The Interface of the symbiotic branching model

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A toy model for two interacting populations

- Consider a population consisting of RED and BLUE particles.
- Start with $\lfloor u_0 N \rfloor$ RED particles and $\lfloor v_0 N \rfloor$ BLUE particles.
- Independently, each particle either gives birth to two children or dies (with equal probability) – at a rate that is proportional to the number of particles of the **opposite** colour.

Speeding up time by N, scaling number of particles down by N gives for $N \to \infty$ a system of coupled SDEs started in (u_0, v_0)

$$\begin{aligned} du_t &= \sqrt{\gamma u_t v_t}\, dB_t^1 \\ dv_t &= \sqrt{\gamma u_t v_t}\, dB_t^2, \end{aligned}$$

- u_t denotes proportion of RED particles, v_t proportion of BLUE particles.
- $\gamma > 0$ is a branching rate and B_t^1, B_t^2 are independent Brownian motions.

Long-term behaviour: A.s. after a finite time, one of the two-populations dies out.

The symbiotic branching SPDE

The *continuous-space symbiotic branching model* $\mathrm{cSBM}(\gamma, \varrho)$ describes the evolution of two interacting populations $u_t(x)$ and $v_t(x)$ on the real line given by

$$\frac{\partial}{\partial t} u_t(x) = \frac{1}{2}\Delta u_t(x) + \sqrt{\gamma u_t(x) v_t(x)}\,\dot{W}^1(t,x),$$
$$\frac{\partial}{\partial t} v_t(x) = \frac{1}{2}\Delta v_t(x) + \sqrt{\gamma u_t(x) v_t(x)}\,\dot{W}^2(t,x),$$

for initial conditions $u_0(\cdot), v_0(\cdot) \geq 0$, where $(\dot{W}^1, \dot{W}^2)$ is a pair of correlated standard Gaussian white noises with correlation parameter $\varrho \in [-1, 1]$. The parameter $\gamma > 0$ is called the branching rate.

- *Branching*: At a rate proportional to population of opposite color.
- *Correlation*: The driving noises are correlated. If $\varrho < 0$, negative correlated branching, $\varrho = 0$ independent and $\varrho > 0$ positively correlated branching.
- *Migration*: $\frac{1}{2}\Delta u(t,x)$ (corresponds to simple random walk in the discrete-space case).

Introduced by [Etheridge, Fleischmann 2004]: Rigorous formulation, existence + uniqueness in terms of a *martingale problem* (see below).

Relation to other models

The symbiotic branching model

$$\frac{\partial}{\partial t} u_t(x) = \tfrac{1}{2}\Delta u_t(x) + \sqrt{\gamma u_t(x) v_t(x)}\, \dot{W}_t^1(x),$$
$$\frac{\partial}{\partial t} v_t(x) = \tfrac{1}{2}\Delta v_t(x) + \sqrt{\gamma v_t(x) u_t(x)}\, \dot{W}_t^2(x)$$

interpolates between several classical models by varying the correlation ϱ:

- $\varrho = -1$: Then $\dot{W}^1 = -\dot{W}^2$. Take $v_0 = 1 - u_0$, then $v_t = 1 - u_t$ for all t

  $$\frac{\partial}{\partial t} u = \tfrac{1}{2}\Delta u + \sqrt{\gamma u(1-u)}\dot{W}.$$

  model. This is the *stepping stone model* ($\rightsquigarrow$ mathematical population genetics).
- $\varrho = 0$: $\dot{W}^1$ and $\dot{W}^2$ independent $\rightsquigarrow$ *mutually catalytic branching.* Introduced by [Dawson, Perkins '98].
- $\varrho = 1$: Then $\dot{W}^1 = \dot{W}^2$. Take $u_0 = v_0$, then one obtains

  $$\frac{\partial}{\partial_t} u = \tfrac{1}{2}\Delta u + \sqrt{\gamma} u \dot{W}.$$

  $\rightsquigarrow$ *parabolic Anderson model*.

The interface in the symbiotic branching model

Throughout this talk: we will start with complementary Heaviside functions as initial conditions:

$$u_0(x) = \mathbb{1}_{\mathbb{R}^-}(x) \quad \text{and} \quad v_0(x) = \mathbb{1}_{\mathbb{R}^+}(x).$$

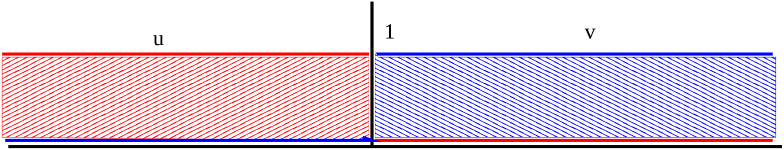
Definition 1

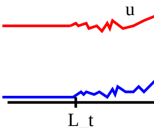
The *interface* between the two populations is defined as

$$\mathrm{Ifc}(t) = \mathrm{cl}\{x : u_t(x)v_t(x) > 0\},$$

Moreover, we define the *right and left end point* of the interface as

$$R_t := \sup\{x : u_t(x)v_t(x) > 0\} \text{ and } L_t = \inf\{x : u_t(x)v_t(x) > 0\}$$





Compact interface property

Theorem 1

Let $(u_0, v_0) = (\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+})$.
(a) [Etheridge, Fleischmann, 2004]*: For any* $\varrho \in [-1, 1]$*, we have almost surely for all* T *sufficiently large*

$$\bigcup_{t \le T} \mathrm{Ifc}(t) \subset [-CT, CT].$$

(b) [Blath, Döring, Etheridge, 2011]*: For all* ϱ *'very close' to* -1*, we have almost surely for all* T *sufficiently large*

$$\bigcup_{t \le T} \mathrm{Ifc}(t) \subset [-C\sqrt{T \log T}, C\sqrt{T \log T}].$$

In (b), the strong restriction on ϱ is due to the technique of the proof, which requires boundedness in t of 35th moments of $u_t(x)$.

$\rightsquigarrow$ Can we do better?

The case $\varrho = -1$: stepping stone Model

Theorem 2 ([Tribe 1995])

Rescale the endpoints of the interface diffusively

$$R_t^{(n)} := \tfrac{1}{n} R_{n^2 t} \quad \text{and} \quad L_t^{(n)} := \tfrac{1}{n} L_{n^2 t}$$

then we have

$$(L_t^{(n)}, R_t^{(n)})_{t \geq 0} \xrightarrow[n\uparrow\infty]{d} (B_t, B_t)_{t \geq 0},$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion.

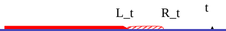
In fact, Tribe shows that for the diffusively rescaled solution

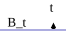
$$u_t^{(n)}(x) := u_{{\color{red}n^2} t}({\color{red}n} x).$$

we have convergence (in distribution) of

$$(u_t^{(n)}(x)dx, (1 - u_t^{(n)}(x))dx)_{t\geq 0} \xrightarrow[n\uparrow\infty]{d} (\mathbb{1}_{\{x \leq B_t\}} dx, \mathbb{1}_{\{x \geq B_t\}} dx)_{t \geq 0}$$

(in the sense of measure-valued processes), where $(B_t)_{t\geq 0}$ is as above.





Extension to $\varrho > -1$: Scaling property

Fix $\varrho > -1$ and consider cSBM(γ, ϱ) for fixed $\gamma > 0$. Rescale solutions diffusively:

$$u_t^{(n)}(x) = u_{n^2 t}(nx) \quad \text{and} \quad v_t^{(n)}(x) = v_{n^2 t}(nx).$$

Lemma 2 ([Etheridge, Fleischmann 2004])

$(u^{(n)}, v^{(n)})$ *satisfies cSBM*$(n\gamma, \varrho)$, *i.e.*

$$\frac{\partial}{\partial t} u_t^{(n)}(x) = \frac{1}{2}\Delta u_t^{(n)}(x) + \sqrt{n\gamma u_t^{(n)}(x) v_t^{(n)}(x)}\, \dot{\tilde{W}}_t^1(x),$$
$$\frac{\partial}{\partial t} v_t^{(n)}(x) = \frac{1}{2}\Delta v_t^{(n)}(x) + \sqrt{n\gamma v_t^{(n)}(x) u_t^{(n)}(x)}\, \dot{\tilde{W}}_t^2(x).$$

Note that complementary Heaviside initial conditions $u_0 = \mathbb{1}_{\mathbb{R}^-}, v_0 = \mathbb{1}_{\mathbb{R}^+}$ are invariant under this rescaling.

- Instead of rescaling space / time equivalent to send $\gamma \to \infty$.
- Verified for a discrete space model: existence of an infinite rate limit $\gamma \to \infty$ by [Klenke, Mytnik 2010-12], [Klenke, Oeler 2010], [Döring, Mytnik 2011-12]
- This suggest there might be an interesting scaling limit for $\varrho > -1$ as well.

But . . .: moment asymptotics

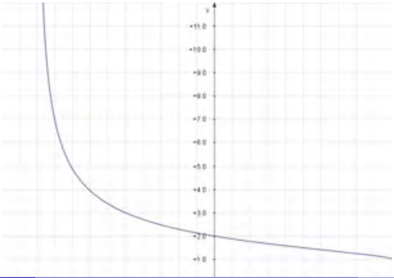
- Unlike in the case $\varrho = -1$, for $\varrho > -1$ local densities are not bounded (uniformly in time).

Varying ϱ affects the moment asymptotics. Define the *critical moment curve*

$$p(\varrho) = \frac{\pi}{\arccos(-\varrho)}.$$

Theorem 3 ([Blath, Döring, Etheridge 2011])

$$\sup_{t \geq 0} \mathbb{E}_{1,1}[u_t(x)^p] < \infty \quad \textit{iff} \quad p < p(\varrho) = \frac{\pi}{\arccos(-\varrho)}.$$



Our main result: a scaling limit

Let $(u_t, v_t)_t$ be a solution of cSBM(γ, ϱ) started with $(u_0, v_0) = (\mathbb{1}_{(-\infty,0]}, \mathbb{1}_{[0,\infty)})$.
Diffusive rescaling:

$$u_t^{(n)}(x) = u_{n^2 t}(nx) \quad \text{and} \quad v_t^{(n)}(x) = v_{n^2 t}(nx).$$

Define measures on $\mathbb{R}$:

$$\mu_t^{(n)}(dx) := u_t^{(n)}(x)dx \quad \text{and} \quad \nu_t^{(n)}(dx) := v_t^{(n)}(x)dx.$$

Theorem 4 (Blath, Hammer, O. 2013)

Suppose $\varrho \in (-1, -\frac{1}{\sqrt{2}})$. The sequence $(\mu_t^{(n)}, \nu_t^{(n)})_{t\geq 0}, n \in \mathbb{N}$ of measure-valued processes converges weakly to a limit $(\mu_t, \nu_t)_{t\geq 0}$ with the following properties:

- *Absolute continuity: For each fixed $t > 0$,*

$$\mu_t(dx) = \mu_t(x)dx, \quad \nu_t(dx) = \nu_t(x)dx \qquad \mathbb{P}\text{-a.s.}$$

- *Separation of types: For each fixed $t > 0$, the densities satisfy*

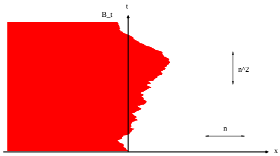
$$\mu_t(\cdot)\nu_t(\cdot) = 0 \qquad \mathbb{P} \otimes \text{Leb-a.s.}$$

The limiting object

- We do have *separation of types* in the limit: The densities $(\mu_t(x), \nu_t(x))_{t\geq 0}$ are singular.
- The absolute continuity result for the limit measures (μ_t, ν_t) follows from [Dawson et. al. 2002].
- Unique characterization of the limit via a martingale problem.
- The limit is not of the form $(\mathbb{1}_{\{x\leq Y_t\}}\, dx, \mathbb{1}_{\{x\geq Y_t\}}\, dx)_{t\geq 0}$ for any semi-martingale $(Y_t)_{t\geq 0}$. So fundamentally different from the case $\varrho = -1$
- The restriction on $\varrho < -\frac{1}{\sqrt{2}}$ is due to the proof of tightness.

We *would like* to know:

- if the limiting process has a 'one-point' interface, i.e. we get the picture:



- an explicit characterization of the limiting object.

More speculation on the latter points at the end.

Main Tool: Dual processes

Two Markov processes (X_t) and (Y_t) are called *dual* with respect to a suitable measurable function F, if

$$\mathbb{E}_{x_0}[F(X_t, y_0)] = \mathbb{E}_{y_0}[F(x_0, Y_t)], \quad t \geq 0.$$

For the symbiotic branching model, at least two different dualities are known:

- a moment duality with a system of coloured Brownian particles, involving their respective collision local times ([Etheridge, Fleischmann 2004]),
- an exponential self-duality useful for (weak) uniqueness-results ([Mytnik 1998]).

Structure of the proof of the scaling limit:

- Tightness (using moment duality),
- Uniqueness (using self-duality).

Tightness via bounding fourth mixed moments

For the proof of tightness in $C_{[0,\infty)}$ we need to get bounds on

$$\mathbb{E}[u_t(x)v_t(x)u_t(z)v_t(z)]$$

that are integrable in x and z and uniform in t.

The *dual process* is a system of 4 (= number of moments) coloured Brownian particles (red/blue)

- starting with a red and blue particle at x and z each.
- colours (c_t^i) change dynamically: particles of the same colour flip into a pair of mixed colours at a rate governed by their collision local time.

Then, if $L_t^=, L_t^{\neq}$ is the total collision time of particles of the same (resp. $\neq$) colour,

$$\mathbb{E}[u_t(x)v_t(x)u_t(z)v_t(z)] = \mathbb{E}_{(x,x,z,z)}\Big[\prod_{i=1}^{4} \mathcal{I}^{c_t^i}(B_t^i)\, e^{\gamma(L_t^= + \varrho L_t^{\neq})}\Big]$$

where

$$\mathcal{I}^{\mathrm{red}} = \mathbb{1}_{\{x\leq 0\}} = u_0(x) \quad \text{and} \quad \mathcal{I}^{\mathrm{blue}} = \mathbb{1}_{\{x\geq 0\}} = v_0(x)$$

Identifying the limit - One step back

Any solution $(u_t, v_t)_{t\geq 0}$ of the symbiotic branching model

$$\begin{aligned} \tfrac{\partial}{\partial t} u_t(x) &= \tfrac{1}{2}\Delta u_t(x) + \sqrt{\gamma u_t(x) v_t(x)}\,\dot{W}^1(t,x), \\ \tfrac{\partial}{\partial t} v_t(x) &= \tfrac{1}{2}\Delta v_t(x) + \sqrt{\gamma u_t(x) v_t(x)}\,\dot{W}^2(t,x), \end{aligned} \tag{1}$$

satisfies the following *martingale problem*:

Integrate against any suitable test function φ, then

$$M(\varphi)_t := \langle u_t, \varphi\rangle - \langle u_0, \varphi\rangle - \tfrac{1}{2}\int_0^t \langle u_s, \Delta\varphi\rangle\, ds,$$

$$N(\varphi)_t := \langle v_t, \varphi\rangle - \langle v_0, \varphi\rangle - \tfrac{1}{2}\int_0^t \langle v_s, \Delta\varphi\rangle\, ds,$$

is a pair of continuous martingales. To guarantee uniqueness, we also need to specify the *correlation structure*

$$\langle\langle M(\varphi)\rangle\rangle_t = \langle\langle N(\varphi)\rangle\rangle_t = \gamma \int_0^t \int u_s(x) v_s(x) \varphi^2(x)\, dx\, ds =: \langle \Lambda_t, \varphi^2\rangle,$$

$$\langle\langle M(\varphi), N(\varphi)\rangle\rangle_t = \varrho\, \langle \Lambda_t, \varphi^2\rangle,$$

here $\langle\langle M, N\rangle\rangle$ is the quadratic (co-)variation of two martingales.

The martingale problem after diffusive rescaling

Recall that we define

$$\mu_t^{(n)}(dx) := u_t^{(n)}(x)dx = u_{n^2 t}(nx)\,dx \quad \nu_t^{(n)}(dx) := v^{(n)}(x)dx = v_{n^2 t}(nx)\,dx,$$

then the measure-valued processes $(\mu_t^{(n)}, \nu_t^{(n)})$ satisfy the same martingale problem:

$$M(\varphi)_t := \langle \mu_t^{(n)}, \varphi\rangle - \langle u_0, \varphi\rangle - \tfrac{1}{2}\int_0^t \langle \mu_s^{(n)}, \Delta\varphi\rangle\, ds,$$

$$N(\varphi)_t := \langle \nu_t^{(n)}, \varphi\rangle - \langle v_0, \varphi\rangle - \tfrac{1}{2}\int_0^t \langle \nu_s^{(n)}, \Delta\varphi\rangle\, ds,$$

are continuous martingales with *correlation structure*

$$\langle\langle M(\varphi)\rangle\rangle_t = \langle\langle N(\varphi)\rangle\rangle_t = \langle \Lambda_t^{(n)}, \varphi^2\rangle,$$

$$\langle\langle M(\varphi), N(\varphi)\rangle\rangle_t = \varrho\,\langle \Lambda_t^{(n)}, \varphi^2\rangle,$$

where

$$\Lambda_t^{(n)}(dx) = \gamma n \int_0^t u_t^{(n)}(x) v_t^{(n)}(x)\, dx.$$

Problem: Not clear what $\Lambda^{(n)}$ converges to! $(\infty \times 0?)$

Uniqueness via separation of types

Problem: Identification of the limit of

$$\Lambda_t^{(n)}(dx) = \gamma n \int_0^t u_t^{(n)}(x) v_t^{(n)}(x)\, dx$$

Solution: Make the measure Λ part of the martingale problem.

Martingale problem

The measure-valued process $(\mu_t, \nu_t)_{t\geq 0}$ satisfies the limiting martingale problem *if there exists* a (sufficiently nice) process $(\Lambda_t)_{t\geq 0}$ such that

$$M(\varphi)_t := \langle \mu_t, \varphi\rangle - \langle \mu_0, \varphi\rangle - \tfrac{1}{2}\int_0^t \langle \mu_s, \Delta\varphi\rangle\, ds,$$
$$N(\varphi)_t := \langle \nu_t, \varphi\rangle - \langle \nu_0, \varphi\rangle - \tfrac{1}{2}\int_0^t \langle \nu_s, \Delta\varphi\rangle\, ds,$$

are continuous martingales with *correlation structure*

$$\langle\langle M(\varphi)\rangle\rangle_t = \langle\langle N(\varphi)\rangle\rangle_t = \langle \Lambda_t, \varphi^2\rangle, \quad \langle\langle M(\varphi), N(\varphi)\rangle\rangle_t \quad = \varrho\, \langle \Lambda_t, \varphi^2\rangle,$$

and moreover if (μ_t, ν_t) satisfies the *'separation-of-types'* (for $(S_t)_t$ heat semigroup):

$$\mathbb{E}_{\mu_0,\nu_0}\left[S_\varepsilon \mu_t(x)\, S_\varepsilon \nu_t(x)\right] \xrightarrow{\varepsilon\downarrow 0} 0.$$

Main Idea for Proof of Uniqueness: Self-Duality

For uniqueness of the martingale problem use self-duality à la [Mytnik 1998]. Define the *self-duality* function:

$$F(\mu,\nu,\varphi,\psi) := \exp\left\{ -\sqrt{1-\varrho}\langle u+v,\varphi+\psi\rangle + i\sqrt{1+\varrho}\langle u-v,\varphi-\psi\rangle \right\}.$$

Aim is to show that

$$\mathbb{E}_{u_0,v_0}\left[F(\mu_t,\nu_t,\tilde{u}_0,\tilde{v}_0)\right] = \mathbb{E}_{\tilde{u}_0,\tilde{v}_0}\left[F(u_0,v_0,\tilde{\mu}_t,\tilde{\nu}_t)\right], \qquad t \geq 0, \tag{2}$$

where

- on the left: (μ_t,ν_t) limit point with $(u_0,v_0) = (\mathbb{1}_{\mathbb{R}^-},\mathbb{1}_{\mathbb{R}^+})$
- on the right $(\tilde{\mu}_t,\tilde{\nu}_t)$ limit points with *rapidly decreasing* initial conditions.

Note that $\tilde{u}_0,\tilde{v}_0$ play role of test functions φ,ψ.

- If (**??**) holds for a sufficiently large class of $(\tilde{u}_0,\tilde{v}_0)$, then self-duality implies uniqueness of one-dimensional marginals of (μ_t,ν_t).
- This is the same self-duality as for the finite rate systems (i.e. no rescaling). and also for the infinite rate *discrete* model, see [Klenke, Mytnik 2010-12], [Döring, Mytnik 2012].
- Main problem: need to construct the dual process for a sufficiently large class of initial conditions $(\tilde{u}_0,\tilde{v}_0)$. $\rightsquigarrow$ use weaker *Meyer-Zhang topology*.

A corollary of the construction of the dual process

Fix absolutely continuous initial conditions with densities which are tempered or rapidly decreasing functions (possibly overlapping).

For each $\gamma > 0$ denote by $(\mu_t^{[\gamma]}, \nu_t^{[\gamma]})_{t \geq 0}$ the solution to the symbiotic branching model with branching rate γ (considered as measure-valued processes).

Theorem 5

Fix $\varrho < 0$. Then, as $\gamma \uparrow \infty$, the sequence of processes $(\mu_t^{[\gamma]}, \nu_t^{[\gamma]})_{t \geq 0}$ converges in law with respect to the Meyer-Zheng topology to the unique solution of the martingale problem satisfying 'separation-of-types':

$$\mathbb{E}_{\mu_0,\nu_0}\left[S_\varepsilon \mu_t(x)\, S_\varepsilon \nu_t(x)\right] \xrightarrow{\varepsilon \downarrow 0} 0,$$

(where S_t denotes the heat semigroup).

- Note that by using a weaker topology we can allow any (strictly) negative ϱ. Reason: we can work with a second moment instead of a fourth moment condition.

Speculation on a more explicit representation of the solution

Case $\varrho = -1$. (Credit to Florian Völlering).

Suppose we start with non-overlapping (bounded) initial conditions u_0, v_0 such that $\operatorname{supp} u_0 = (-\infty, 0]$, $\operatorname{supp} v_0 = [0, \infty)$.

- Observation: $u_t + v_t$ solves the (deterministic) heat equation with initial condition $u_0 + v_0$.
- Define w_t as the solution of the heat equation with initial condition $u_0 + v_0$.
- Define $\mathcal{I}_t$ as the solution of

  $$d\mathcal{I}_t = dB_t - 2\frac{w_t'(\mathcal{I}_t)}{w_t(\mathcal{I}_t)}dt,$$

  for $(B_t)_{t\geq 0}$ a standard Brownian motion.
- Then,

  $$\mu_t(x) = w_t(x)\mathbb{1}_{\{x\leq \mathcal{I}_t\}} \quad \text{and} \quad \nu_t(x) = w_t(x)\mathbb{1}_{\{x\geq \mathcal{I}_t\}},$$

  solves the limiting (infinite rate) symbiotic branching model.

Trying to identify the limit for $\varrho > -1$

Identify the correlation Λ.

- In analogy to [Dawson et al, 2003] on work on the 2-dimensional SBM, we can identify Λ via 'intersection local times', i.e. a way of smoothing out at the interface to measure overlap.
- However, this approach does not help with an explicit identification (and also does not help with proving uniqueness).

Rescaling of discrete infinite-rate model

- Recall that [Klenke, Mytnik 2010,2012] and [Döring, Mytnik 2012] give an explicit representation of the solution of the *discrete-space* infinite rate SBM in terms of an infinite system of jump-type SDEs.
- (Work in progress): For Heaviside initial conditions, we can show that this system simplifies (only need 2 driving noises).
- Goal: rescale the discrete system and obtain a solution of the continuous space infinite-rate model.

Outlook

- **Parameter range for ϱ:**
  - So far we have convergence in $C_{[0,\infty)}$ for $\varrho < -\frac{1}{\sqrt{2}}$ and in the weaker Meyer-Zhang sense for $\varrho < 0$. Is $\varrho = 0$ critical?
- **Width of the interface**
  - [Mueller, Tribe 1997] show for $\varrho = -1$ (stepping stone model) that the width of the interface (even without rescaling) has a stationary distribution.
  - For $\varrho > -1$, we can recover an approximate version of that result. Aim: extend to the exact interface.
- **Shape of the interface**
  - Microscopic description of the interface (e.g. for $\varrho = -1$). Is there a "nice" description of the stationary interface?