Random evolution of population subject to competition

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joint work with Anton Wakolbinger
(and also with Mamadou Ba, Vi Le)

Contents

Finite population

- Consider a continuous–time population model, where each individual gives birth at rate λ, and dies at an exponential time with parameter μ.
- We superimpose a death rate due to interaction equal to $f^-(k)$ (resp. a birth rate due to interaction equal to $f^+(k)$) while the total population size is k.
- In fact since we want to couple the models for all possible initial population sizes, we need to introduce a pecking order (e.g. from left to right) on our ancestors at time 0, which is passed on to the descendants, and so that any daughter is placed on the right of her mother.
- In all what follows, we assume that $f \in C(\mathbb{R}_+; \mathbb{R})$, $f(0) = 0$ and for some fixed $a > 0$, $f(x + y) - f(x) \leq ay$, for all $x, y \geq 0$.

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- We want that the individual i interacts only with those individuals who sit on her left. Let $\mathcal{L}_i(t)$ denote the number of individuals alive at time t who sit on the left of i.
- Then we decide that i gives birth at rate $\lambda + [f(\mathcal{L}_i(t)) - f(\mathcal{L}_i(t) - 1)]^+$, and dies at rate $\mu + [f(\mathcal{L}_i(t)) - f(\mathcal{L}_i(t) - 1)]^-$.
- Summing up, we conclude that the size of the population X_t^m, starting from $X_0^m = m$, jumps

$$\text{from } k \quad \text{to } \begin{cases} k+1, & \text{at rate } \lambda k + \sum_{\ell=1}^{k} [f(\ell) - f(\ell-1)]^+ \\ k-1, & \text{at rate } \mu k + \sum_{\ell=1}^{k} [f(\ell) - f(\ell-1)]^- \end{cases}$$

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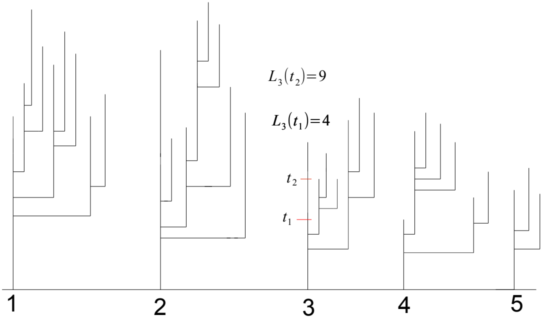
$L_3(t_2)=9$
$L_3(t_1)=4$
t_2
t_1
1
2
3
4
5

- In case f linear, we have a branching process, and for each $t > 0$, $\{X_t^m,\ m \geq 1\}$ has independent increments.
- In the general case, we don't expect that for fixed t, $\{X_t^m,\ m \geq 1\}$ is a Markov chain.
- However, $\{X_t^m,\ t \geq 0\}_{m \geq 1}$ is a path–valued Markov chain. We can specify the transitions as follows.
- For $1 \leq m < n$, the law of $\{X_t^n - X_t^m, t \geq 0\}$, given $\{X_t^\ell,\ t \geq 0,\ 1 \leq \ell \leq m\}$ and given that $X_t^m = x(t), t \geq 0$, is that of the time–inhomogeneous jump Markov process whose rate matrix $\{Q_{k,\ell}(t),\ k,\ell \in \mathbb{Z}_+\}$ satisfies

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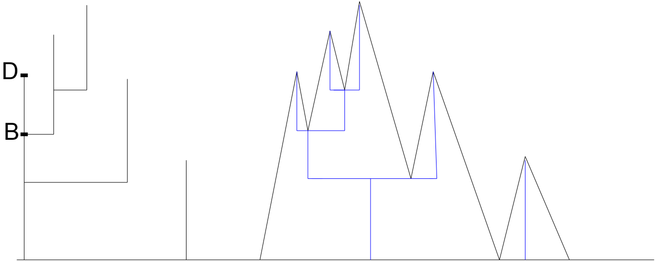
Exploration process of the forest of genealogical trees



- Call $\{H^m_s,\ s \ge 0\}$ the zigzag curve in the above picture (with slope ± 2), and define the local time accumulated by H^m at level t up to time s by

$$L^m_s(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{t \le H^m_r < t+\varepsilon} dr.$$

- H^m is piecewise linear, with slopes ± 2. While the slope is 2, the rate of appearance of a maximum is

$$\mu + [f(\lfloor L^m_s(H^m_s) \rfloor + 1) - f(\lfloor L^m_s(H^m_s) \rfloor)]^- ,$$

and the rate of appearance of a minimum while the slope is -2 is

$$\lambda + [f(\lfloor L^m_s(H^m_s) \rfloor + 1) - f(\lfloor L^m_s(H^m_s) \rfloor)]^+ .$$

- Let $S^m = \inf\{s > 0,\ L^m_s(0) \ge m\}$ the time needed for H^m_s to explore the genealogical trees of m ancestors. If we assume that the population goes extinct in finite time, we have the Ray–Knight type result (see next figure)

$$\{X^m_t,\ t \ge 0,\ m \ge 1\} = \{L^m_{S^m}(t),\ t \ge 0, m \ge 1\}.$$

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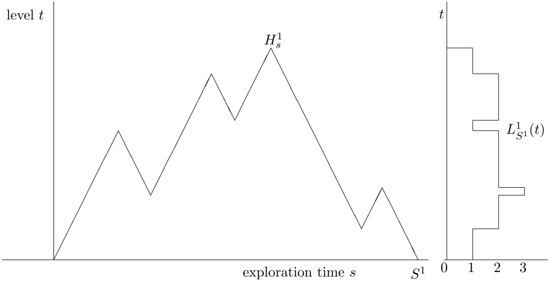
$$\{X^m_t,\ t \geq 0,\ m \geq 1\} \equiv \{L^m_{S^m}(t),\ t \geq 0, m \geq 1\}.$$

How to recover X^m from H^m ?



Renormalization

- Let $N \geq 1$. Suppose that for some $x > 0$, $m = \lfloor Nx \rfloor$, $\lambda = 2N$, $\mu = 2N$, replace f by $f_N = Nf(\cdot/N)$. We define $Z_t^{N,x} = N^{-1} X_t^{\lfloor Nx \rfloor}$.
- We have

Theorem

As $N \to \infty$,

$$\{Z_t^{N,x},\ t \geq 0, x \geq 0\} \Rightarrow \{Z_t^x,\ t \geq 0, x \geq 0\}$$

in $D([0,\infty); D([0,\infty);\mathbb{R}_+))$ equipped with the Skorohod topology of the space of càlàg functions of x, with values in the Polish space $D([0,\infty);\mathbb{R}_+)$, equipped with the adequate metric.

- $\{Z_t^x,\ t \geq 0, x \geq 0\}$ solves for each $x > 0$ the Dawson–Li type SDE

$$Z_t^x = x + \int_0^t f(Z_s^x)ds + 2\int_0^t \int_0^{Z_s^x} W(ds,du),$$

where $W(ds,du)$ is a space–time white noise.

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How to check tightness ?

- Our assumptions on f are pretty minimal. In order to check tightness for x fixed, we establish the two bounds

$$\sup_{N\ge 1}\sup_{0\le t\le T}\mathbb{E}\left(Z_t^{N,x}\right)^2<\infty,\ \sup_{N\ge 1}\sup_{0\le t\le T}\mathbb{E}\left(-\int_0^t Z_s^{N,x}f(Z_s^{N,x})ds\right)<\infty,$$

  and exploit Aldous' criterion.

- Concerning the tightness "in the x direction", we establish the following bound : for any $0\le x<y<z$ with $y-x\le 1$, $z-y\le 1$,

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Continuous population models

- For each fixed $x > 0$, there exists a standard BM B_t such that

$$Z_t^x = x + \int_0^t f(Z_s^x)ds + 2\int_0^t \sqrt{Z_s^x} dB_s.$$

However, B depends upon x in a non obvious way, and the good way of coupling the evolution of Z^x for various x's, which is compatible with the above coupling in the discrete case, is to use the Dawson–Li formulation

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(Sub)criticality

- We will say that Z^x is (sub)critical if

$$T_0^x = \inf\{t > 0;\ Z_t^x = 0\} < \infty \ \text{ a.s.}$$

$$\text{Let } \Lambda(f) = \int_1^\infty \exp\left(-\frac{1}{2}\int_1^u \frac{f(r)}{r} dr\right) du.$$

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A generalized Ray–Knight theorem

- We assume now that $f \in C^1(\mathbb{R}_+;\mathbb{R})$, and there exists $a > 0$ such that $f'(x) \le a$, for all $x \ge 0$. Suppose that we are in the (sub)critical case. We consider the SDE

- $$H_s = B_s + \frac{1}{2}\int_0^s f'(L_r^z(H_r))dr + \frac{1}{2}L_s(0),$$

  where $L_s(0)$ denotes the local time accumulated by the process H at level 0 up to time s. We define $S_x = \inf\{s > 0,\ L_s(0) > x\}$.

- We have

Theorem

The laws of the two random fields $\{L_{S_x}(t);\ t \ge 0, x \ge 0\}$ and $\{Z_t^x;\ t \ge 0, x \ge 0\}$ coincide.

The proof exploits ideas from Norris, Rogers, Williams (1987) who prove the other Ray–Knight theorem in a similar context.

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The laws of the two random fields $\{L_{S_x}(t);\ t \ge 0, x \ge 0\}$ and $\{Z_t^x;\ t \ge 0, x \ge 0\}$ coincide.

The proof exploits ideas from Norris, Rogers, Williams (1987) who prove the other Ray–Knight theorem in a similar context.

Effect of the competition on the height and length of the forest of genealogical trees

The finite population case

- We assume again that $f \in C(\mathbb{R}_+;\mathbb{R})$, $f(0)=0$ and for some fixed $a>0$, $f(x+y)-f(x) \le ay$, for all $x,y \ge 0$. We assume in addition that for some $b>0$, $f(x)<0$ for all $x \ge b$. Define $H^m = \inf\{t>0,\ X_t^m = 0\}$, $L^m = \int_0^{H^m} X_t^m dt$.
- We have

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1. *If* $\int_b^\infty |f(x)|^{-1}dx = \infty$, *then* $\sup_m H^m = \infty$ *a.s.*
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Assume in addition that $g(x) = f(x)/x$ *satisfies* $g(x+y)-g(x) \le ay$.

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Intuitive idea

- The reason why the above works is essentially because, if $g : \mathbb{R}_+ \to \mathbb{R}_+$ satifies

$$\int_0^\infty \frac{1}{g(x)} dx < \infty$$

then the solution of the ODE

$$\dot{x}(t) = g(x), \quad x(0) = x > 0$$

explodes in finite time.

- Reversing time, we conclude that the ODE

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The path–valued Markov process

Our assumptions

- For the rest of this talk, we assume again that f is continuous, $f(0) = 0$, $f(x+y) - f(x) \le ay$ for some $a > 0$, all $x, y \ge 0$, and moreover

  •(Sub)criticality: $\int_1^\infty \exp(-\int_1^u (2r)^{-1} f(r)dr)du = +\infty$;
  •(1/2–Hölder): For all $M > 0$, there exists C_M s.t.$|f(x+y) - f(x)| \le C_M\sqrt{y}$, for all $0 \le x \le M$, $0 \le y \le 1$.
- We define E to be the subset of $C([0,+\infty);[0,+\infty))$ consisting of those functions φ such that whenever $\zeta(\varphi) := \inf\{t > 0, \varphi(t) = 0\}$ is finite, then $\varphi(t) = 0$ for any $t \ge \zeta(\varphi)$. E is equipped with the topology of uniform convergence on compacts.
- From now on, we choose a version of the solution of the SDE

$$Z_t^x = x + \int_0^t f(Z_s^x)ds + 2\int_0^t \int_0^{Z_s^x} W(ds, du)$$

s.t. the mapping $x \to Z_\cdot^x$ is right–continuous and increasing from $[0,\infty)$ into E.

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Coupling with a Feller CSBP

- Consider, with the same space–time white noise W, the two SDEs

$$Y_t^x = x + a\int_0^t Y_s^x ds + 2\int_0^t \int_0^{Y_s^x} W(ds, du),$$
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- It follows readily from a comparison theorem due to Dawson, Li that $Z_t^x \le Y_t^x$ a.s., for all $x > 0$ and $t \ge 0$.

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- We recall that for each $t > 0$, $x > 0$, the mapping $\xi \in [0, x] \mapsto Y_t^\xi$ has a finite number of (positive) jumps, and is constant between those jumps.
- For each $t > 0$, $x > 0$, let

$$D_t = \{\xi > 0;\ Y_t^\xi > Y_t^{\xi-}\}, \text{ and}$$

$$A_t^x(Z) = \bigcup_{\xi \le x, \xi \in D_t} (Y_t^{\xi-}, Y_t^{\xi-} + Z_t^\xi - Z_t^{\xi-}].$$

- One can construct a random field $\{\tilde{Z}_t^x,\ x > 0, t \ge 0\}$ such that $t \mapsto \tilde{Z}_t^x$ is continuous, $x \mapsto \tilde{Z}_t^x$ is right–continuous, $\{\tilde{Z}_t^x,\ x > 0, t \ge 0\}$ has the same law as $\{Z_t^x,\ x > 0, t \ge 0\}$, $\{\tilde{Z}_t^x,\ x > 0, t \ge 0\}$ solves the SDE

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- Moreover $\mathbb{P}(\tilde{Z}_t^{x+y} - \tilde{Z}_t^x \le Y_t^{x+y} - Y_t^x, \forall t \ge 0) = 1$, for all $x, y > 0$.

A stronger coupling

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A corollary

- It follows readily from the above coupling

Corollary

For any $t > 0$, $x \mapsto Z_t^x$ has finitely many jumps on any compact interval, and is constant between those jumps.

- It is also not too hard to show that

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For any $s > 0$,

$$\mathbb{P}\left(\bigcup_{t>s}\{x,\ Z_t^x \neq Z_t^{x-}\} \subset \{x,\ Z_s^x \neq Z_s^{x-}\} \text{ for all } x > 0\right) = 1.$$

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The critical Feller diffusion as a sum of excursions

- For the rest of the talk Y_t^x denotes the critical Feller diffusion, solution of

$$Y_t^x = x + 2\int_0^t \int_0^{Y_s^x} W(ds, du).$$

- We can write Y^x as the solution of the SDE

$$Y_\cdot^x = \int_{[0,x]\times E} uN(dy, du),$$

where N is a Poisson random measure on $\mathbb{R}_+ \times E$ with mean measure $dy \times \mathbb{Q}(du)$, where $\mathbb{Q}$ is the excursion measure of the critical Feller diffusion, in the sense of Pitman–Yor.

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Z^x as a sum of excursion

- The above Corollary implies that similarly $x \to Z^x$ can be decomposed as a sum of excursions. Call $N_Z(dy, du)$ the corresponding point process, which is such that for all $x > 0$,

$$Z^x = \int_{[0,x]\times E} u N_Z(dy, du).$$

- We want to characterize the point process N_Z. Let $F(x, y) = f(x + y) - f(x)$ and

$$L(Z, u) = \exp\left(-\frac{1}{4}\int_0^{\zeta(u)} \frac{F(Z_s, u_s)}{u_s} du_s - \frac{1}{8}\int_0^{\zeta(u)} \frac{F^2(Z_s, u_s)}{u_s} ds\right).$$

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Main result

- Our main result says that the predictable intensity of N_Z is

$$L(Z^y, u)\mathbb{Q}(du)dy.$$

- This is equivalent to

Theorem

The path–valued process $\{Z_\cdot^x,\ x > 0\}$ can be decomposed as

$$Z^x = \int_{[0,x]\times E} uL(Z^\xi, u)\mathbb{Q}(du)d\xi + M^x,$$

where M^x is an E–valued càdlàg $\mathcal{G}^x$–martingale.

- Here $\mathcal{G}^x = \sigma\{Z_t^\xi,\ 0 \le \xi \le x, t > 0\}$.

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Indication of proof 1

- The last identity is proved as follows. We want to establish that for any $t > 0$,

$$Z_t^x = \int_{[0,x]\times E} L(Z^y, u)u(t)\mathbb{Q}(du)dy + M^x(t).$$

- Clearly if x is a dyadic number, then for n large enough

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Indication of proof 2

- But
$$y^{-1}\mathbb{E}\left(L(Z^x, U^y)U_t^y \middle| \mathcal{G}^x\right) = \mathbb{E}_{\mathbb{Q}_{y,t}}\left(L(Z^x, U^y) \middle| \mathcal{G}^x\right),$$
where under $\mathbb{Q}_{y,t}$
$$U_r = y + 4t \wedge r + 2\int_0^t \sqrt{U_s}dB_s.$$
- Finally we can take the limit as $y \to 0$ in the last identity, yielding
$$y^{-1}\mathbb{E}\left(L(Z^x, U^y)U_t^y \middle| \mathcal{G}^x\right) \to \mathbb{E}_{\mathbb{Q}_{0,t}}\left(L(Z^x, U) \middle| \mathcal{G}^x\right).$$
It just remain to verify that
$$\mathbb{E}_{\mathbb{Q}_{0,t}}\left(L(Z^x, U) \middle| \mathcal{G}^x\right) = \int_E L(Z^x, u)u(t)\mathbb{Q}(du),$$
where $\mathbb{Q}$ is the above excursion measure.

Indication of proof 2

- But
$$y^{-1}\mathbb{E}\left(L(Z^x, U^y)U_t^y \middle| \mathcal{G}^x\right) = \mathbb{E}_{\mathbb{Q}_{y,t}}\left(L(Z^x, U^y) \middle| \mathcal{G}^x\right),$$
where under $\mathbb{Q}_{y,t}$
$$U_r = y + 4t \wedge r + 2\int_0^t \sqrt{U_s}dB_s.$$
- Finally we can take the limit as $y \to 0$ in the last identity, yielding
$$y^{-1}\mathbb{E}\left(L(Z^x, U^y)U_t^y \middle| \mathcal{G}^x\right) \to \mathbb{E}_{\mathbb{Q}_{0,t}}\left(L(Z^x, U) \middle| \mathcal{G}^x\right).$$
It just remain to verify that
$$\mathbb{E}_{\mathbb{Q}_{0,t}}\left(L(Z^x, U) \middle| \mathcal{G}^x\right) = \int_E L(Z^x, u)u(t)\mathbb{Q}(du),$$
where $\mathbb{Q}$ is the above excursion measure.

The infinitesimal generator

We deduce from the above statement

Corollary

For bounded $g : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $z \in E$, *put* $\Phi_g(z) := e^{-\langle g, z\rangle}$. *Then, for this class of functions,*

$$A\Phi_g(z) := \Phi_g(z) \int_E \left(e^{-\langle g,u\rangle} - 1\right) L(z,u)\mathbb{Q}(du)$$

gives the generator of Z *in the sense that for all* $g : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\Phi_g(Z^x) - \Phi_g(Z^0) - \int_{[0,x]\times E} A\Phi_g(Z^\xi)d\xi,\ x \geq 0 \qquad \textit{is a } \mathcal{G}^x\textit{–martingale.}$$

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THANK YOU FOR
YOUR ATTENTION !