Variable speed branching Brownian motion

Lisa Hartung
(with Anton Bovier)

Institute for Applied Mathematics Bonn

EURANDOM, Eindhoven, 2014

Outline

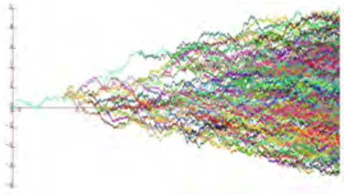
1. Definition variable speed BBM
2. Extremal Process of variable speed BBM
3. Elements of the proof:
   - 1. Step: Extremal Process of two-speed BBM
   - 2. Step: Gaussian Comparison

Definition BBM

- Start a Brownian motion x in 0.
- After an exponential holding time T the particle splits into k offspring (according to a specified probability law).
- Each of these performs independent Brownian motion starting at x(T).
- The new particles are subject of the same splitting rule.

Definition BBM

- Start a Brownian motion x in 0.
- After an exponential holding time T the particle splits into k offspring (according to a specified probability law).
- Each of these performs independent Brownian motion starting at x(T).
- The new particles are subject of the same splitting rule.



Picture by Matt Roberts, Bath

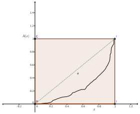
Variable speed BBM

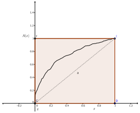
Let $A : [0,1] \to [0,1]$ be increasing. Define

$$\Sigma^2(s) = tA(s/t).$$

Brownian motion with speed function Σ^2

$$B_s^{\Sigma} = B_{\Sigma^2(s)}.$$

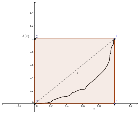


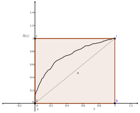


Variable speed BBM

Let $A : [0,1] \to [0,1]$ be increasing. Define

$$\Sigma^2(s) = tA(s/t).$$





Brownian motion with speed function Σ^2

$$B_s^{\Sigma} = B_{\Sigma^2(s)}.$$

Variable speed BBM:
same splitting rules, but if a particle splits at time $s < t$:
law of movement independent copies of $\{B_r^{\Sigma} - B_s^{\Sigma}\}_{t \geq r \geq s}$

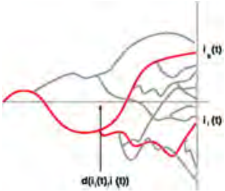
Example for Gaussian process labelled by tree

Example for Gaussian process labelled by tree

- A time-homogeneous tree. Label individuals at time t as $\mathbf{i}_1(t), \dots, \mathbf{i}_{n(t)}(t)$.

Example for Gaussian process labelled by tree

- A time-homogeneous tree. Label individuals at time t as $\mathbf{i}_1(t), \ldots, \mathbf{i}_{n(t)}(t)$.
- Canonical tree-distance: $d(\mathbf{i}_\ell(t), \mathbf{i}_k(t)) \equiv$ time of most recent common ancestor of $\mathbf{i}_\ell(t)$ and $\mathbf{i}_k(t)$

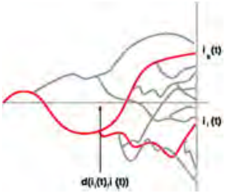


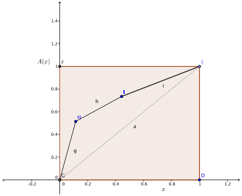
Example for Gaussian process labelled by tree

- A time-homogeneous tree. Label individuals at time t as $\mathbf{i}_1(t), \ldots, \mathbf{i}_{n(t)}(t)$.
- Canonical tree-distance: $d(\mathbf{i}_\ell(t), \mathbf{i}_k(t)) \equiv$ time of most recent common ancestor of $\mathbf{i}_\ell(t)$ and $\mathbf{i}_k(t)$
- For fixed time horizon t, define Gaussian process, $(x_k^t(s), k \leq n(t), s \leq t)$, with covariance

$$\mathbb{E} x_k^t(r) x_\ell^t(s) = tA(t^{-1} d(\mathbf{i}_k(r), \mathbf{i}_\ell(s)))$$

for $A : [0,1] \to [0,1]$, increasing.





Question: Extreme value theory

- Is there a rescaling $u_t(x)$, such that

$$\mathbb{P}\left(\max_{k\leq n(t)} x_k(t) \leq u_t(x)\right) \to F(x)?$$

Question: Extreme value theory

- Is there a rescaling $u_t(x)$, such that

$$\mathbb{P}\left(\max_{k\leq n(t)} x_k(t) \leq u_t(x)\right) \to F(x)?$$

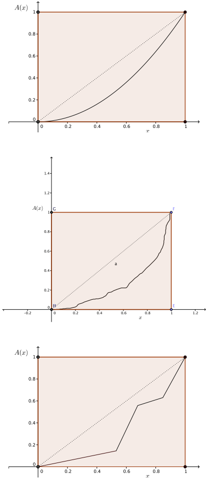
- Is there a limiting extremal process, $\mathcal{P}$, such that

$$\sum_{k\leq n(t)} \delta_{u_t^{-1}(x_k(t))} \to \mathcal{P}?$$

Extremal Process of variable speed BBM

Assumptions on $A : [0,1] \to [0,1]$:

- increasing, $A(0) = 0,\ A(1) = 1$
- below the identity: $A(x) < x$ for $x \in (0,1)$
- $A'(0) = \sigma_b^2 < 1$
- $A'(1) = \sigma_e^2 > 1$



Description of the extremal process

Description of the extremal process

Poisson Point Process: $\mathcal{P}_Y = \sum_{i \in \mathbb{N}} \delta_{p_i} \equiv \mathrm{PPP}\left(C(\sigma_e) Y_{\sigma_b} e^{-\sqrt{2}x} dx \right)$,

where Y_{σ_b} is the limit of a martingale that only depends on σ_b!

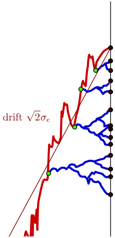
Description of the extremal process

Poisson Point Process: $\mathcal{P}_Y = \sum_{i\in\mathbb{N}} \delta_{p_i} \equiv \mathrm{PPP}\left(C(\sigma_e) Y_{\sigma_b} e^{-\sqrt{2}x} dx \right)$,
where Y_{σ_b} is the limit of a martingale that only depends on σ_b!

Cluster process: $\{\bar{x}_k(t)\}_{k\le n(t)}$ standard BBM,

$$\Delta(t) \equiv \sum_k \delta_{\bar{x}_k(t)-\max_{j\le n(t)} \bar{x}_j(t)}.$$

conditioned on the event $\left\{\max_{j\le n(t)} \bar{x}_j(t) > \sqrt{2}\sigma_e t\right\}$
converges in law to point process, Δ.

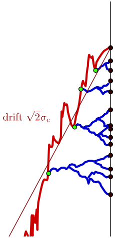


Description of the extremal process

Poisson Point Process: $\mathcal{P}_Y = \sum_{i\in\mathbb{N}} \delta_{p_i} \equiv \mathrm{PPP}\left(C(\sigma_e) Y_{\sigma_b} e^{-\sqrt{2}x} dx \right)$,
where Y_{σ_b} is the limit of a martingale that only depends on σ_b!

Cluster process: $\{\bar{x}_k(t)\}_{k\le n(t)}$ standard BBM,

$$\Delta(t) \equiv \sum_k \delta_{\bar{x}_k(t) - \max_{j\le n(t)} \bar{x}_j(t)}.$$

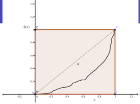


conditioned on the event $\left\{\max_{j\le n(t)} \bar{x}_j(t) > \sqrt{2}\sigma_e t\right\}$
converges in law to point process, Δ.

$$\mathcal{E}_{\sigma_b,\sigma_e} \equiv \sum_{i,j\in\mathbb{N}} \delta_{p_i + \sigma_e \Delta_j^{(i)}}, \qquad \Delta^{(i)} \textbf{ iid copies of } \Delta$$

Convergence of Extremal process

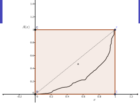
Convergence of Extremal process



Theorem (Bovier, H. '13, '14)

Assume that $A(x) < x, \forall x \in (0,1)$, $A'(0) = \sigma_b^2 < 1$, $A'(1) = \sigma_e^2 > 1$. *Let* $\tilde{m}(t) = \sqrt{2}t - \frac{1}{2\sqrt{2}} \ln t$.

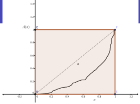
Convergence of Extremal process



Theorem (Bovier, H. '13, '14)

Assume that $A(x) < x, \forall x \in (0,1)$, $A'(0) = \sigma_b^2 < 1$, $A'(1) = \sigma_e^2 > 1$. *Let* $\tilde{m}(t) = \sqrt{2}t - \frac{1}{2\sqrt{2}} \ln t$. *Then*

- $\mathbb{P}\left(\max_{k \leq n(t)} x_k(t) - \tilde{m}(t) \leq x\right) \to \mathbb{E} e^{-C(\sigma_e) Y_{\sigma_b} e^{-\sqrt{2}x}}$



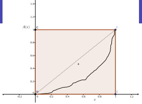
Convergence of Extremal process

Theorem (Bovier, H. '13, '14)

Assume that $A(x) < x, \forall x \in (0,1)$, $A'(0) = \sigma_b^2 < 1$, $A'(1) = \sigma_e^2 > 1$. *Let* $\tilde{m}(t) = \sqrt{2}t - \frac{1}{2\sqrt{2}} \ln t$. *Then*

- $\mathbb{P}\left(\max_{k \le n(t)} x_k(t) - \tilde{m}(t) \le x\right) \to \mathbb{E} e^{-C(\sigma_e) Y_{\sigma_b} e^{-\sqrt{2}x}}$
- $\sum_{k \le n(t)} \delta_{x_k(t) - \tilde{m}(t)} \to \mathcal{E}_{\sigma_b, \sigma_e} = \sum_{i,j} \delta_{p_i + \sigma_e \Delta_j^{(i)}}$

Convergence of Extremal process



Theorem (Bovier, H. '13, '14)

Assume that $A(x) < x, \forall x \in (0,1)$, $A'(0) = \sigma_b^2 < 1$, $A'(1) = \sigma_e^2 > 1$. *Let* $\tilde{m}(t) = \sqrt{2}t - \frac{1}{2\sqrt{2}} \ln t$. *Then*

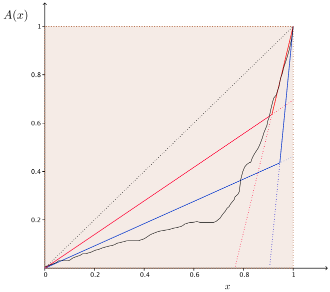
- $\mathbb{P}\left(\max_{k \le n(t)} x_k(t) - \tilde{m}(t) \le x\right) \to \mathbb{E} e^{-C(\sigma_e) Y_{\sigma_b} e^{-\sqrt{2}x}}$
- $\sum_{k \le n(t)} \delta_{x_k(t) - \tilde{m}(t)} \to \mathcal{E}_{\sigma_b, \sigma_e} = \sum_{i,j} \delta_{p_i + \sigma_e \Delta_j^{(i)}}$

Universality of limiting objects:

Only depend on the slope of A at 0 and 1!

- Poisson point process: depends on σ_b^2 through RANDOM VARIABLE Y_{σ_b} and on σ_e through a constant $C(\sigma_e)$.
- Cluster process: Only depends on σ_e !

Idea:
Use comparison for Laplace transforms with two-speed process; only good approximation of covariance near 0 and 1 needed.



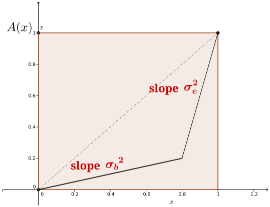
$\Rightarrow$ **Proof in two steps:**

1. **Extremal Process of two speed BBM**
2. **Gaussian Comparison**

Two-speed BBM

Let $\sigma_b^2 < 1$ and $\sigma_e^2 > 1$. Consider the two-speed BBM with speed

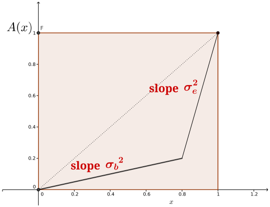
$$\sigma^2(s) = \begin{cases} \sigma_b^2, & \text{for } 0 < s \leq \frac{1-\sigma_e^2}{\sigma_b^2-\sigma_e^2} t, \\ \sigma_e^2, & \text{for } \frac{1-\sigma_e^2}{\sigma_b^2-\sigma_e^2} t < s \leq t, \end{cases}$$



Two-speed BBM

Let $\sigma_b^2 < 1$ and $\sigma_e^2 > 1$. Consider the two-speed BBM with speed

$$\sigma^2(s) = \begin{cases} \sigma_b^2, & \text{for } 0 < s \leq \frac{1-\sigma_e^2}{\sigma_b^2-\sigma_e^2} t, \\ \sigma_e^2, & \text{for } \frac{1-\sigma_e^2}{\sigma_b^2-\sigma_e^2} t < s \leq t, \end{cases}$$



Theorem (Bovier, H. '13)

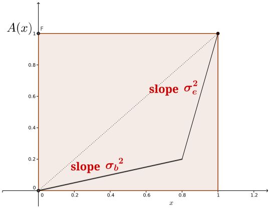
Then

- $\mathbb{P}\left(\max_{k\leq n(t)} x_k(t) - \tilde{m}(t) \leq x\right) \to \mathbb{E}e^{-C(\sigma_e)Y_{\sigma_b}e^{-\sqrt{2}x}}$

Two-speed BBM

Let $\sigma_b^2 < 1$ and $\sigma_e^2 > 1$. Consider the two-speed BBM with speed

$$\sigma^2(s) = \begin{cases} \sigma_b^2, & \text{for } 0 < s \leq \frac{1-\sigma_e^2}{\sigma_b^2-\sigma_e^2} t, \\ \sigma_e^2, & \text{for } \frac{1-\sigma_e^2}{\sigma_b^2-\sigma_e^2} t < s \leq t, \end{cases}$$

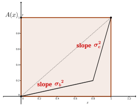


Theorem (Bovier, H. '13)

Then

- $\mathbb{P}\left(\max_{k\leq n(t)} x_k(t) - \tilde{m}(t) \leq x\right) \to \mathbb{E} e^{-C(\sigma_e) Y_{\sigma_b} e^{-\sqrt{2}x}}$
- $\sum_{k\leq n(t)} \delta_{x_k(t)-\tilde{m}(t)} \to \mathcal{E}_{\sigma_b,\sigma_e} = \sum_{i,j} \delta_{p_i+\sigma_e \Delta_j^{(i)}}$

Step 1.1: Localization



Localisation of the particles reaching extreme levels

- at the time of the speed change in a narrow ($\sqrt{t}$) gate around $\sqrt{2}bt\sigma_b^2$
- stay in a tube $\sqrt{2}\sigma_b^2 s \pm O(s^\gamma)$, $\frac{1}{2} < \gamma < 1$ for $s < bt$

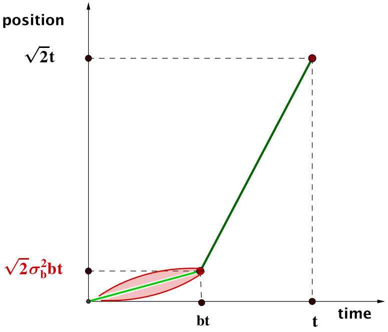
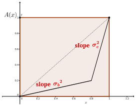


Figure: Path of extremal particle

Step 1.1: Localization



Localisation of the particles reaching extreme levels

- at the time of the speed change in a narrow ($\sqrt{t}$) gate around $\sqrt{2}bt\sigma_b^2$
- stay in a tube $\sqrt{2}\sigma_b^2 s \pm O(s^\gamma), \frac{1}{2} < \gamma < 1$ for $s < bt$

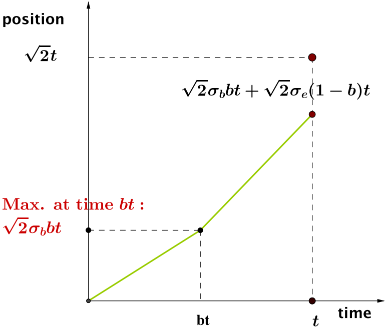
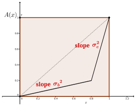


Figure: Path of extremal particle

Step 1.1: Localization



Localisation of the particles reaching extreme levels

- at the time of the speed change in a narrow ($\sqrt{t}$) gate around $\sqrt{2}bt\sigma_b^2$
- stay in a tube $\sqrt{2}\sigma_b^2 s \pm O(s^\gamma), \frac{1}{2} < \gamma < 1$ for $s < bt$

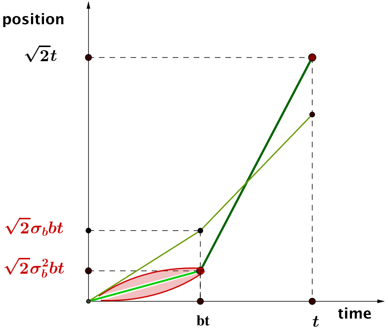
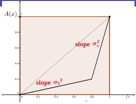


Figure: Path of extremal particle

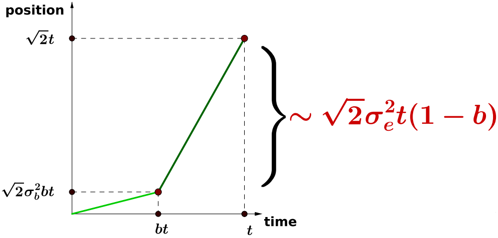


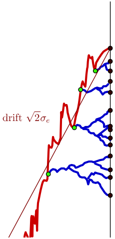
Step 1.2: FKPP-equation

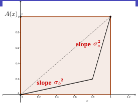
$$\partial_t u(x,t) = \frac{1}{2}\partial_x^2 u(x,t) + u - u^2$$

Asymptotics of solutions of the FKPP equation at very large values ahead of the travelling wave:
$x = \sqrt{2}(\sigma_e - 1)t + o(1)$

$$u(t, \sqrt{2}t + x) \sim C(\sigma_e) t^{-\frac{1}{2}} e^{-\sqrt{2}x - x^2/2t}$$





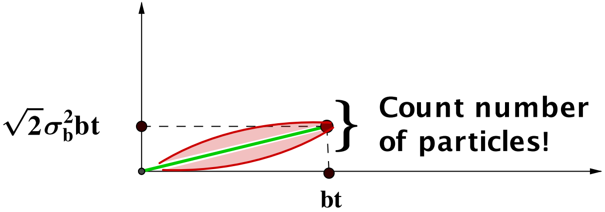


Step 1.3: Martingale Convergence

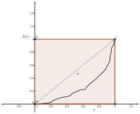
Let $\bar{x}_k(s), k \leq n(s)$ be particles of a standard BBM .
Show convergence of the McKean martingale

$$Y_{\sigma_b}(s) \equiv \sum_{i=1}^{n(s)} e^{-s(1+\sigma_b^2)+\sqrt{2}\sigma_b \bar{x}_i(s)}.$$

For $\sigma_b < 1$ $Y_{\sigma_b}(s)$ is uniformly integrable!
Shown by truncated second moment method.



Step 2: Convergence of Extremal Process for general A



Tightness of extremal process: ✓

Convergence of finite dimensional distributions:

for $u \in \mathbb{R}$,

$$\mathcal{N}_u(t) = \sum_{i=1}^{n(t)} \mathbb{1}_{x_i(t) - \tilde{m}(t) > u}.$$

Lemma

For all $k \in \mathbb{N}$ and $u_1, \ldots, u_k \in \mathbb{R}$

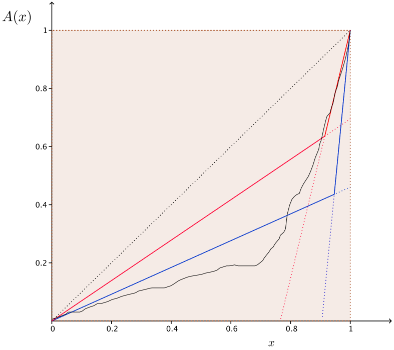
$$\{\mathcal{N}_{u_1}(t), \ldots, \mathcal{N}_{u_k}(t)\} \xrightarrow{d} \{\mathcal{N}_{u_1}, \ldots, \mathcal{N}_{u_k}\}$$

as $t \uparrow \infty$.

Step 2.1: Gaussian Comparison

2)For general A that satisfies assumption:

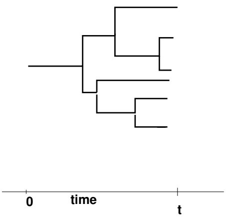
To establish convergence of finite dimensional distributions use Gaussian comparison! Only good approximation at 0 and 1 needed!



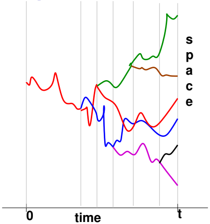
Step 2.1: Definition of auxiliary two-speed processes

Define two-speed BBM's with the same underlying Galton Watson tree:

$$(\overline{y}_1, \dots, \overline{y}_{n(t)})$$
$$(\underline{y}_1, \dots, \underline{y}_{n(t)})$$



Galton–Watson tree



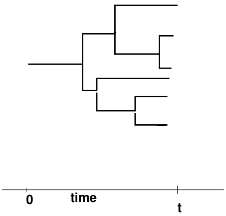
Branching Bownian motion

- first order Taylor expansion around 0 and upper respectively lower bound the remainder!
- the same at 1 and bound remainder from below respectively above!

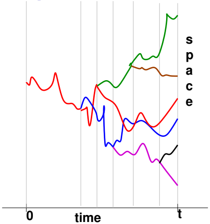
Step 2.1: Definition of auxiliary two-speed processes

Define two-speed BBM's with the same underlying Galton Watson tree:

$$\left(\overline{y}_1, \dots, \overline{y}_{n(t)}\right)$$
$$\left(\underline{y}_1, \dots, \underline{y}_{n(t)}\right)$$



Galton–Watson tree



Branching Bownian motion

- first order Taylor expansion around 0 and upper respectively lower bound the remainder!
- the same at 1 and bound remainder from below respectively above!

Lemma

The extremal processes of $\left(\overline{y}_1, \dots, \overline{y}_{n(t)}\right)$ *and* $\left(\underline{y}_1, \dots, \underline{y}_{n(t)}\right)$ *BOTH converge to* $\mathcal{E}_{\sigma_b,\sigma_e}$.

Step 2.3: Gaussian Comparison

We want to compare the Laplace functionals of original process and $(\overline{y}_1, \dots, \overline{y}_{n(t)})$!
$\Rightarrow$ function of particle positions at time t!
Compare the difference

$$\mathbb{E}_B \left(f(x_1(t), \dots, x_{n(t)}(t)) \right) - \mathbb{E}_B \left(f(\overline{y}_1(t), \dots, \overline{y}_{n(t)}(t)) \right),$$

where $\mathbb{E}_B$ denotes expectation w.r.t. particle movement (tree fixed).

Step 2.3: Gaussian Comparison

We want to compare the Laplace functionals of original process and $(\bar{y}_1, \ldots, \bar{y}_{n(t)})$!
$\Rightarrow$ function of particle positions at time t!
Compare the difference

$$\mathbb{E}_B \left(f(x_1(t), \ldots, x_{n(t)}(t)) \right) - \mathbb{E}_B \left(f(\bar{y}_1(t), \ldots, \bar{y}_{n(t)}(t)) \right),$$

where $\mathbb{E}_B$ denotes expectation w.r.t. particle movement (tree fixed).
Using the interpolating process with speed function

$$\Sigma_h^2(s) = h\Sigma^2(s) + (1-h)\overline{\Sigma}^2(s).$$

this is equal to

$$\mathbb{E}_B \left(\int_0^1 \frac{d}{dh} f(x^h(t)) dh \right).$$

Step 2.3: Gaussian Comparison

Now as in the normal Gaussian comparison, we would get

$$\sum_{\substack{i,j=1\\ i\neq j}}^{n(t)} \left[\mathbb{E}_B(x_i(t)x_j(t)) - \mathbb{E}_B(\overline{y}_i(t)\overline{y}_j(t))\right] \mathbb{E}_B\left(\frac{\partial^2 f(x^h(t))}{\partial x_i \partial x_j}\right)$$

Looks like a second moment! Would like to take expectation w.r.t tree structure and simple bounds...!

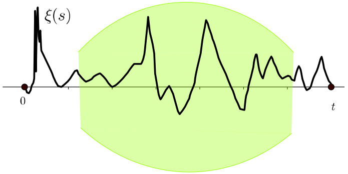
Second moment type computation

BUT that has to be done in a clever way:

Introduce localization [Needs justification!!!]

$$\sum_{\substack{i,j=1\\ i\neq j}}^{n(t)} \left[\mathbb{E}_B(x_i(t)x_j(t)) - \mathbb{E}_B(\bar{y}_i(t)\bar{y}_j(t))\right] \mathbb{E}_B\left(\mathbb{1}_{x_i^h \in \mathcal{T}^{\gamma}_{t,\bar{I},\Sigma_h^2}} \frac{\partial^2 f(x^h(t))}{\partial x_i \partial x_j}\right)$$

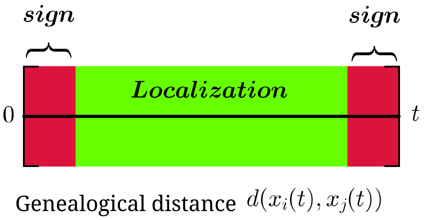
Localization of Brownian bridge



Motion of single particle
= Time change of BM

$$B_s^{\Sigma} = B_{\Sigma^2(s)}.$$

Monotonicity around 0 and t



Green region: Using localization show that terms in sum are $o(1)$.

Monotonicity around 0 and t

Genealogical distance $d(x_i(t), x_j(t))$

Green region: Using localization show that terms in sum are $o(1)$.

Red region:

$$\mathbb{E}_B(x_i(t)x_j(t)) - \mathbb{E}_B(\overline{y}_i(t)\overline{y}_j(t)) < 0$$

and

$$\frac{\partial^2 f(x^h(t))}{\partial x_i \partial x_j} \geq 0!$$

$\Rightarrow$ Upper and lower bound on corresponding terms in the sum!

Thank you for your attention!