1^{(n)} + \mathbf{R}^{(n)}\mathbf{R}^{(n+1)}\mathbf{Q}_2^{(n+1)} = \mathbf{O} \quad (n = 1, 2, \ldots).$$

⇕

$$\begin{aligned}
b_0^{(n)} r_0^{(n)} + \nu_1 r_1^{(n)} + \widetilde{r}_0^{(n+1)} r_K^{(n)} &= 0, && i = 0,\\
\lambda r_{i-1}^{(n)} + b_i^{(n)} r_i^{(n)} + \nu_{i+1} r_{i+1}^{(n)} + \widetilde{r}_i^{(n+1)} r_K^{(n)} &= 0, && i = 1, 2, \ldots, K-1,\\
\lambda r_{K-1}^{(n)} + \left(b_K^{(n)} + \widetilde{r}_K^{(n+1)}\right) r_K^{(n)} &= -p\lambda, && i = K,
\end{aligned}$$

$$\begin{aligned}
\widetilde{r}_0^{(n)} &= n\mu\bar{r} r_0^{(n)}, && i = 0,\\
\widetilde{r}_i^{(n)} &= n\mu r r_{i-1}^{(n)} + n\mu\bar{r} r_i^{(n)}, && i = 1, 2, \ldots, K-1,\\
\widetilde{r}_K^{(n)} &= n\mu r r_{K-1}^{(n)} + n\mu(\bar{r} + r\bar{q}) r_K^{(n)}, && i = K.
\end{aligned}$$

Censored Markov chain

$$Q^{\le n-1} = \begin{pmatrix} Q_1^{(0)} & Q_0^{(0)} & O & \cdots & O \\ Q_2^{(1)} & Q_1^{(1)} & Q_0^{(1)} & \ddots & O \\ O & Q_2^{(2)} & Q_1^{(2)} & \ddots & \vdots \\ \vdots & O & \ddots & \ddots & O \\ \vdots & \ddots & \ddots & Q_2^{(n-2)} & Q_0^{(n-2)} \\ O & \cdots & O & Q_2^{(n-1)} & \widehat{Q}^{(n-1)} \end{pmatrix}$$

$Q^{\le n-1}$: Infinitesimal generator of the censored Markov chain on levels $\{0,1,\ldots,n-1\}$

$$\widehat{Q}^{(n-1)} = Q_1^{(n-1)} + R^{(n)} Q_2^{(n)}$$

$$(Q_2^{(n-1)} + \widehat{Q}^{(n-1)})e = 0$$

Look at the last row!

Nontrivial equation: $$r_0^{(n)} + r_1^{(n)} + \cdots + r_{K-1}^{(n)} + (\bar{r} + r\bar{q}) r_K^{(n)} = \frac{\lambda p}{n\mu}$$

$$\mathbf{Q}_0^{(n-1)} + \mathbf{R}^{(n)}\mathbf{Q}_1^{(n)} + \mathbf{R}^{(n)}\mathbf{R}^{(n+1)}\mathbf{Q}_2^{(n+1)} = \mathbf{O} \ (n = 1, 2, \ldots).$$

This talk → Taylor series expansion for $r_{K-k}^{(n)}$ from this two equations.

$q < 1$ or $r < 1$

Retrial customers may give up

Our results: One term expansion

Lemma 1 *We have* $\lim_{n\to\infty} n^{k+1} r_i^{(n)} = 0$ *for* $i = 0, 1, \ldots, K-k-1$ *and*

$$r_{K-k}^{(n)} = \gamma_1^{(k)} \frac{1}{n^{k+1}} + o(\frac{1}{n^{k+1}}), \qquad k = 0, 1, \ldots, K,$$

where

$$\gamma_1^{(0)} = \frac{\lambda p}{\mu(\bar{r} + r\bar{q})}, \qquad \gamma_1^{(k)} = \frac{\nu_{K-k+1}}{\mu} \gamma_1^{(k-1)}, \qquad k = 1, 2, \ldots, K.$$

Lemma 2 *We have*

$$r_{K-k}^{(n)} = \gamma_1^{(k)} \frac{1}{n^{k+1}} + O(\frac{1}{n^{k+2}}),$$

where $O(x)$ *denotes* $\lim_{x\to 0} |O(x)/x| = C \geq 0.$

Mathematical induction

Step 1: check for *k*=0

$$\lim_{n\to\infty} n r_i^{(n)} = 0, i = 0,1,\ldots,K-1, and\ r_{K-0}^{(n)} = \frac{\lambda p}{\bar{r}+r\bar{q}}\frac{1}{n} + o\left(\frac{1}{n}\right).$$

Lemma 1 is true for $k = 0$.

Step 2: assumption for *k*-1

We assume that Lemma 1 is true for $k := k-1, i.e.$

$$\lim_{n\to\infty} n^k r_i^{(n)} = 0\ (i \le K-k), \qquad r_{K-(k-1)}^{(n)} = \gamma_1^{(k-1)}\frac{1}{n^k} + o\left(\frac{1}{n^k}\right).$$

Step 3: prove for *k*

We prove that Lemma 1 is true for k , $i.e.$

$$\lim_{n\to\infty} n^{k+1} r_i^{(n)} = 0, \quad r_{K-k}^{(n)} = \gamma_1^{(k)}\frac{1}{n^{k+1}} + o\left(\frac{1}{n^{k+1}}\right),$$

$$i = 0,1,\ldots,K-k-1.$$

Proof of Lemma 1 (Step 1, k=0)

$$\frac{1}{\bar{r}+r\bar{q}}\sum_{i=0}^{K-1} r_i^{(n)} + r_K^{(n)} = \frac{\lambda \mathrm{p}}{(\bar{r}+r\bar{q})\mu n} \quad \Rightarrow \quad \begin{array}{ll} \lim_{n\to\infty} r_k^{(n)} = 0, & k=0,1,\dots,K \\ n\mu r_k^{(n)} \le \lambda p, & k=0,1,\dots,K-1 \end{array}$$

$$\underbrace{-\lambda r_0^{(n)}}_{\to 0} - n\mu r_0^{(n)} + \underbrace{v_1 r_1^{(n)}}_{\to 0} + \underbrace{(n+1)\mu r_K^{(n)}}_{\text{bounded}}\underbrace{\bar{r} r_0^{(n+1)}}_{\to 0} = 0$$

$$\boxed{\lim_{n\to\infty} n r_0^{(n)} = 0}$$

$$\underbrace{\lambda r_0^{(n)}}_{\to 0} - (\lambda + n\mu + v_1) r_1^{(n)} + \underbrace{v_2 r_2^{(n)}}_{\to 0} + \underbrace{(n+1)\mu[r r_0^{(n+1)} + \bar{r} r_1^{(n+1)}]}_{\to 0} r_K^{(n)} = 0$$

$$\boxed{\lim_{n\to\infty} n r_1^{(n)} = 0} \quad \text{Similarly} \quad \boxed{\lim_{n\to\infty} n r_{K-1}^{(n)} = 0}$$

$$\frac{1}{\bar{r}+r\bar{q}}\sum_{i=0}^{K-1} \lim_{n\to\infty} n r_i^{(n)} + \lim_{n\to\infty} n r_K^{(n)} = \frac{\lambda \mathrm{p}}{(\bar{r}+r\bar{q})\mu} \quad \Rightarrow \quad \boxed{r_{K-0}^{(n)} = \frac{\lambda \mathrm{p}}{(\bar{r}+r\bar{q})\mu n} + o\left(\frac{1}{n}\right)}$$

Step 2: Lemma 1 is true for *k*-1 (assumption)

Step 2: assumption for *k*-1

We assume that Lemma 1 is true for $k := k-1$, $i.e.$

$$\lim_{n\to\infty} n^k r_i^{(n)} = 0 \ (i \le K-k), \qquad r_{K-(k-1)}^{(n)} = \gamma_1^{(k-1)} \frac{1}{n^k} + o\left(\frac{1}{n^k}\right).$$

Key equation

$$r_{K-k}^{(n)} = \frac{\lambda r_{K-k-1}^{(n)}}{n\mu} + \frac{\nu_{K-k+1} r_{K-k+1}^{(n)}}{n\mu} + \frac{\tilde{r}_{K-k}^{(n+1)} r_K^{(n)}}{n\mu} - \frac{\lambda + \nu_{K-k}}{n\mu} r_{K-k}^{(n)}$$

- $\lim_{n\to\infty} n^{k+1} r_i^{(n)} = 0, i = 0,1,\ldots,K-k-1$ (Easy)
- Key equation → one term expansion for $r_{K-k}^{(n)}$

One term expansion (cont.)

Key equation

$$r_{K-k}^{(n)} = \frac{\lambda r_{K-k-1}^{(n)}}{n\mu} + \frac{\nu_{K-k+1} r_{K-k+1}^{(n)}}{n\mu} + \frac{\tilde{r}_{K-k}^{(n+1)} r_K^{(n)}}{n\mu} - \frac{\lambda + \nu_{K-k}}{n\mu} r_{K-k}^{(n)}$$

$$\frac{\lambda r_{K-k-1}^{(n)}}{n\mu} = \frac{\lambda r_{K-k-1}^{(n)} n^k}{n^{k+1}\mu} = o(\frac{1}{n^{k+1}}), \qquad \leftarrow \lim_{n\to\infty} n^k r_i^{(n)} = 0 \ (i \le K-k)$$

$$\frac{\nu_{K-k+1} r_{K-k+1}^{(n)}}{n\mu} = \frac{\nu_{K-k+1}}{\mu} \gamma_1^{(k-1)} \frac{1}{n^{k+1}} + o(\frac{1}{n^{k+1}}),$$ One term expansion for $r_{K-(k-1)}^{(n)}$

$$\frac{\tilde{r}_{K-k}^{(n+1)} r_K^{(n)}}{n\mu} = \frac{n+1}{n} \frac{r n^k r_{K-k-1}^{(n+1)} + \bar{r} n^k r_{K-k}^{(n+1)}}{n^k} \frac{n r_K^{(n)}}{n} = o(\frac{1}{n^{k+1}}), \qquad \leftarrow \lim_{n\to\infty} n^k r_i^{(n)} = 0$$

$$\frac{\lambda + \nu_{K-k}}{n\mu} r_{K-k}^{(n)} = \frac{\lambda + \nu_{K-k}}{n^{k+1}\mu} r_{K-k}^{(n)} n^k = o(\frac{1}{n^{k+1}}). \qquad \leftarrow \lim_{n\to\infty} n^k r_i^{(n)} = 0 \ (i \le K-k)$$

$$\Rightarrow \quad r_{K-k}^{(n)} = \gamma_1^{(k)} \frac{1}{n^{k+1}} + o\left(\frac{1}{n^{k+1}}\right), \text{ where } \gamma_1^{(k)} = \frac{\nu_{K-k+1}}{\mu} \gamma_1^{(k-1)}.$$

Two term expansion

- We have two term expansion as follows

Lemma 3

$$r_{K-k}^{(n)} = \gamma_1^{(k)} \frac{1}{n^{k+1}} - \gamma_2^{(k)} \frac{1}{n^{k+2}} + O(\frac{1}{n^{k+3}}),$$

where

$$\gamma_2^{(0)} = \frac{\nu_K \lambda p}{\mu^2(\bar{r} + r\bar{q})^2},$$

$$\gamma_2^{(k)} = \frac{\nu_{K-k+1}}{\mu}\gamma_2^{(k-1)} + \left(\frac{\lambda + \nu_{K-k}}{\mu} - \frac{\lambda p \bar{r}}{\mu(\bar{r} + r\bar{q})}\right)\gamma_1^{(k)}, \qquad k = 1, 2, \ldots, K.$$

Mathematical induction

Step 1: check for *k*=0

$$r_K^{(n)} = \gamma_1^{(0)}\frac{1}{n} - \gamma_2^{(0)}\frac{1}{n^2} + O\left(\frac{1}{n^3}\right).$$ Lemma 3 is true for $k = 0$.

Step 2: assumption for *k*-1

We assume that Lemma 3 is true for $k := k - 1, i.e.$

$$r_{K-(k-1)}^{(n)} = \gamma_1^{(k-1)}\frac{1}{n^k} - \gamma_2^{(k-1)}\frac{1}{n^{(k+1)}} + O\left(\frac{1}{n^{k+2}}\right).$$

Step 3: proof for *k*

We prove that Lemma 3 is true for k , $i.e.$

$$r_{K-k}^{(n)} = \gamma_1^{(k)}\frac{1}{n^{k+1}} - \gamma_2^{(k)}\frac{1}{n^{(k+2)}} + O\left(\frac{1}{n^{k+3}}\right).$$

Step 1 (check for *k*=0)

Lemma 3 for *k* = 0

$$r_0^{(n)} + r_1^{(n)} + \cdots + r_{K-1}^{(n)} + (\bar{r} + r\bar{q})r_K^{(n)} = \frac{\lambda p}{n\mu}$$

$$r_K^{(n)} = \gamma_1^{(0)}\frac{1}{n} - \frac{1}{\bar{r}+r\bar{q}}\sum_{i=0}^{K-1} r_i^{(n)} \overset{\text{Lemma 2}}{=} \gamma_1^{(0)}\frac{1}{n} - \gamma_2^{(0)}\frac{1}{n^2} + O(\frac{1}{n^3})$$

$$\gamma_2^{(0)} = \frac{\gamma_1^{(1)}}{\bar{r}+r\bar{q}} = \frac{\nu_K \lambda p}{\mu^2(\bar{r}+r\bar{q})^2}$$

$r_{K-0}^{(n)}$ has two term expansion!

Step 2: Lemma 3 is true for *k*-1 (assumption)

Step 2: assumption for *k*-1 $(k \geq 1)$

We assume that Lemma 3 is true for $k := k-1, i.e.$

$$r_{K-(k-1)}^{(n)} = \gamma_1^{(k-1)} \frac{1}{n^k} - \gamma_2^{(k-1)} \frac{1}{n^{(k+1)}} + o\left(\frac{1}{n^{k+2}}\right).$$

Key equation

$$r_{K-k}^{(n)} = \frac{\lambda r_{K-k-1}^{(n)}}{n\mu} + \frac{\nu_{K-k+1} r_{K-k+1}^{(n)}}{n\mu} + \frac{\tilde{r}_{K-k}^{(n+1)} r_K^{(n)}}{n\mu} - \frac{\lambda + \nu_{K-k}}{n\mu} r_{K-k}^{(n)}$$

- Key equation → two term expansion for $r_{K-k}^{(n)}$

Key equation

$$r^{(n)}_{K-k} = \frac{\lambda r^{(n)}_{K-k-1}}{n\mu} + \frac{\nu_{K-k+1} r^{(n)}_{K-k+1}}{n\mu} + \frac{\tilde{r}^{(n+1)}_{K-k} r^{(n)}_{K}}{n\mu} - \frac{\lambda + \nu_{K-k}}{n\mu} r^{(n)}_{K-k}$$

$$r^{(n)}_{K-k-1} = \gamma_1^{(k+1)} \frac{1}{n^{k+2}} + O(\frac{1}{n^{k+3}}),$$ ← Lemma 2 (one term expansion)

$$r^{(n)}_{K-k} = \gamma_1^{(k)} \frac{1}{n^{k+1}} + O(\frac{1}{n^{k+2}}),$$ ← Lemma 2 (one term expansion)

$$r^{(n)}_{K-k+1} = \gamma_1^{(k-1)} \frac{1}{n^{k}} - \gamma_2^{(k-1)} \frac{1}{n^{k+1}} + O(\frac{1}{n^{k+2}}).$$ ← Two term expansion

$$\frac{\tilde{r}^{(n+1)}_{K-k} r^{(n)}_{K}}{n\mu} = \bar{r}\gamma_1^{(0)}\gamma_1^{(k)} \frac{1}{n^{k+2}} + O(\frac{1}{n^{k+3}}).$$ ← Lemma 2 (one term)

- Substituting these formulae to Key equation.
- Two term expansion for $r^{(n)}_{K-k}$

Two term expansion

- We obtain two term expansion for $r_{K-k}^{(n)}$

Lemma 3

$$r_{K-k}^{(n)} = \gamma_1^{(k)} \frac{1}{n^{k+1}} - \gamma_2^{(k)} \frac{1}{n^{k+2}} + O(\frac{1}{n^{k+3}}),$$

where

$$\gamma_2^{(0)} = \frac{\nu_K \lambda p}{\mu^2(\bar{r} + r\bar{q})^2},$$

$$\gamma_2^{(k)} = \frac{\nu_{K-k+1}}{\mu}\gamma_2^{(k-1)} + \left(\frac{\lambda + \nu_{K-k}}{\mu} - \frac{\lambda p \bar{r}}{\mu(\bar{r} + r\bar{q})}\right)\gamma_1^{(k)}, \qquad k = 1, 2, \ldots, K.$$

Main result

- *m* term expansion

Theorem 1 *For* $m \geq 3$, *we have*

$$r_{K-k}^{(n)} = \sum_{i=1}^{m} \gamma_i^{(k)} (-1)^{i+1} \frac{1}{n^{k+i}} + O(\frac{1}{n^{k+m+1}}),$$

where $\gamma_m^{(k)}$ *is recursively defined as follows*

$$\gamma_m^{(0)} = \frac{1}{\bar{r} + r\bar{q}} \sum_{k=1}^{K} \gamma_{m-k}^{(k)} (-1)^{k+1},$$

$$\gamma_m^{(k)} = \frac{\nu_{K-k+1}}{\mu} \gamma_m^{(k-1)} + \frac{\lambda}{\mu} \gamma_{m-2}^{(k+1)} + \frac{\lambda + \nu_{K-k}}{\mu} \gamma_{m-1}^{(k)} + \sum_{j=0}^{m-2} \varphi_j^{(k)} \gamma_{m-j-1}^{(0)} (-1)^{m-j}, \qquad k = 1, 2, \ldots, K.$$

Further definition

Furthermore, $\varphi_j^{(k)}$ *is defined by*

$$\varphi_j^{(k)} = \begin{cases} \bar{r}\beta_0^{(k)}, & j = 0 \\ r\alpha_j^{(k)} + \bar{r}\beta_j^{(k)}, & j \geq 1, \end{cases}$$

where

$$\alpha_j^{(k)} = \sum_{i=1}^{j} \gamma_i^{(k+1)} (-1)^{j+1} \frac{(k+j)_{j-i}}{(j-i)!}, \qquad j \in \mathbb{N},$$

$$\beta_j^{(k)} = \sum_{i=1}^{j+1} \gamma_i^{(k)} (-1)^{j} \frac{(k+i-1)_{j+1-i}}{(j+1-i)!}, \qquad j \in \mathbb{Z}_+.$$

Derivation

- The derivation for m (≥ 3) term expansion of $r_{K-k}^{(n)}$ ($k = 0,1,\ldots,K$)is similar to that for the case $m = 1,2$

- The key tool is the following expansion

$$\left(\frac{n}{n+1}\right)^a = \left(1+\frac{1}{n}\right)^{-a} = \sum_{j=0}^{\infty} \frac{(a)_j}{j!}(-1)^j \frac{1}{n^j}, \qquad a > 0.$$

Asymptotic formulae for the stationary distribution

Theorem 2 *We have*

$$C_1^{(0)} \frac{1}{n!} (\gamma_1^{(0)})^n n^{-\frac{\nu_K}{\mu(\bar{r}+r\bar{q})}} \le \pi_{n,K} \le C_2^{(0)} \frac{1}{n!} (\gamma_1^{(0)})^n n^{-\frac{\nu_K}{\mu(\bar{r}+r\bar{q})}},$$

where $C_1^{(0)}$ and $C_2^{(0)}$ are positive numbers independent of n.

Corollary 1 *There exist $C_1^{(k)} > 0$ and $C_2^{(k)} > 0$ independent of n such that*

$$C_1^{(k)} \frac{1}{n!} (\gamma_1^{(0)})^n n^{-\frac{\nu_K}{\mu(\bar{r}+r\bar{q})}-k} \le \pi_{n,K-k} \le C_2^{(k)} \frac{1}{n!} (\gamma_1^{(0)})^n n^{-\frac{\nu_K}{\mu(\bar{r}+r\bar{q})}-k}, \qquad n \to \infty.$$

for $k = 1, 2, \ldots, K$.

Using the three term expansion for $r_K^{(n)}$

Numerical results

$\mu = 1,\ K = c = 5,\ r = 0.5,\quad p = 0.7,\ q = 0.7$

Table 1 Relative error of $\boldsymbol{r}^{(N)}$ for the case $\bar{r} + r\bar{q} > 0$ ($N = 100$).

Traffic intensity (ρ)	First order	Second order	Third order
0.1	0.078979804	0.006347302	0.000512522
0.2	0.078922701	0.006528123	0.000548023
0.3	0.078865830	0.006708717	0.000584347
0.4	0.078809192	0.006889085	0.000621491
0.5	0.078752783	0.007069227	0.000659455
0.6	0.078696602	0.007249146	0.000698238
0.7	0.078640650	0.007428842	0.000737837
0.8	0.078584923	0.007608316	0.000778252
0.9	0.078529420	0.007787571	0.000819482

Relative error: $\left|\left|r^{(N)} - \hat{r}^{(N)}\right|\right| / ||r^{(N)}||$

- Exact $r^{(N)}$ (Phung-Duc et al. 2010)

Numerical results

$$\mu = 1,\ K = c = 5,\ r = 0.5, \quad p = 0.7,\ q = 0.7$$

Table 2 Relative error of $\boldsymbol{r}^{(N)}$ for the case $\bar{r} + r\bar{q} > 0$ ($N = 1000$).

Traffic intensity (ρ)	First order	Second order	Third order
0.1	0.007711805	0.000061185	0.000000491
0.2	0.007711190	0.000062962	0.000000525
0.3	0.007710574	0.000064739	0.000000560
0.4	0.007709959	0.000066516	0.000000596
0.5	0.007709344	0.000068292	0.000000633
0.6	0.007708729	0.000070068	0.000000671
0.7	0.007708115	0.000071844	0.000000709
0.8	0.007707500	0.000073620	0.000000748
0.9	0.007706887	0.000075395	0.000000788

- Relative error is quite small!!

Part II: $q = r = 1$

Retrial customers never give up

One term expansion

Lemma 5 *We have* $\lim_{n\to\infty} n^k r_i^{(n)}$ $(i = 0, 1, \ldots, K-k-1)$ *and*

$$r_{K-k}^{(n)} = \theta_0^{(k)} \frac{1}{n^k} + o(\frac{1}{n^k}), \qquad k = 1, 2, \ldots, K.$$

where

$$\theta_0^{(1)} = \frac{\lambda p}{\mu}, \qquad \theta_0^{(k)} = \frac{\nu_{K-k+1}}{\mu} \theta_0^{(k-1)}, \qquad k = 2, 3, \ldots, K.$$

Lemma 6 *We have the following result*

$$r_{K-k}^{(n)} = \theta_0^{(k)} \frac{1}{n^k} + O(\frac{1}{n^{k+1}}), \qquad k \in \mathbb{Z}_+.$$

Theorem 4 *We have*

$$r^{(n)}_{K-k} = \sum_{i=0}^{m} \theta_i^{(k)} (-1)^i \frac{1}{n^{k+i}} + O(\frac{1}{n^{k+m+1}}), \qquad m \in \mathbb{N},$$

where $\theta_i^{(k)}$ *is recursively defined as follows.*

$$\begin{aligned}
\theta_m^{(1)} &= \sum_{i=2}^{\min(K,m+1)} \theta_{m+1-i}^{(i)} (-1)^i, \\
\theta_m^{(k)} &= \frac{\nu_{K-k+1}}{\mu} \theta_m^{(k-1)} + \frac{\lambda}{\mu} \theta_{m-2}^{(k+1)} + \frac{\lambda + \nu_{K-k}}{\mu} \theta_{m-1}^{(k)} \\
&\quad + \sum_{j=0}^{m-1} \Phi_j^{(k)} \theta_{m-j-1}^{(0)} (-1)^{j+1}, \qquad k = 2, 3, \ldots, K,
\end{aligned}$$

where

$$\Phi_j^{(k)} = \sum_{i=0}^{j} \theta_i^{(k+1)} (-1)^j \frac{(k+i)_{j-i}}{(j-i)!}.$$

Further definitions

Furthermore, we have

$$\theta_m^{(0)} = -\frac{\lambda}{\nu_K}\theta_{m-1}^{(1)} + \frac{\mu}{\nu_K}\sum_{j=1}^{m}\widetilde{\Phi}_j^{(0)}\theta_{m-j}^{(0)}(-1)^j,$$

where

$$\widetilde{\Phi}_j^{(0)} = \sum_{i=1}^{j}\theta_i^{(1)}\frac{(i)_{j-i}}{(j-i)!}(-1)^j, \qquad j = 1, 2, \ldots, m.$$

Conclusion

- Conclusion
  - Multiserver retrial queues with two type of nonpersistent customers
  - Level-dependent QBD formulation
  - Taylor series expansion for the rate matrices
  - Asymptotic analysis
  - Numerical examples
- Future work
  - More dense rate matrices
  - Work in progress!!