Does noisy environment facilitate extinction, or stabilize population?

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PRL **101**, 268103 (2008);
PRL **107**, 180603 (2011);

Eindhoven, 2014

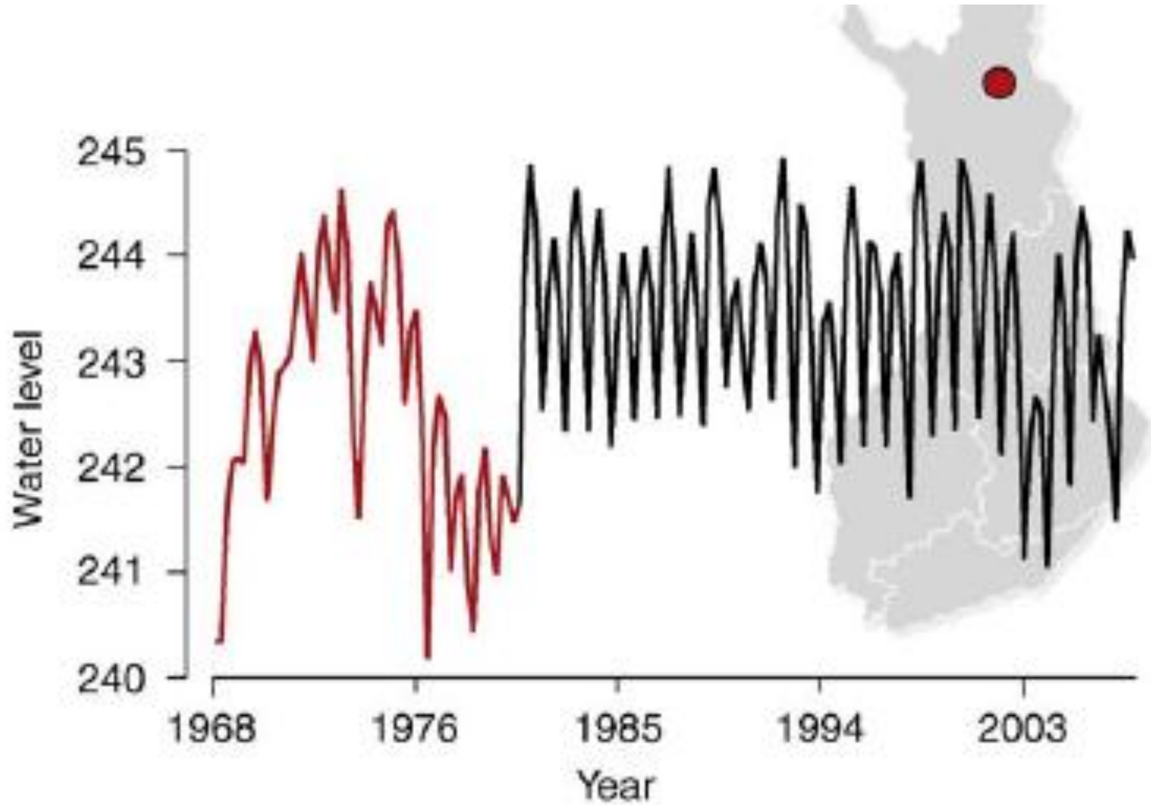
Noise in reaction models

- ✓ Internal, due to discreteness of agents (demographic, shot…)
- ✓ External, due to rate variations (environmental, bath…)

Both affect large deviations statistics

Environmental noise

Temporal variation in the water level (m above sea level) at Lokka reservoir (67849’N, 27844’E) in northern Finland during 1968–2008.



Extinction in time-varying environment

Extinction rates of populations of different sizes, compared with those expected in a constant environment

(a) *Bird populations in the California Channel Islands* (*Jones & Diamond*, 1976)				
Number of breeding pairs per population	4–10	11–30	31–60	over 60
Proportion of populations dying out within 80 years	0·30	0·09	0·15	0·02
(b) *Theoretical populations in a constant environment* ($r = 0{\cdot}2$)				
Equilibrium population size (individuals)	$12\frac{1}{2}$	25	50	100
Chance of dying out within 80 generations	0·73	0·18	0·002	5×10^{-7}

J. theor. Biol. (1981) **90, 213–239**

Logistic model

$A \rightarrow 0$ rate $d_n = d$

$A \rightarrow 2A$ rate $b_n = b - cn$

rate equation : $\dot{n} = (b-d)n - cn^2$

stationary solution : $N = \dfrac{b-d}{c}$

$$\dot{P}_n = d_{n+1}P_{n+1} - d_n P_n + b_{n-1}P_{n-1} - b_n P_n$$
$$= (e^{\partial_n} - 1)d_n P_n + (e^{-\partial_n} - 1)b_n P_n = \hat{H}(n, \partial_n)P_n$$

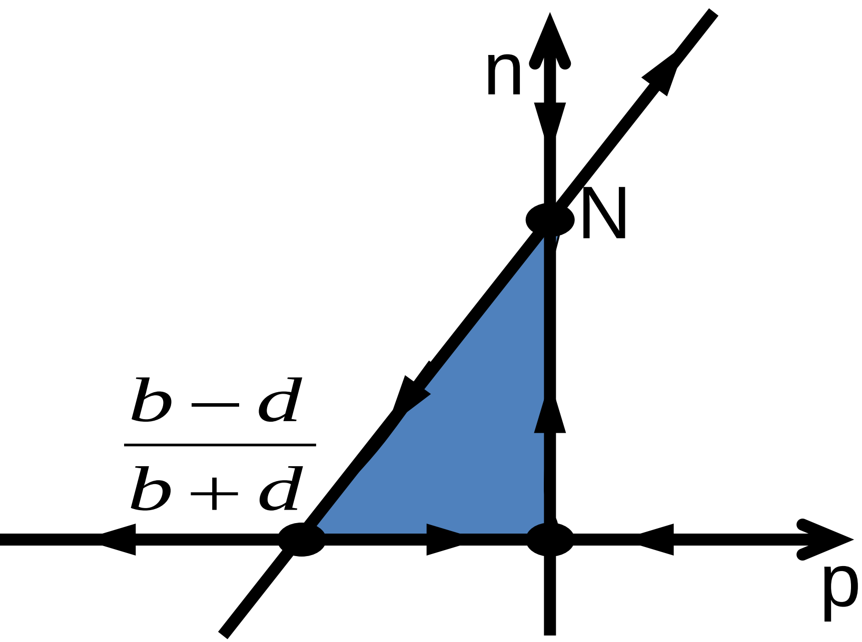
"Hamiltonian": $H(n,p) = (e^p - 1)d(n) + (e^{-p} - 1)b(n)$

"Quantum mechanics" of populations

evolution operator: $$\hat{U}_{n_f n_i} = \int_{n_i}^{n_f} D[n,p]\, e^{\int_i^f dt[p\dot{n} - H(n,p)]}$$

large fluctuations – stationary trajectories (WKB)

Dykman, et.al. (1994)
Freidlin, Wentzel (1969)



Elgart, AK (2004).

$$\tau_{\text{ext}} = e^{\frac{1}{2}\frac{b-d}{b+d}N}$$

exponential with the system size N

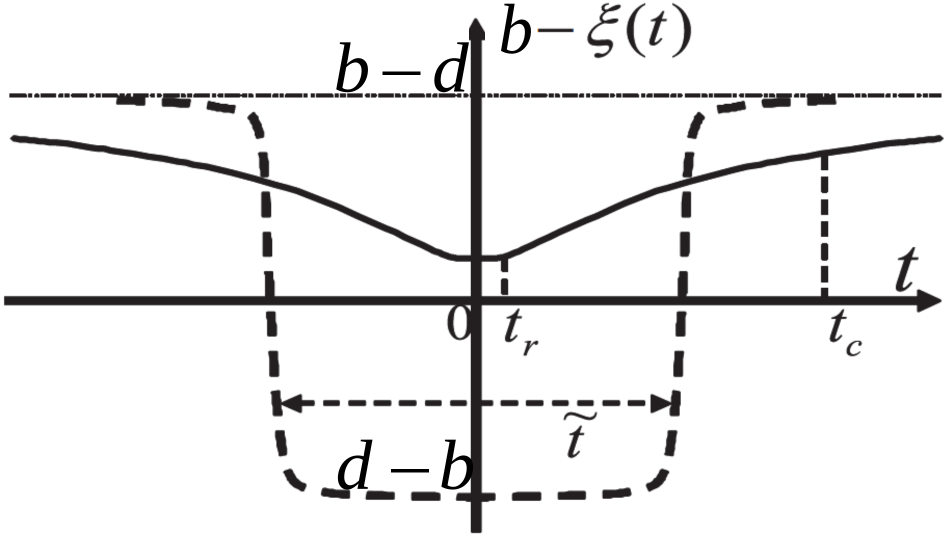
Extinction in the presence of noise

$$b \to b - \xi(t)$$

$$\mathcal{P}[\xi(t)] \propto \exp\{-S[\xi(t)]\}$$

$$S[\xi(t)] = \frac{1}{4v} \int dt(t_c \dot{\xi}^2 + t_c^{-1} \xi^2)$$

optimal noise realisation :

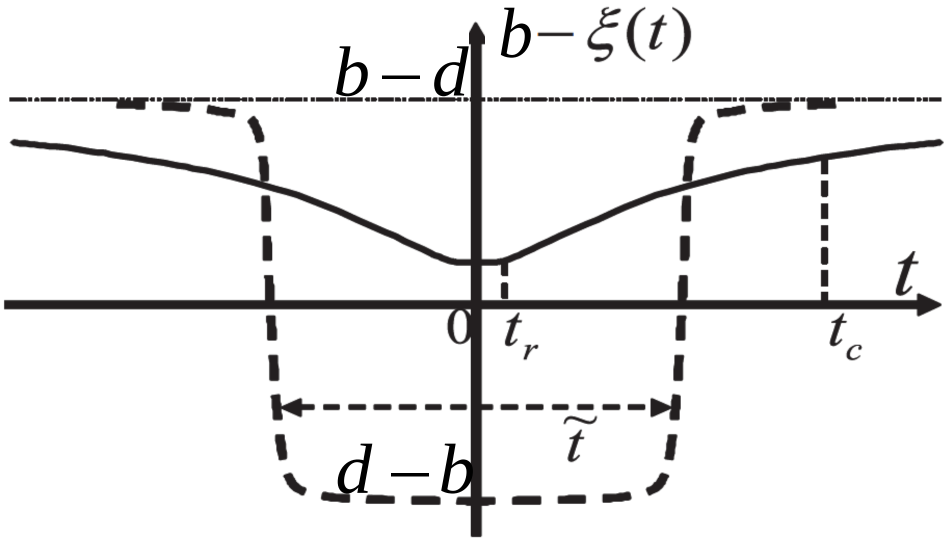


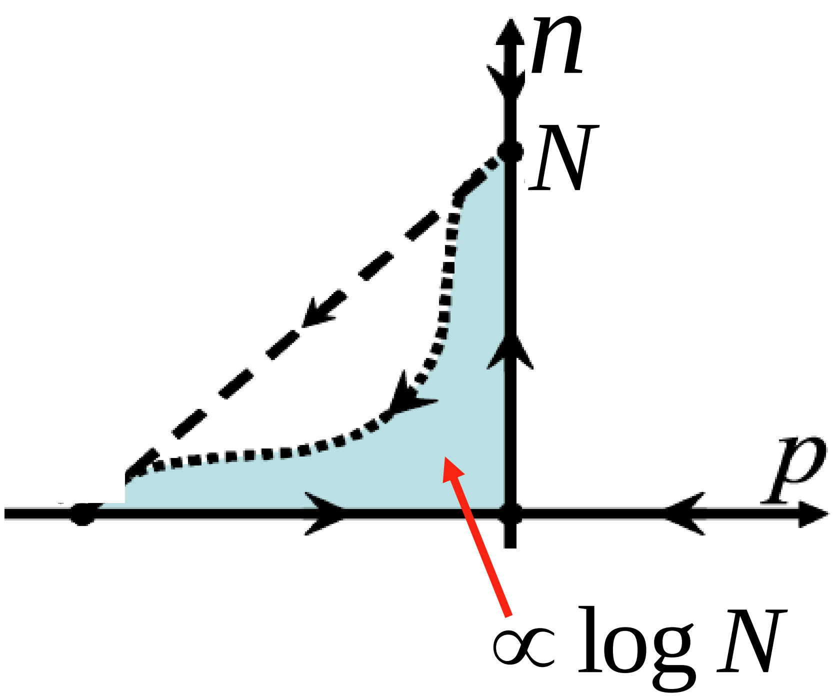
$$\dot{n} = \partial_p H(n, p, \xi);$$

$$\dot{p} = -\partial_n H;$$

$$t_c^2 \ddot{\xi} + \xi = 2 v t_c \partial_\xi H$$

Accelerated extinction



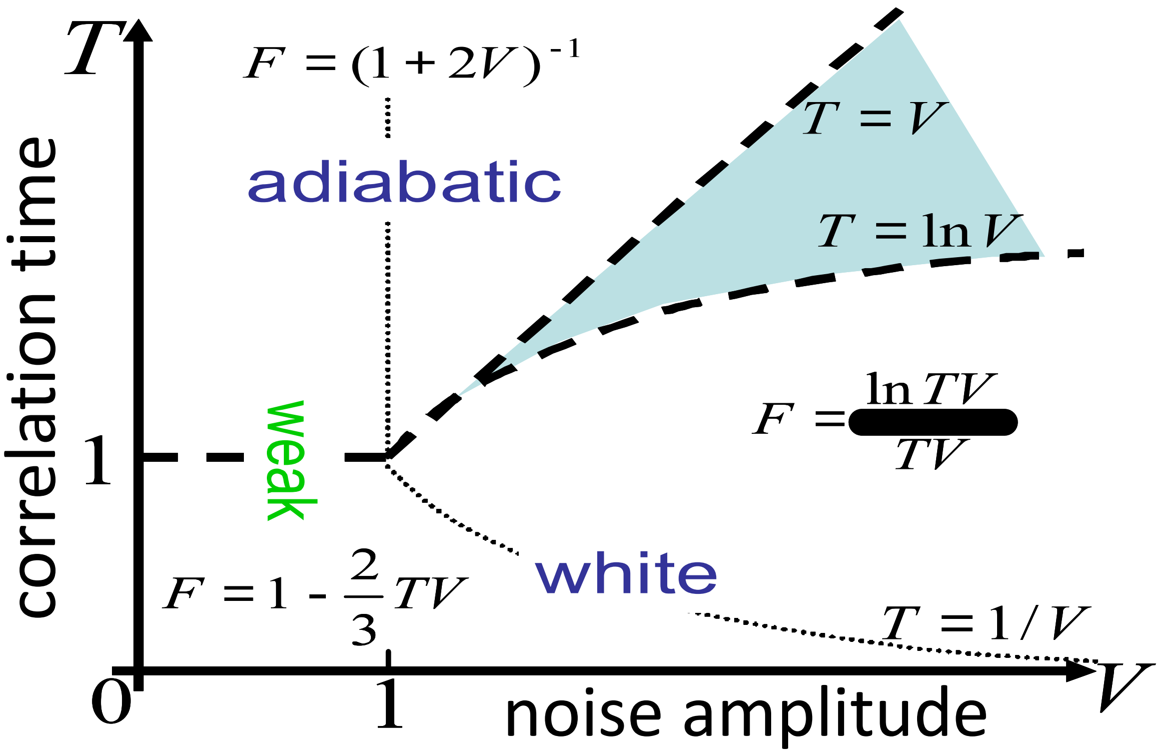


$$\tau_{\text{ext}} \propto N^{\gamma} \qquad \gamma = \frac{b-d}{v t_c}$$

power-law with the system size N

Leigh (1981)
AK, Meerson, Shklovskii (2008)

Phase Diagram



AK, Meerson, Shklovskii (2008)

$$\ln \tau_\xi = F(V, T) \ln \tau_0$$

However: Noise may also stabilize populations

Parker, Meerson, AK (2011)

Well-known analogs in **equilibrium** context:

- ✓ Coleman-Weinberg effect in quantum gauge theories
- ✓ Order by disorder phenomena in classical phase transitions

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Radiative Corrections as the Origin of Spontaneous Symmetry Breaking*

Sidney Coleman

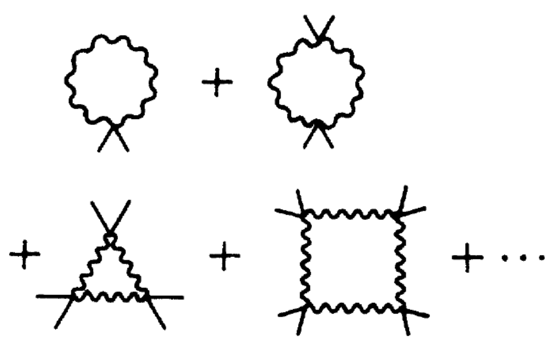
and

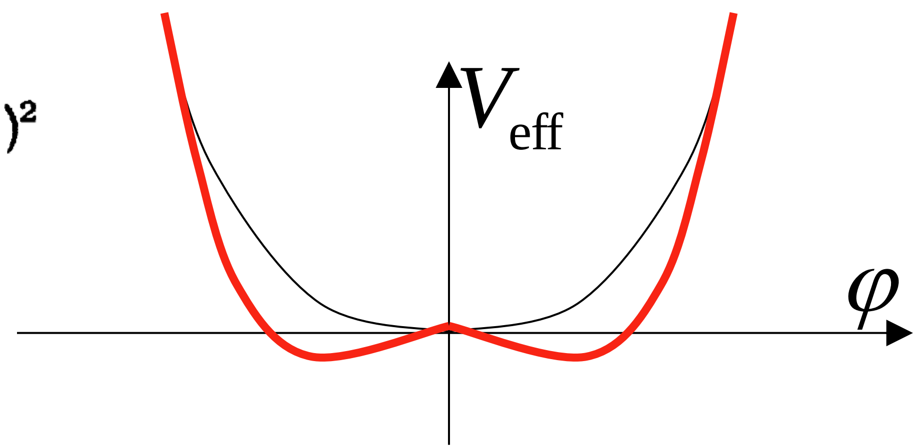
Erick Weinberg
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 8 November 1972)

We investigate the possibility that radiative corrections may produce spontaneous symmetry breakdown in theories for which the semiclassical (tree) approximation does not indicate such breakdown. Massless scalar electrodynamics does not remain massless, nor does it remain electrodynamics;

$$\mathcal{L} = -\tfrac{1}{4}(F_{\mu\nu})^2 + \tfrac{1}{2}(\partial_\mu\varphi_1 - eA_\mu\varphi_2)^2 + \tfrac{1}{2}(\partial_\mu\varphi_2 + eA_\mu\varphi_1)^2 - \frac{\lambda}{4!}(\varphi_1{}^2 + \varphi_2{}^2)^2$$





Classical: order by disorder

[19] J. Villain, R. Bidaux, J. P. Carton, and R. Conte, J. Phys. (Paris) **41**, 1263 (1980).

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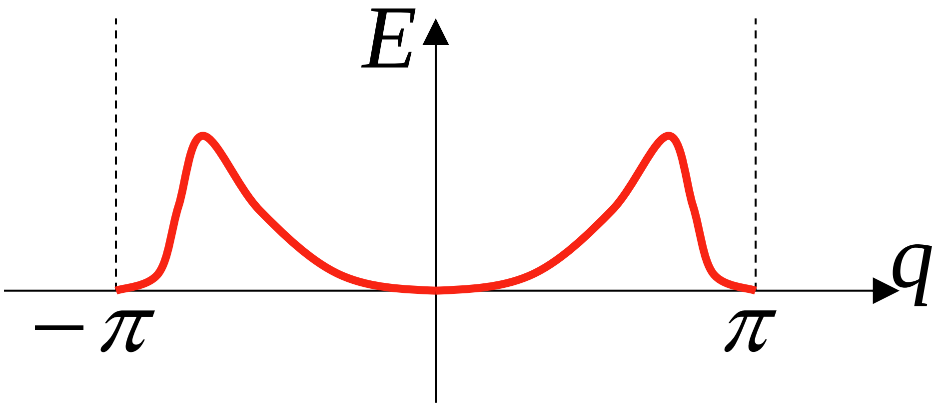
Ordering Due to Disorder in a Frustrated Vector Antiferromagnet

Christopher L. Henley

In many continuous spin systems, competing interactions give nontrivial degeneracies of the classical ground states. Degeneracy-breaking free-energy terms arise from thermal (or quantum) fluctuations,



Increasing temperature drives the system into a stable **ordered** state

Neutral genetic drift

$$A+B \leftrightarrow 2A \qquad \mathrm{A+B \leftrightarrow 2B}$$

fast mutations, conserving total number

$$S \rightarrow 0 \qquad \mathrm{2A \rightarrow 2A+S} \qquad \mathrm{2B \rightarrow 2B+S}$$

S is either A or B; slow non-conserving processes

$x = n_A + n_B$ slow variable, close to bifurcation

$y = n_A - n_B$ fast variable, mean-field $y \rightarrow 0$

The model

$$x = n_A + n_B \qquad y = n_A - n_B$$

$$\dot{y} = -2y + \xi_y(t)$$

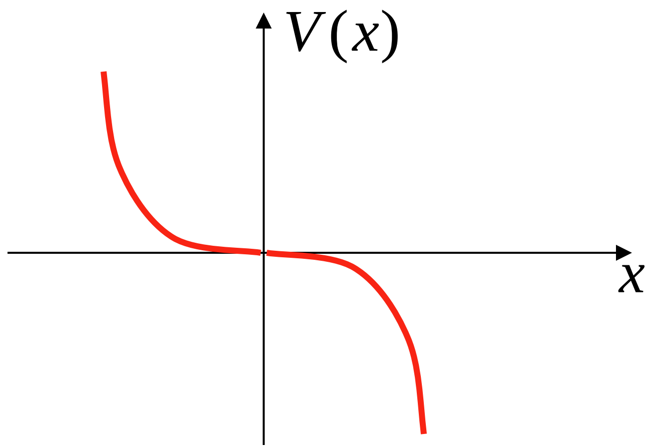
entropic drift + strong fluctuations

$$\dot{x} = -V'(x) - y^2 + \xi_x(t)$$

drift, feedback,
weak fluctuations

$$\langle \xi_{x(y)}(t)\xi_{x(y)}(t') \rangle = 2T_{x(y)}\delta(t - t')$$

$$T_x << T_y$$

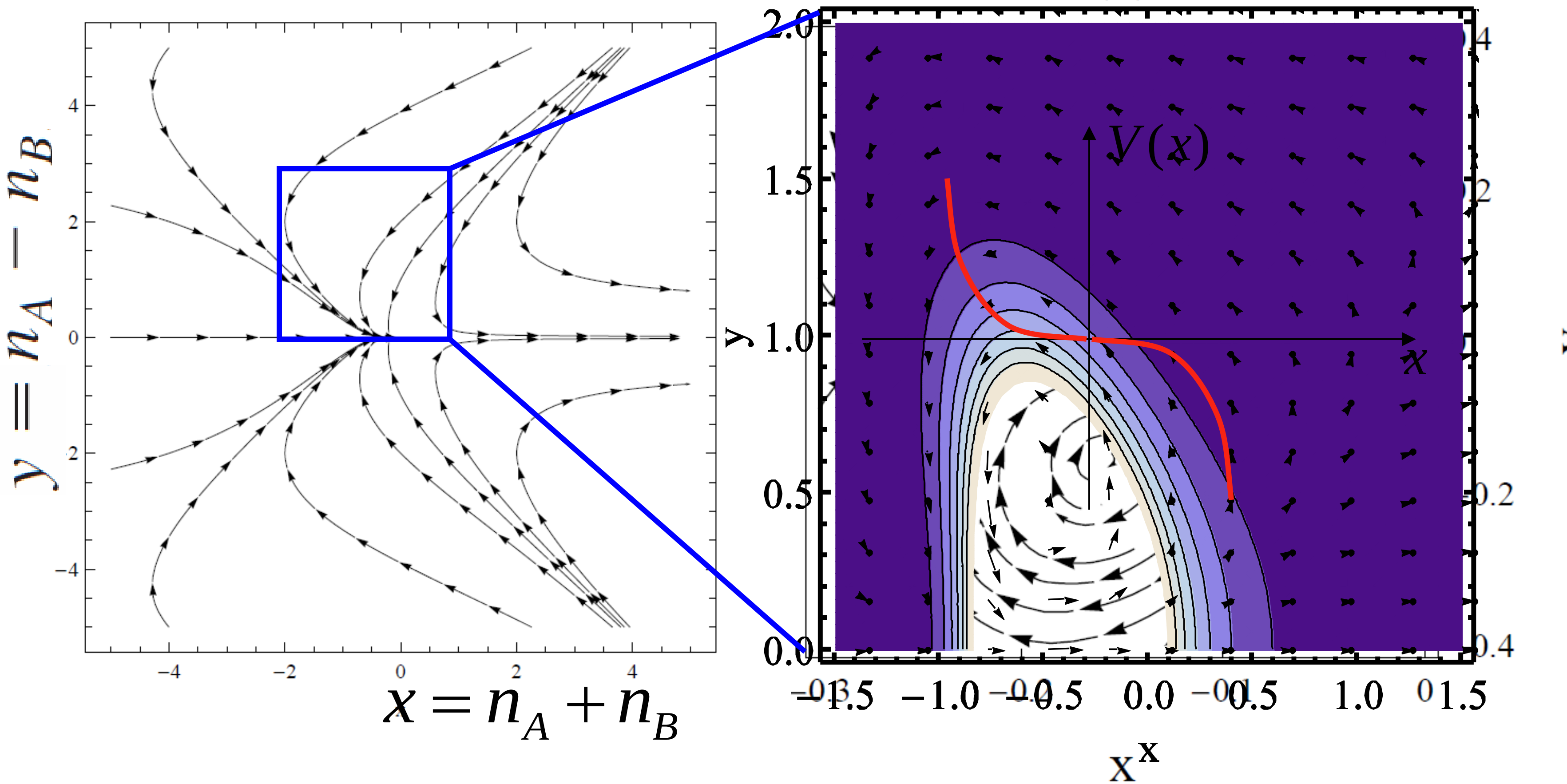


$$V(x) \propto -x^3$$

"massless"

Deterministic and stochastic evolution

$$\dot{x} = -V'(x) - y^2 + \xi_x(t), \qquad \dot{y} = -2y + \xi_y(t)$$



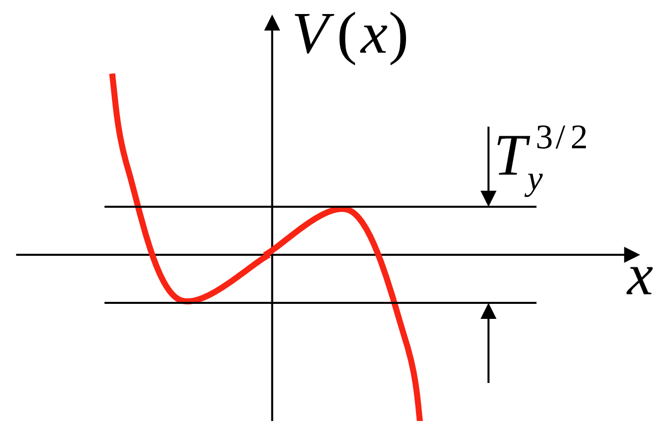
Lifetime

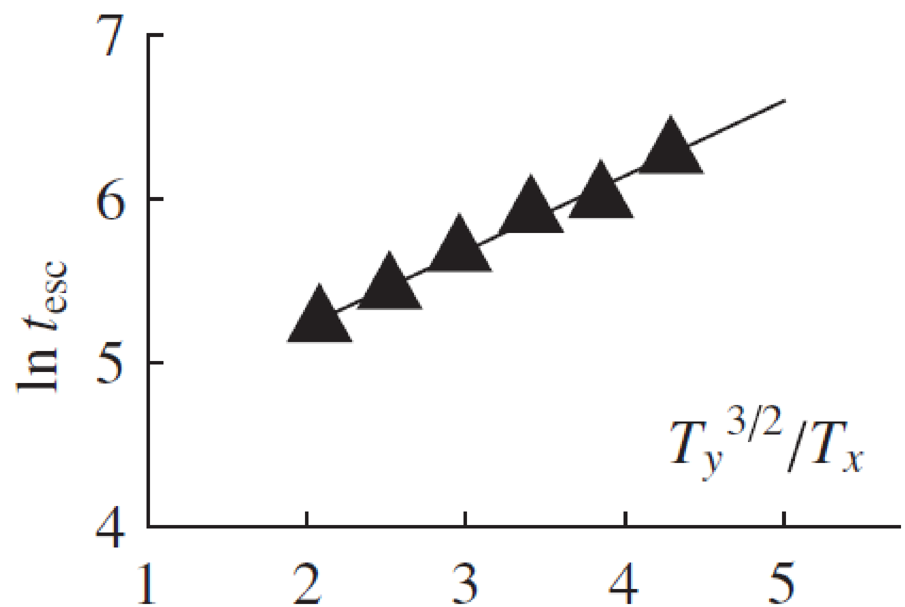
$$\dot{x} = -V'(x) - \underbrace{\langle y^2 \rangle}_{T_y} + \xi_x(t)$$

$$V(x) \rightarrow -x^3 + T_y x$$

$$\tau_{\text{esc}} = e^{\frac{T_y^{3/2}}{T_x}}$$

y-fluctuations lead to exponential stability





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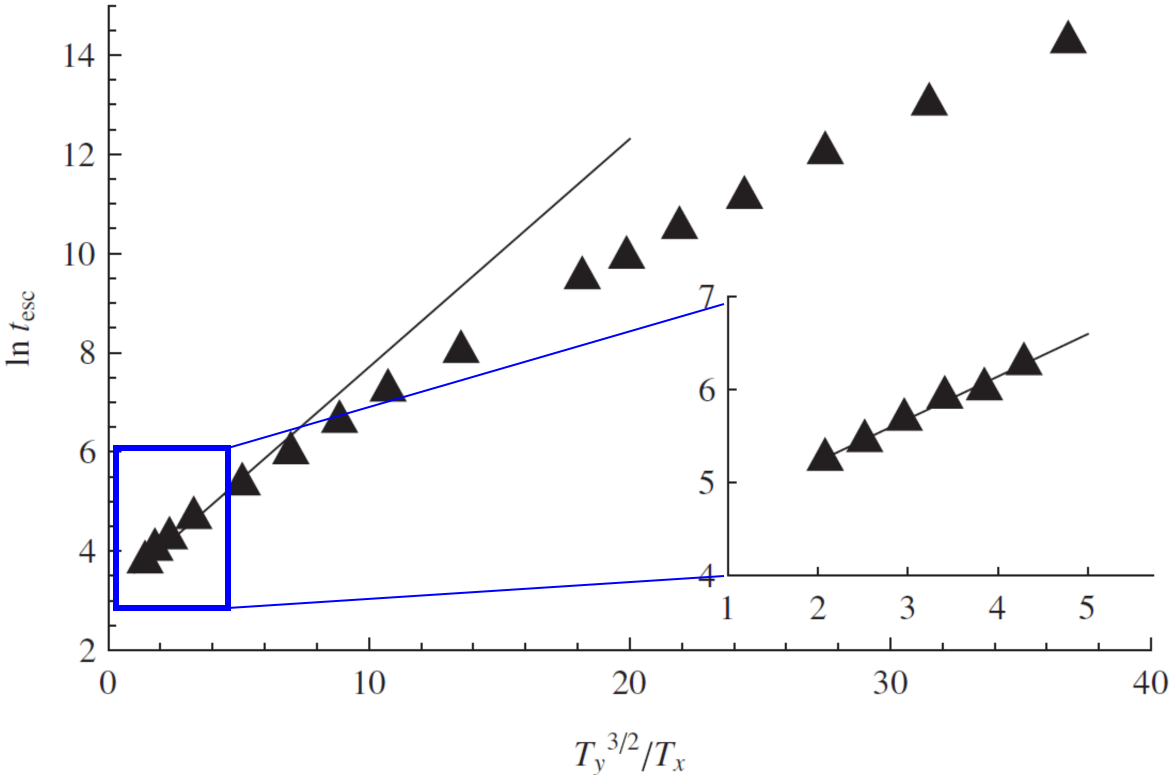
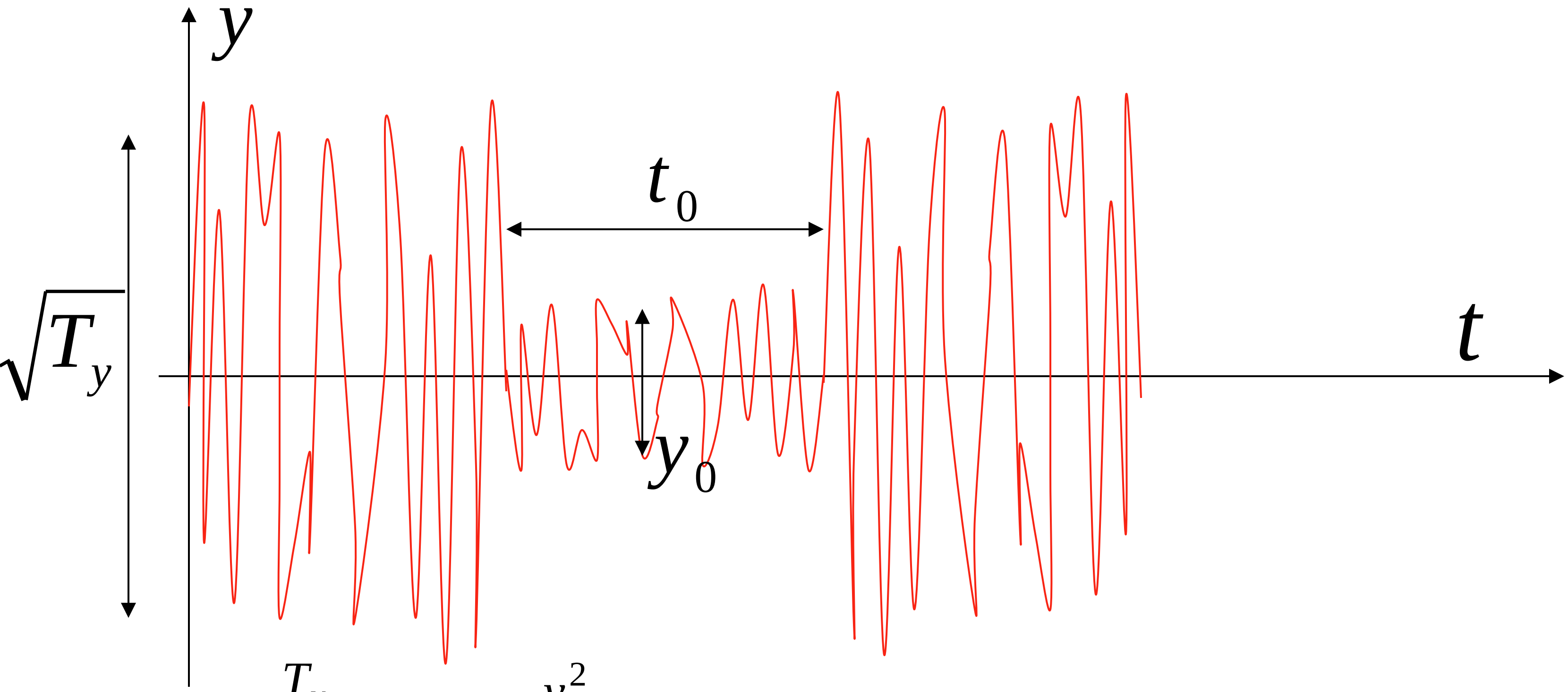


14
12
10
8
6
4
2
0
10
20
30
40
$\ln t_{esc}$
$T_y^{3/2}/T_x$
7
6
5
4
1
2
3
4
5

Rare events

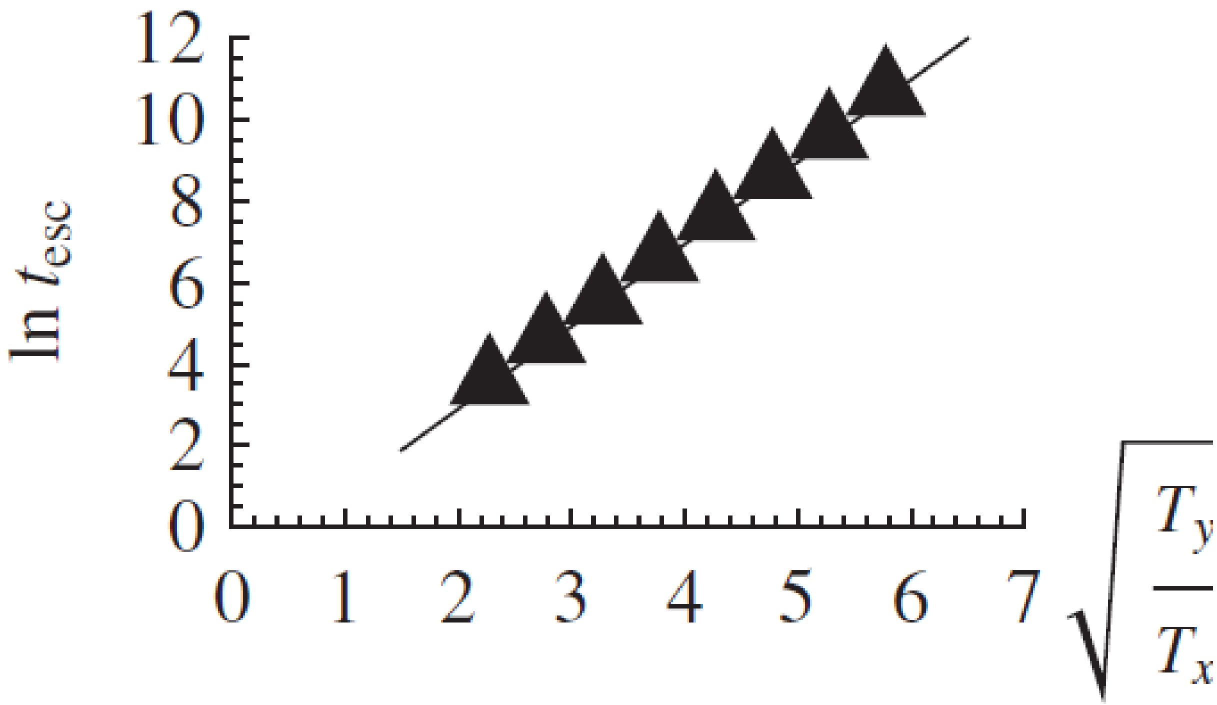


$$e^{-\frac{T_y}{y_0^2}t_0} \times e^{-\frac{y_0^2}{T_x t_0}}$$

$$\text{optimization}: \quad \frac{y_0^2}{t_0} \propto \sqrt{T_y T_x}$$

$$\tau_{\text{esc}} = e^{\sqrt{\frac{T_y}{T_x}}}$$

!!!

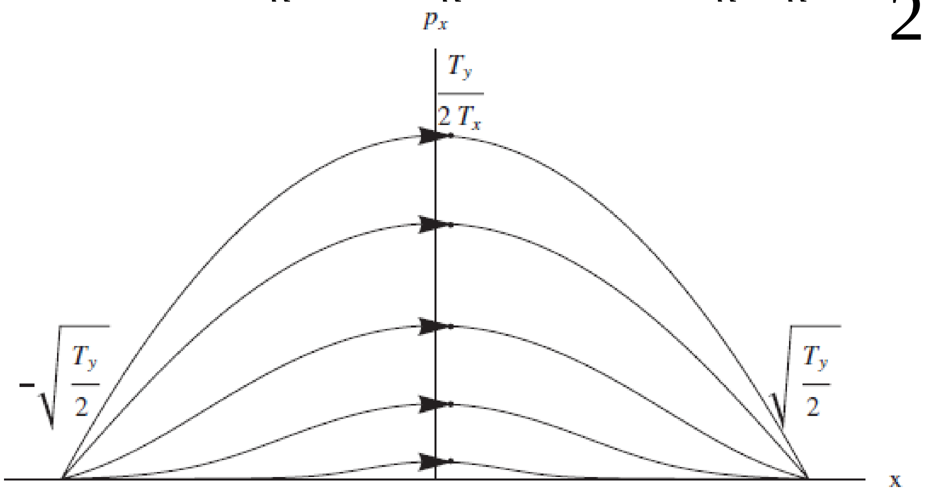

12
10
8
6
4
2
0
ln t_{esc}
0 1 2 3 4 5 6 7
$\sqrt{\frac{T_y}{T_x}}$

Theory

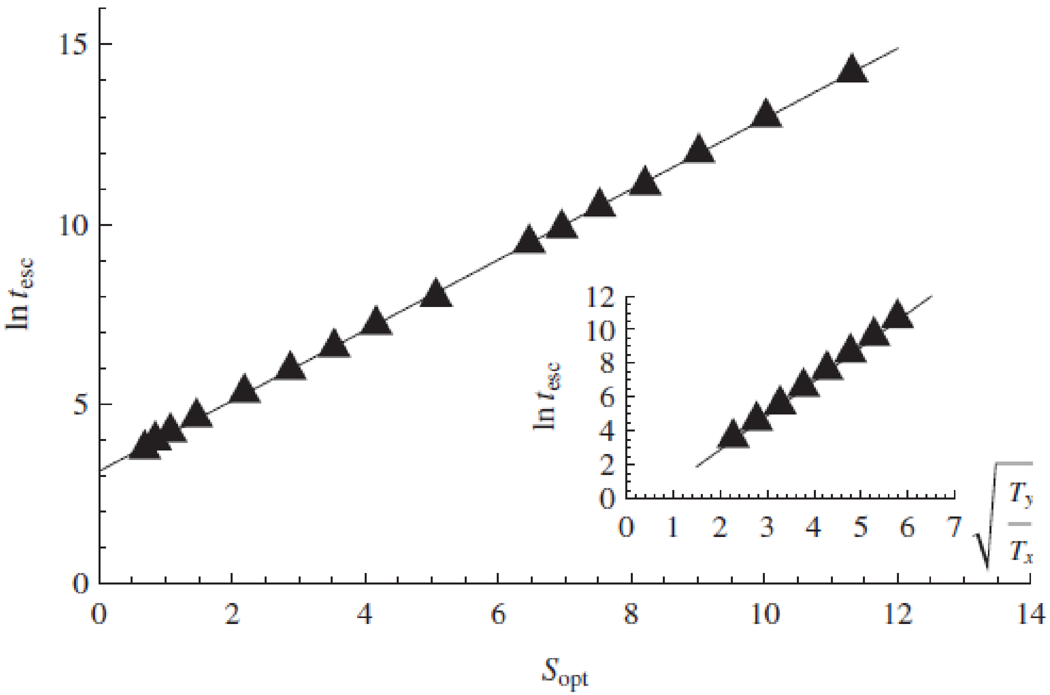
$$S = p_x\dot{x} + p_y\dot{y} + p_x\left[V'(x) + y^2\right] + 2p_y y + T_x p_x^2 + T_y p_y^2$$

$$= p_x\dot{x} + p_x V'(x) + T_x p_x^2 + \begin{pmatrix} y & p_y \end{pmatrix} \begin{bmatrix} p_x(t) & -\partial_t + 1 \\ \partial_t + 1 & T_y \end{bmatrix} \begin{pmatrix} y \\ p_y \end{pmatrix}$$

$$\to p_x\dot{x} + p_x V'(x) + T_x p_x^2 - \frac{1}{2}\mathrm{Tr}\log\begin{bmatrix} p_x(t) & -\partial_t + 1 \\ \partial_t + 1 & T_y \end{bmatrix}$$



Results



$$\log \tau_{esc} \propto \begin{cases} T_y^{3/2} / T_x\,; & T_y < \sqrt{T_x} \\ \sqrt{T_y / T_x}\,; & T_y > \sqrt{T_x} \end{cases}$$

Last transparency of the day!

Environmental noise may greatly accelerate the extinction. It may even change exponential scaling of the extinction time to a power-law.

Noise may trap population in an exponentially long-lived quasi-stationary state. (Coleman-Weinberg, or “order by disorder”mechanism.)