Is there more biodiversity in non-homogeneous environments than in homogeneous ones?

(in progress + S. Kliem, ALEA, Lat. Am. J. Probab. Math. Stat. 11 (1), 43–140. 2014.)

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Joint work with Andreas Greven, Frank den Hollander

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Challenge

Need

- mathematical models for (**forwards in time**) evolution of spatially structured populations, **and**
- consistent (**backwards in time**) models for genealogies (data, inference!).

This talk

- Introduce a new model for **structured populations**, based on interacting **Cannings processes**.
- Explain **duality techniques** to study the large space-time scale behaviour of the system ($\rightsquigarrow$ **backwards in time models**).
- Derive the **renormalisation transformation** that connects the behaviour on successive space-time scales.
- Consider this model in **inhomogeneous environment**.

Renormalisation program

- **Probabilistic** part (construction, duality, scaling).
- **Analytic** part (renormalisation mapping, its orbits, attractors).

Substantial literature on renormalisation of **diffusive spatial models** (T. Cox, D. Dawson, A. Greven, F. den Hollander, R. Sun, J. Swart, J. Vaillancourt, et al.)

This talk:

- Universality for a class of **non-diffusive** spatio-temporal models with **jumps**.

Geographically structured colonies of individuals

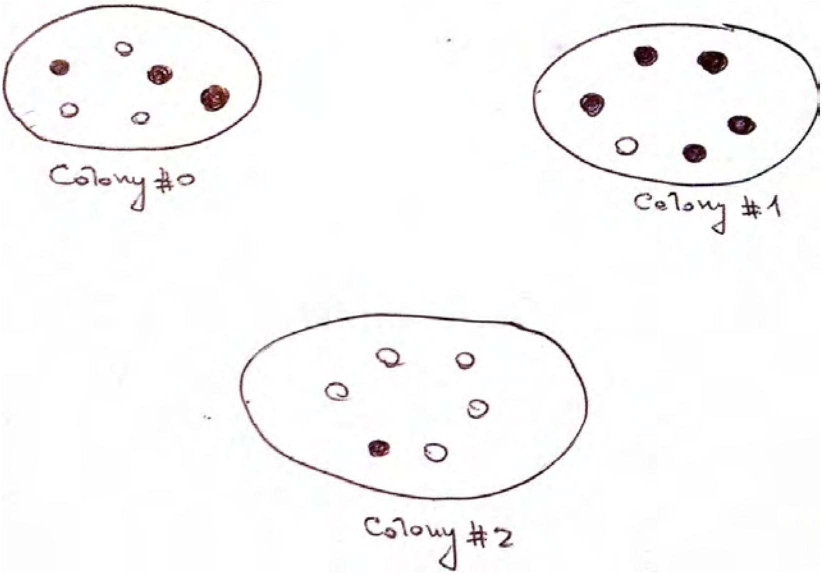


Figure: Colonies with individuals

Migration

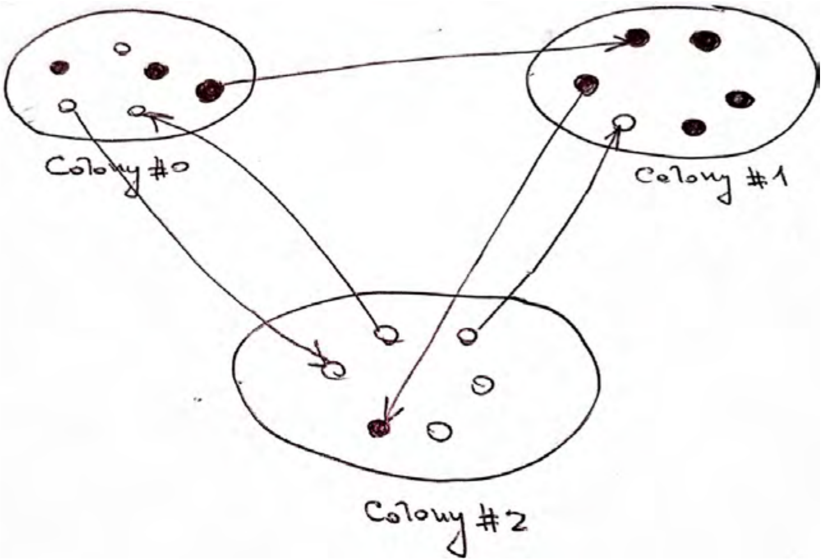


Figure: Random walk between the colonies

More geographical structure...

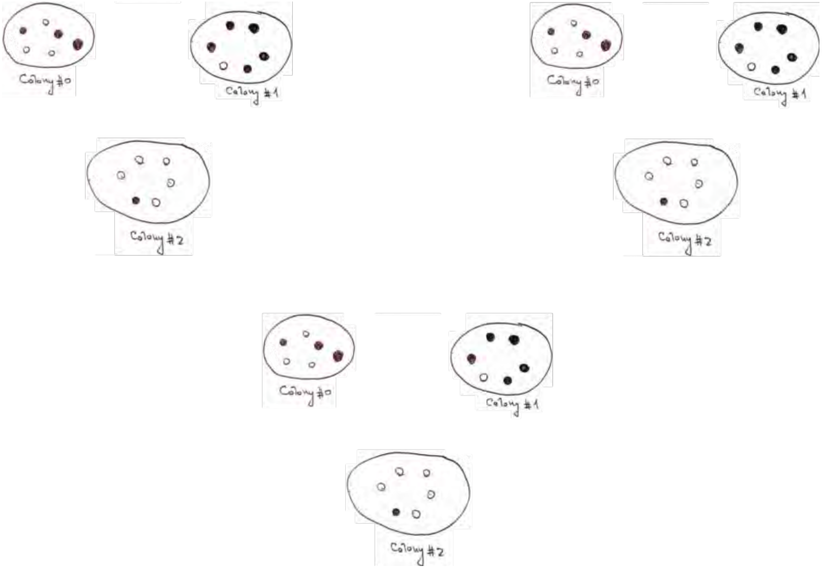
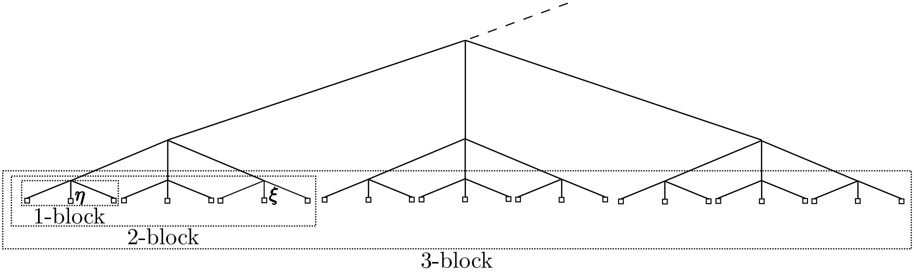


Figure: Another level of spatial structure

Hierarchical geography (Felsenstein, Sawyer)



- **Hierarchical group**:
  $\Omega_N = \left\{ \eta = (\eta^l)_{l\in\mathbb{N}_0} \in \{0,1,\ldots,N-1\}^{\mathbb{N}_0} : \; \sum_{l\in\mathbb{N}_0} \eta^l < \infty \right\}$.
- Branching parameter $N \in \mathbb{N}$ (regular tree).
- **Distance:** $d(\eta,\zeta) = \min\{k \in \mathbb{N}_0 : \; \eta^l = \zeta^l$, for all $l \geq k\}$, $\eta, \zeta \in \Omega_N$.
- **Topology:** $B_k(\eta) = \{\zeta \in \Omega_N : \; d(\eta,\zeta) \leq k\}$, $\eta \in \Omega_N$, $k \in \mathbb{N}_0$

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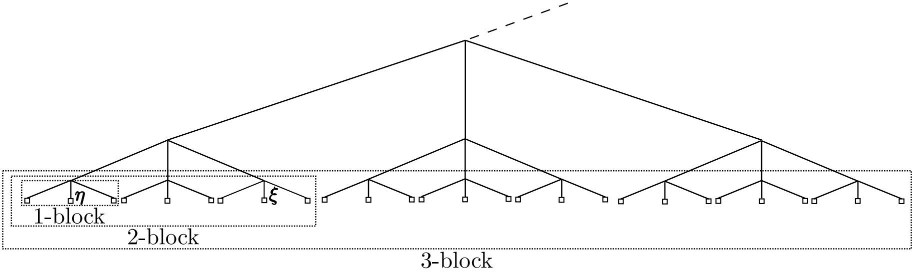

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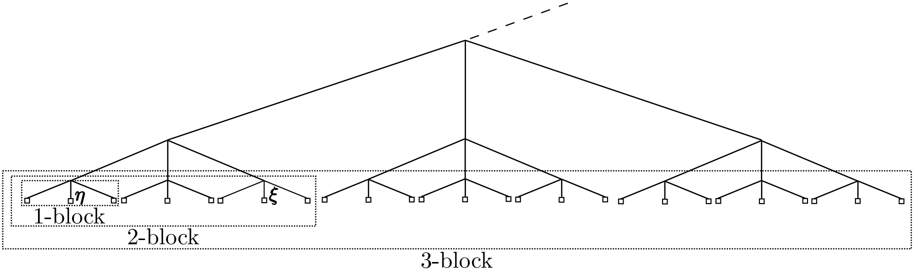

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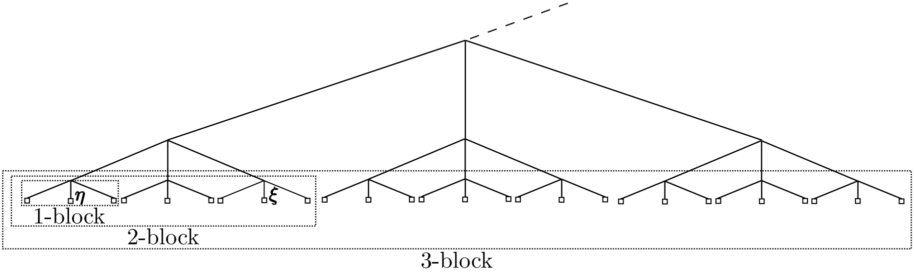

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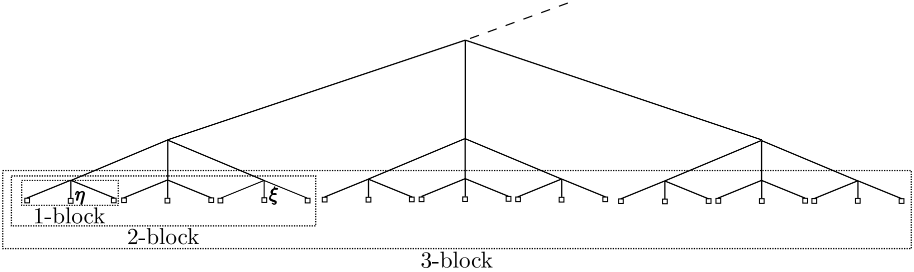
Migration on the hierarchical space (Dawson, Gorostiza, Wakolbinger)



Hierarchical random walk:

- **Migration rates:** $\underline{c} := (c_k)_{k\in\mathbb{N}_0} \in (0,N)^{\mathbb{N}_0}$
- each indiv. at $\eta \in \Omega_N$ jumps unif. in $B_k(\eta)$ at rate c_{k-1}/N^{k-1}

Reproduction within a colony (Cannings)

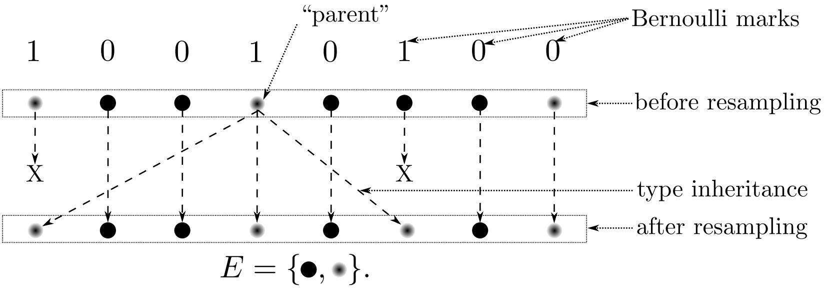
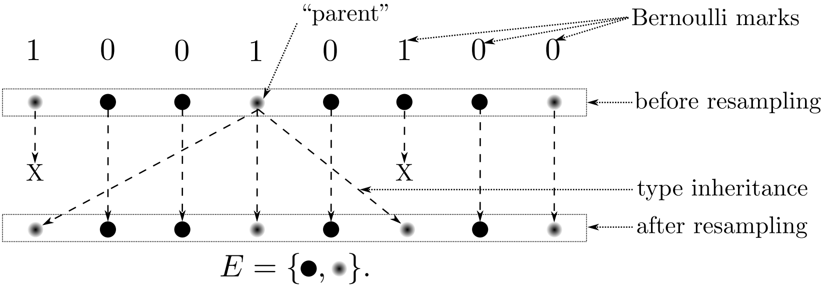


Figure: Resampling

Cannings model (discrete time):

- M fixed (# of individuals).
- **Exchangeable** collection of r.v. $\{\nu_i^{(M)} \in [0:M] : i \in [1:M]\}$.
- $\sum_{i=1}^{M} \nu_i^{(M)} = M$.

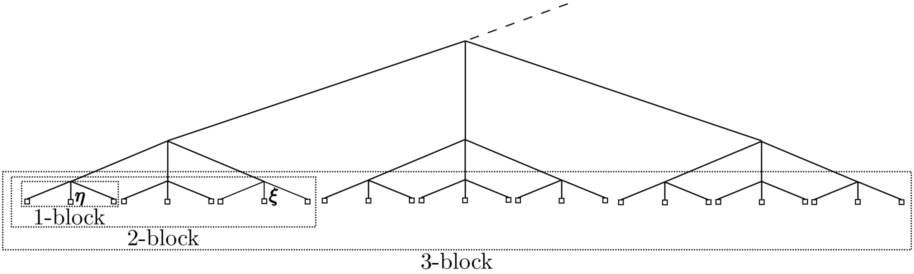
Λ-Cannings model (continous time, continuous mass limit)



A large universality class ($M \to \infty$, Sagitov'1999, Möhle-Sagitov'2001):

- Driven by **PPP** on $\mathbb{R}_+ \times [0,1]$ with $\mathrm{d}t \otimes \Lambda(\mathrm{d}r)/r^2$, where $\Lambda \in \mathscr{M}_{\text{finite}}([0,1])$, $\Lambda(\{0\}) = 0$.
- **Resampling** $(r\delta_1 + (1-r)\delta_0)^{\otimes M}$ (**Bernoulli experiment**).
- For $M \to \infty$, study the **distribution of types**: $X(t) := \frac{1}{M}\sum_{i=1}^{M} \delta_{T(i,t)} \in \mathscr{M}_1(E)$ in a colony.

Non-local resampling-reshuffling



Catastrophies in $B_k(\xi)$:

- Driven by **PPP** on $\mathbb{R}_+ \times [0,1]$ with $\mathrm{d}t \otimes N^{-2k}\Lambda_k(\mathrm{d}r)/r^2$, where $\Lambda_k \in \mathscr{M}_1([0,1])$, $\Lambda(\{0\}) = 0$.
- **Reshuffle** the individuals in $B_k(\xi)$.
- **Resample** the individuals in $B_k(\xi)$ using Λ_k.

Summary (so far)

Hierarchically interacting $(\underline{c}, \underline{\Lambda})$-Cannings process

$$X^{(\Omega_N)} = \left(X^{(\Omega_N)}(t)\right)_{t\geq 0} \quad \text{with} \quad X^{(\Omega_N)}(t) = \left\{X_\eta^{(\Omega_N)}(t)\right\}_{\eta\in\Omega_N} \in \mathscr{M}_1(E)^{\Omega_N}.$$

Competition between:

- **Migration** $\underline{c} = (c_k)_{k\in\mathbb{Z}_+}$ (spatial movement)
  vs.
  Resampling $\underline{\Lambda} = (\Lambda_k)_{k\in\mathbb{Z}_+}$ (reproduction under constrained resources).
  plus
- (Hierarchy of) **slow** and **fast time scales**.

N.B. Important features:

- **Non-diffusive behaviour**: PPP driven jumps.
- **Strongly correlated global updates:** non-local reshuffling-resampling.

Long-time behaviour of the spatial process

$$\boxed{\mathbf{Q}: \mathscr{L}\left[X^{(\Omega_N)}(t)\right] \underset{t\to+\infty}{\Longrightarrow} \mathbf{?}}$$

Biodiversity?

Duality with a spatial coalescent with non-local coalescence

Relate $X = \{X_t\}_{t\in\mathbb{R}_+}$ with a **simpler process**. Find H and $Y = \{Y_t\}_{t\in\mathbb{R}_+}$:

$$\mathbb{E}_{X_0}[H(X_t, Y_0)] = \mathbb{E}_{Y_0}[H(X_0, Y_t)], \text{ for all } (X_0, Y_0), \quad t \in \mathbb{R}_+$$

- Backwards in time dynamics of the **coalescing lineages**.
- **Spatial Λ-coalescent** with non-local coalescence: Y_t.
- At start, infinitely many singleton **families**.
- Families move around according to the HRW.
- At coalescence event, $k \geq 2$ families in $B_k(\eta)$ coalesce. Then, all families in B_k are reshuffled.
- Driven by **PPP** $\mathrm{d}t \otimes \mathrm{d}\eta \otimes \Big(N^{-2k}\mathrm{d}k \Big[\Lambda_k(\mathrm{d}r)\big(r\delta_1 + (1-r)\delta_0\big)^{\otimes\mathbb{N}}\Big](\mathrm{d}\omega)\Big)$.

Biodiversity dichotomy: clustering vs. coexistence

Dichotomy seen backwards in time:

- Single family in the long run $\rightsquigarrow$ no **biodiversity** (clustering).
- More then one family $\rightsquigarrow$ **coexistence**.

Exchangeability $\rightsquigarrow$ enough to consider **two coalescing random walks** $(Z_t^1, Z_t^2)_{t \geq 0}$ on Ω_N with migration coefficients $(c_k + \lambda_{k+1} N^{-(k+1)})_{k \in \mathbb{N}_0}$ and coalescence at rates $(\lambda_k = \Lambda_k([0,1]))_{k \in \mathbb{N}_0}$. Consider the time-$t$ accumulated hazard for coalescence of this pair:

$$H_N(t) = \sum_{k \in \mathbb{N}_0} \lambda_k N^{-k} \int_0^t \mathbb{1}\left\{ d(Z_s^1, Z_s^2) \leq k \right\} \mathrm{d}s.$$

Lemma

- $\lim_{t \to \infty} H_N(t) = \infty$ *a.s.* $\rightsquigarrow$ *no **biodiversity** (clustering).*
- $\lim_{t \to \infty} H_N(t) < \infty$ *a.s* $\rightsquigarrow$ ***coexistence**.*

Biodiversity dichotomy: criterion

Theorem (Migration – resampling-reshuffling tradeoff)

- $\sum_{k\in\mathbb{N}_0}(1/c_k)\sum_{l=0}^{k}\Lambda_l([0,1])=\infty \rightsquigarrow$ *no **biodiversity** (clustering).*
- $\sum_{k\in\mathbb{N}_0}(1/c_k)\sum_{l=0}^{k}\Lambda_l([0,1])<\infty$ *a.s.* $\rightsquigarrow$ ***coexistence**.*

Large space-time scale analysis: $N \to \infty$, hierarchical mean-field limit

- Analyse the system scale by scale.
- **"Separate" slow and fast time scales.**
- **Renormalise.**
- **Macroscopic observables:**

$$Y_{\eta,k}^{(N)}(tN^k) = \frac{1}{N^k} \sum_{\zeta \in B_k(\eta)} X_\zeta^{(\Omega_N)}(tN^k), \qquad \eta \in \Omega_N, k \in \mathbb{Z}_+$$

  (block averages of order $k \in \mathbb{Z}_+$).
- **Single scale** (mean-field) $\rightsquigarrow$ **propagation of chaos** and appearance of **McKean-Vlasov process.**
- **Multiple scales** simultaneously: $\rightsquigarrow$ Markov **interaction chain**.
- All this in the **hierarchical mean-field limit:**

$$\Omega_N \uparrow \Omega_\infty, \quad N \to +\infty.$$

McKean-Vlasov limiting object

Algebra of test functions: $\mathscr{B} \subseteq C_b(\mathscr{M}_1(E), \mathbb{R})$ with $G \in \mathscr{B}$:

$$G(x) = \int_{E^n} x^{\otimes n}(\mathrm{d}u)\, \varphi(u), \qquad x \in \mathscr{M}_1(E), n \in \mathbb{N}, \varphi \in C_{\mathrm{b}}(E^n, \mathbb{R}).$$

Generator:

$$\begin{aligned}(L_\theta^{c,d,\Lambda} G)(x) &= c \int_E (\theta - x)(\mathrm{d}a)\, \frac{\partial G(x)}{\partial x}[\delta_a] \leftarrow \textbf{[drift]} \\ &+ d \int_E \int_E Q_x(\mathrm{d}u, \mathrm{d}v)\, \frac{\partial^2 G(x)}{\partial x^2}[\delta_u, \delta_v] \leftarrow \textbf{[Fleming-Viot diffision]} \\ &+ \int_{[0,1]} \Lambda^*(\mathrm{d}r) \int_E x(\mathrm{d}a) \left[G\big((1-r)x + r\delta_a\big) - G(x)\right] \leftarrow \textbf{[jumps]}, \quad G \in \mathscr{B},\end{aligned}$$

where

$$Q_x(\mathrm{d}u, \mathrm{d}v) = x(\mathrm{d}u)\, \delta_u(\mathrm{d}v) - x(\mathrm{d}u)\, x(\mathrm{d}v).$$

C^Λ-processes with immigration-emigration:

$$Z_\theta^{c,d,\Lambda} = \big(Z_\theta^{c,d,\Lambda}(t)\big)_{t \geq 0}, \quad Z_\theta^{c,d,\Lambda}(0) = \theta.$$

Asymptotic behaviour of the macroscopic observables

- **Volatility constants:** $\underline{d} = (d_k)_{k\in\mathbb{Z}_+}$,

$$d_0 = 0, \qquad d_{k+1} = \frac{c_k(\lambda_k/2 + d_k)}{c_k + (\lambda_k/2 + d_k)}, \quad k \in \mathbb{Z}_+,$$

  where $\lambda_k = \Lambda_k([0;1])$.
- **N.B.** (inhomogeneous) **Möbius transformation**.

Theorem (behaviour of the macroscopic observables)

Let $X^{(\Omega_N)}(0)$ be i.i.d. with the single-site mean $\theta \in \mathscr{P}(E)$ For every $k \in \mathbb{Z}_+$, uniformly in $\eta \in \Omega_\infty$,

$$\mathscr{L}\left[\left(Y^{(N)}_{\eta,k}(tN^k)\right)_{t\geq 0}\right] \xrightarrow[N\to+\infty]{} \mathscr{L}\left[\left(Z^{c_k,d_k,\Lambda_k}_{\theta}(t)\right)_{t\geq 0}\right].$$

Ergodic behaviour of $X^{(N)}$, $N < \infty$

Set

$$m_k := \frac{\lambda_k/2 + d_k}{c_k}.$$

Theorem (Clustering vs. coexistence criterion)

- **[Clustering]** *(= formation of large mono-type regions), if* $\sum_{k\in\mathbb{Z}_+} m_k = \infty$ ***vs.***
- **[Local coexistence]** *(= convergence to multi-type equilibria), if* $\sum_{k\in\mathbb{Z}_+} m_k < \infty$.

N.B. $\sum_{k\in\mathbb{Z}_+} m_k = \infty$ vs. $< \infty \Leftrightarrow \sum_{k\in\mathbb{N}_0}(1/c_k)\sum_{l=0}^{k}\lambda_l = \infty$ vs. $< \infty$.

- Recurrent migration $\rightsquigarrow$ clustering.
- $\exists$ transient migrations and strong enough reshuffling-resampling $\sum_{l\in\mathbb{N}_0}\lambda_l = \infty \rightsquigarrow$ clustering.

Dichotomy for $N < \infty$

Theorem (Clustering vs. coexistence criterion)

The following dichotomy holds:

(a) [Local coexistence] *If* $\sum_{k\in\mathbb{Z}_+} m_k < \infty$, *then for every* $\theta \in \mathscr{P}(E)$ *and every* $X^{(\Omega_N)}(0)$ *whose law is stationary and ergodic w.r.t. translations in* Ω_N *and has a single-site mean* θ,

$$\mathscr{L}\left[X^{(\Omega_N)}(t)\right] \underset{t\to+\infty}{\Longrightarrow} \nu_\theta^{(\Omega_N),\underline{c},\underline{\lambda}} \in \mathscr{P}(\mathscr{P}(E)^{\Omega_N})$$

for some unique law $\nu_\theta^{(\Omega_N),\underline{c},\underline{\lambda}}$ *that is stationary and ergodic w.r.t. translations in* Ω_N *and has single-site mean* θ.

(b) [Clustering] *If* $\sum_{k\in\mathbb{Z}_+} m_k = \infty$, *then, for every* $\theta \in \mathscr{P}(E)$,

$$\mathscr{L}\left[X^{(\Omega_N)}(t)\right] \underset{t\to+\infty}{\Longrightarrow} \int_0^1 \theta(\mathrm{d}u)\delta_{(\delta_u)^{\Omega_N}} \in \mathscr{P}(\mathscr{P}(E)^{\Omega_N}).$$

Inhomogeneous iterates of the Möbius transformation: universality classes

Polynomial case. Regular variation at infinity.

$$c_k \sim L_c(k)k^a, \quad a \in \mathbb{R}, \qquad \lambda_k \sim L_\lambda(k)k^b, \quad b \in \mathbb{R}, \qquad k \to +\infty,$$

Denote $\lim_{k\to\infty} \frac{\lambda_k}{c_k} = K \in [0,\infty]$, $\lim_{k\to\infty} k^2 \frac{c_k}{\lambda_k} = L \in [0,\infty]$.

Theorem (Scaling of the volatility)

(a) *If* $K \in (0,\infty)$, *then*

$$\lim_{k\to\infty} \frac{d_k}{c_k} = M \text{ with } M = \tfrac{1}{2}K[-1+\sqrt{1+(4/K)}] \in (0,1).$$

(b) *If* $K = \infty$, *then* $\lim_{k\to\infty} \frac{d_k}{c_k} = 1$.

(c) *If* $K = 0$, $L = \infty$, *then* $\lim_{k\to\infty} \frac{d_k}{\sqrt{c_k\lambda_k}} = 1$.

(d) *If* $K = 0$, $L < \infty$, *then* $\lim_{k\to\infty} \sigma_k d_k = M' \in [1,\infty)$ *with* $M' = 1 \Leftrightarrow L = 0$.

Multi-scale analysis

- **Scale of the age of the system** $N^j t$.
- **Interaction chain** $(M_k^{(j)})_{k=-(j+1),\dots,0}$, $j \in \mathbb{Z}_+$, initial state at time $-(j+1)$,

$$M_{-(j+1)}^{(j)} = \theta \in \mathscr{M}_1(E),$$

- **Equilibrium of the McKean-Vlasov process** $Z_x^{c,d,\Lambda}$: $\nu_x^{c,d,\Lambda}$
- **Transition** $-(k+1) \rightsquigarrow -k$, $k = j,\dots,0$:

$$K_k(x,\cdot) = \nu_x^{c_k,d_k,\Lambda_k}(\cdot), \quad x \in \mathscr{M}_1(E), \quad k \in \mathbb{Z}_+,$$

Theorem (multi-scale behaviour)

Let $(t_N)_{N\in\mathbb{N}}$ *be such that* $\lim_{N\to\infty} t_N = \infty$ *and* $\lim_{N\to\infty} t_N/N = 0$.
Then, for every $j \in \mathbb{Z}_+$*, uniformly in* $\eta \in \Omega_\infty$ *and* $u_k \in (0,\infty)$, $k = 0,\dots,j$,

$$\mathscr{L}\left[\left(Y_{\eta,k}^{(N)}(N^j t_N + N^k u_k)\right)_{k=j,\dots,0}\right] \underset{N\to+\infty}{\Longrightarrow} \mathscr{L}\left[\left(M_{-k}^{(j)}\right)_{k=j,\dots,0}\right],$$

$$\mathscr{L}\left[Y_{\eta,j+1}^N(N^j t_N)\right] \underset{N\to+\infty}{\Longrightarrow} \delta_\theta.$$

Inhomogeneous environment

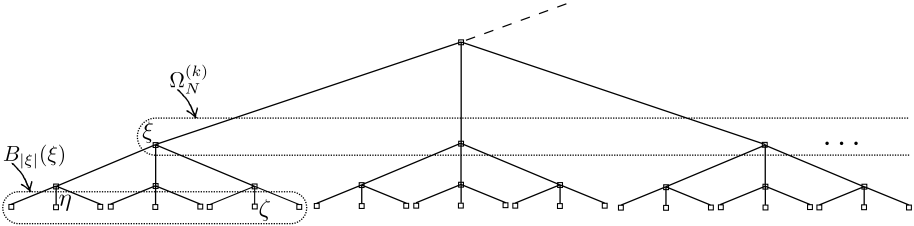


Figure: $\Omega_N^{\mathbb{T}}$ with $N = 3$, $\xi \in \Omega_N^{(k)} \subset \Omega_N^{\mathbb{T}}$, $|\xi| = k = 2$, $\eta, \zeta \in B_{|\xi|}(\xi)$. The elements of $\Omega_N^{\mathbb{T}}$ are the vertices of the tree (indicated by $\square$'s).

Spatially inhomogeneous reshuffling-resampling:

$$\underline{\Lambda}(\omega) = \left\{\Lambda^{\xi}(\omega)\colon\ \xi \in \Omega_N^{\mathbb{T}}\right\}$$

Model via a **random environment** ω.

Assumptions on the random environment

Assume

$$\Lambda^{\xi}(\omega) = \lambda_{|\xi|}\chi^{\xi}(\omega)$$

where

- $\underline{\lambda} = (\lambda_k)_{k\in\mathbb{N}_0}$ is a deterministic seq., and
- $\{\chi^{\xi}(\omega)\colon\ \xi \in \Omega_N^{\mathbb{T}}\}$ is an $\mathscr{M}_f([0,1])^{\Omega_N^{\mathbb{T}}}$-valued **random field** that is stationary under translations.

Denote the **total mass** of χ by

$$\mathscr{A}^{\xi}(\omega) = \chi^{\xi}(\omega)([0,1]).$$

Assume that

$$\mathbb{E}[\mathscr{A}^{\xi}(\omega)] = 1, \qquad \mathbb{E}[(\mathscr{A}^{\xi}(\omega))^2] \in (0,\infty),$$

and let the terminal σ-algebra generated by χ be trivial.

Long time behaviour: $t \to \infty$

Theorem (Equilibrium)

Fix $N \in \mathbb{N} \setminus \{1\}$. Suppose that $X^{(\Omega_N)}(\omega;0)$ is a random field (w.r.t. ω) that is stationary and ergodic under the law $\mathbb{P}$ with mean single-coordinate measure $\theta \in \mathscr{P}(E)$. Then for $\mathbb{P}$-a.s. ω there exists a $\nu_\theta(\omega) \in \mathscr{P}(\mathscr{P}(E)^{\Omega_N})$, the equilibrium measure given ω, such that

$$\lim_{t\to\infty} \mathscr{L}\left[X^{(\Omega_N)}(\omega;t)\right] = \nu_\theta(\omega),$$

where

$$\int_{\mathscr{P}(E)^{\Omega_N}} x_0\, \nu_\theta(\omega)(\mathrm{d}x) = \theta.$$

Moreover, $\omega \mapsto \nu_\theta(\omega)$ is stationary and ergodic under the law $\mathbb{P}$.

$N < \infty$

Theorem (Dichotomy for finite system)

Fix $N \in \mathbb{N} \setminus \{1\}$.

(a) *Let* $\mathscr{C} = \{\omega\colon$ *coexistence given* ω *occurs*$\}$. *Then* $\mathbb{P}(\mathscr{C}) \in \{0,1\}$.

(b) $\mathbb{P}(\mathscr{C}) = 1$ *if and only if*

$$\sum_{k\in\mathbb{N}_0} \frac{1}{c_k + N^{-1}\lambda_{k+1}} \sum_{l=0}^{k} \lambda_l < \infty.$$

Hierarchical mean-field limit: $N \to \infty$

Macro-colony averages

$$Y^{(\Omega_N)}_{\eta,k}(\omega;t) = \frac{1}{N^k} \sum_{\zeta \in B_k(\eta)} X^{(\Omega_N)}_{\zeta}(\omega;t), \qquad \eta \in \Omega_N.$$

Theorem (Hierarchical mean-field limit and renormalisation)

Suppose that for each N the random field $X^{(\Omega_N)}(\omega;0)$ is the restriction to Ω_N of a random field $X(\omega)$ indexed by $\Omega_\infty = \bigoplus_{\mathbb{N}} \mathbb{N}$ that is i.i.d. with single-component mean $\theta \in \mathscr{P}(E)$. Then, for $\mathbb{P}$-a.s. ω and for every $k \in \mathbb{N}$ and $\eta \in \Omega_\infty$,

$$\lim_{N\to\infty} \mathscr{L}\left[\left(Y^{(\Omega_N)}_{\eta,k}(\omega;tN^k)\right)_{t\geq 0}\right] = \mathscr{L}\left[\left(Z_\theta^{c_k,d_k,\Lambda^{\mathrm{MC}_k(\eta)}(\omega)}(t)\right)_{t\geq 0}\right],$$

where $\mathrm{MC}_k(\eta) \in \Omega_\infty^{(k)}$ is the unique site at height k above $\eta \in \Omega_\infty$, i.e., the label of the block (= macro-colony) of radius k in Ω_∞ around $\eta \in \Omega_\infty$ (see Fig. 5*). The same is true for $k = 0$ when $Z_\theta^{c_0,d_0,\Lambda^\eta(\omega)}(0) = X^{(\Omega_N)}(\omega;0)$ instead of $Z_\theta^{c_0,d_0,\Lambda^\eta(\omega)}(0) = \theta$.*

Volatilities in the inhomogeneous case: $d_k = ?$

In the theorem $\underline{d} = (d_k)_{k\in\mathbb{N}_0}$ is the sequence of **volatility constants** defined recursively as

$$d_{k+1} = \mathbb{E}_{\mathscr{L}_{\mathscr{A}}}\left[\frac{c_k(\mu_k\mathscr{A} + d_k)}{c_k + (\mu_k\mathscr{A} + d_k)}\right], \quad k \in \mathbb{N}_0,$$

where

- $\mu_k = \frac{1}{2}\lambda_k$
- $\mathscr{A}$ is the $(0,\infty)$-valued random variable whose law $\mathscr{L}_{\mathscr{A}}$ is the same as that of $\mathscr{A}^0(\omega)$ under $\mathbb{P}$.
- $\mathbb{E}_{\mathscr{L}_{\mathscr{A}}}$ is expecation w.r.t. $\mathscr{L}_{\mathscr{A}}$.

The right-hand side is the **average of a random Möbius transformation that depends on $\mathscr{A}$**.

Theorem (Randomness lowers volatility)

If $d_0^0 = d_0 = d_0^*$, then $d_k^0 < d_k < d_k^*$ for all $k \in \mathbb{N}$.

Scaling regimes

Theorem (Scaling of the Fleming-Viot volatility: polynomial coefficients)

(a) *If* $K = \infty$, *then* $\lim_{k\to\infty} d_k/c_k = 1$.

(b) *If* $K \in (0,\infty)$, *then* $\lim_{k\to\infty} d_k/c_k = M$ *with* $M \in (0,1)$ *the unique solution of the equation*

$$M = \mathbb{E}_{\mathscr{L}_{\mathscr{A}}}\left[\frac{(K\mathscr{A} + M)}{1 + (K\mathscr{A} + M)}\right].$$

(c) *If* $K = 0$ *and* $L = \infty$, *then* $\lim_{k\to\infty} d_k/\sqrt{c_k \mu_k} = 1$.

(d) *If* $K = 0$, $L \in [0,\infty)$ *and* $a \in (-\infty, 1)$, *then* $\lim_{k\to\infty} \sigma_k d_k = M^*$ *with* $M^* \in [1,\infty)$ *given by*

$$M^* = \tfrac{1}{2}\left[1 + \sqrt{1 + 4L/(1-a)^2}\right],$$

where $\sigma_k = \sum_{l=0}^{k-1}(1/c_l)$.

Summary

- Constructed the the hierarchically interacting Cannings processes in random environment.
- **Dichotomy**: the **clustering vs. local coexistence dichotomy** in the long-time behaviour in terms of $\underline{c}, \underline{\lambda}$. Also for **finite** N.
- The dichotomy is not affected by the random environment.
- Identified its **space-time scaling behaviour** in the hierarchical mean-field limit $N \to \infty$. Volatilities decrease in the inhomogeneous environment. Changes in the scaling regimes. Clusters grow slower.
- Fluctuations in the environment reduce clustering $\rightsquigarrow$ **increased biodiversity**.

Outlook:

- Other geographical spaces. Continuum limit to the geographic space $\mathbb{R}^2$.

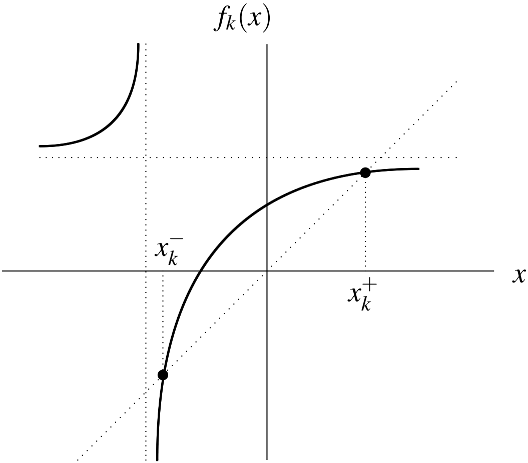


Figure: The Möbius-transformation $x \mapsto f_k(x)$.

R.J. Kooman (1998)