# Combinatorial aspects of tree-like structures
# II. Asymptotic analysis

Christoph Richard
Universität Erlangen-Nürnberg
Population Genetics and Statistical Physics in Synergy
EURANDOM, 27. 08. 2014

# plan:

singularity analysis of $q$-functional equations

- universal asymptotic behaviour
- corrections to asymptotic behaviour
- DE for Laplace transform of area limit law generating function

combinatorical framework analogues of

- CLTs
- Edgeworth expansions
- Feynman-Kac formulae

relation to scaling functions from statistical physics

first a trivial but instructive example...

# area limit law of rectangles: direct approach

setup and notation

- $p_{m,n}$ number of rectangles of half-perimeter $m$, area $n$
- uniform fixed perimeter ensemble
- $\widetilde{X}_m$ random variable of area

$$\mathbb{P}(\widetilde{X}_m = n) = \frac{p_{m,n}}{\sum_n p_{m,n}}$$

moment method

$$\mathbb{E}[\widetilde{X}_m^k] = \frac{1}{m-1} \sum_{l=1}^{m-1} (l(m-l))^k \sim m^{2k} \int_0^1 (x(1-x))^k \mathrm{d}x = \frac{(k!)^2}{(2k+1)!} m^{2k}$$

- $\mu_m = \mathbb{E}[\widetilde{X}_m] \sim m^2/6$
- $\sigma_m^2 = \mathbb{V}[\widetilde{X}_m] \sim m^4/180$
- no concentration, i.e. $\lim_{m\to\infty} \sigma_m/\mu_m \neq 0$

# area limit law of rectangles: direct approach

normalised area variable

$$X_m = \frac{2}{3}\frac{\widetilde{X}_m}{\mu_m} = 4\frac{\widetilde{X}_m}{m^2}$$

- $M_k := \lim_{m\to\infty} \mathbb{E}[X_m^k]$ obey Carleman condition $\sum_k M_{2k}^{-\frac{1}{2k}} = \infty$
- uniquely define law with moments $M_k$ for random variable $X$
- moment generating function

$$M(t) = \mathbb{E}[e^{-tX}] = \sum_{k=0}^{\infty} \frac{\mathbb{E}[X^k]}{k!}(-t)^k = \frac{1}{2}\sqrt{\frac{\pi}{t}}e^t \mathrm{erf}\left(\sqrt{t}\right)$$

- density by inverse Laplace transform

$$p(x) = \frac{1}{2\sqrt{1-x}} \cdot 1_{(0,1)}(x)$$

- $\beta_{1,1/2}$ distribution

# limit law via generating functions

- half-perimeter and area generating function

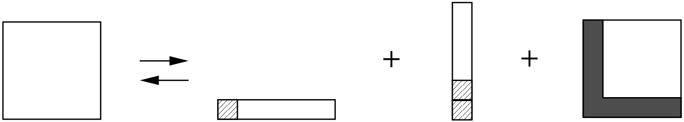
$$P(x,q) = \sum_{m,n} p_{m,n} x^m q^n$$

- factorial area moments

$$\mathbb{E}[(\widetilde{X}_m)_k] = \frac{\sum_n (n)_k p_{m,n}}{\sum_n p_{m,n}} = \frac{[x^m] \left. \frac{\partial^k}{\partial q^k} P(x,q) \right|_{q=1}}{[x^m] P(x,1)}$$

lower factorial $(a)_k = a \cdot (a-1) \cdot \ldots \cdot (a-k+1)$

# limit law via generating functions



- decomposition induces linear $q$-difference equation

$$P(x,q) = \frac{x^2 q}{1-qx} + \frac{x^3 q^2}{1-qx} + x^2 q P(qx,q)$$

- extract area moment generating functions

$$g_k(x) = \frac{1}{k!}\,\frac{\partial^k}{\partial q^k} P(x,q)\Big|_{q=1}$$

- "moment pumping" (Flajolet 97)
  compute $g_k(x)$ recursively from functional equation, by repeated differentiation w.r.t. $q$ and then setting $q{=}1$

# limit law via generating functions

- the first few area moment gf's

$$g_0(x) = \frac{x^2}{(1-x)^2}, \qquad g_1(x) = \frac{x^2}{(1-x)^4},$$

$$g_2(x) = \frac{2x^3}{(1-x)^6}, \qquad g_3(x) = \frac{6x^4}{(1-x)^8},$$

$$g_4(x) = \frac{x^4(1+22x+x^2)}{(1-x)^{10}}, \qquad g_5(x) = \frac{12x^5(1+8x+x^2)}{(1-x)^{12}}$$

- asymptotic behaviour

$$g_k(x) \sim \frac{k!}{(1-x)^{2k+2}} \qquad (x \to 1)$$

- asymptotic behaviour of area moments

$$\frac{\mathbb{E}[(X_m)_k]}{k!} \sim [x^m]\frac{k!}{(1-x)^{2k+2}} = \frac{k!}{(2k+1)!} \sim \frac{\mathbb{E}[(X_m)^k]}{k!}$$

## an observation

For $X$ with law $\beta_{1,1/2}$ consider the double Laplace transform

$$F(s) := \int_0^\infty e^{-st}\mathbb{E}[e^{-t^2X}]t\,\mathrm{d}t$$

- with $\mathrm{Ei}(z) = \int_1^\infty e^{-tz}/t\,\mathrm{d}t$ the exponential integral, we have

$$F(s) = \mathrm{Ei}(s^2)e^{s^2}$$

- asymptotic expansion

$$F(s) \sim \sum_{k\geq 0}(-1)^k k! s^{-(2k+2)} \qquad (s \to \infty)$$

- coefficents $k!$ are amplitudes of $g_k(x)$ at singularity
- $F(s)$ analytic for $\Re(s) \geq s_0$ uniquely determined by its asymptotic behaviour

## strategy

Feynman-Kac type approach

- reconstruct limit law $X$ from double Laplace transform $F(s)$
- obtain $F(s)$ from amplitudes of $g_k(x)$ at singularity
- functional equation induces differential equation for $F(s)$

Edgeworth type expansions

- subleading corrections $X_\ell$ via double Laplace transforms $F_\ell(s)$
- differential equations for $F_\ell(s)$

statistical physics terminology:

- $F(s)$ scaling function, $F_\ell(s)$ correction-to-scaling functions
- scaling ansatz ("method of dominant balance")

# scaling ansatz

area moment generating functions

$$g_k(x) = \sum_{\ell \geq 0} \frac{f_{k,\ell}}{(1-x)^{2k+2-\ell}}$$

generating functions for amplitudes $f_{k,\ell}$

- $F(s)$ generating function of $(f_{k,0})_k$
- $F_\ell(s)$ generating function of $(f_{k,\ell})_k$
- compute these from functional equation

## scaling ansatz

formal manipulation:

$$
\begin{aligned}
P(x,q) &= \sum_k (-1)^k g_k(x) \cdot (1-q)^k \\
&= \sum_k (-1)^k \sum_\ell \frac{f_{k,\ell}}{(1-x)^{2k+2-\ell}} \cdot (1-q)^k \\
&= \frac{1}{1-q} \sum_\ell \sum_k (-1)^k \frac{f_{k,\ell}}{\left(\frac{1-x}{\sqrt{1-q}}\right)^{2k+2-\ell}} \left(\sqrt{1-q}\right)^\ell \\
&= \frac{1}{1-q} F\left(\frac{1-x}{\sqrt{1-q}}, \sqrt{1-q}\right)
\end{aligned}
$$

with $F(s,\epsilon) = \sum_\ell F_\ell(s)\epsilon^\ell$ and

$$
F_\ell(s) = \sum_k (-1)^k \frac{f_{k,\ell}}{s^{2k+2-\ell}}
$$

## scaling ansatz

- introduce $F(s, \epsilon)$ into functional equation via

$$P(x, q) = \frac{1}{1-q} F\left( \frac{1-x}{(1-q)^{1/2}}, (1-q)^{1/2} \right)$$

- introduce variables $s, \epsilon$ via $x = 1 - s\epsilon$ and $q = 1 - \epsilon^2$
- expand functional equation in powers of $\epsilon$
- order $\epsilon^0$ yields first order differential equation

$$sF_0'(s) + 2 - 2s^2 F_0(s) = 0$$

- order $\epsilon^\ell$ yields DE for $F_\ell(s)$

## remarks

limit distributions for rectangles

- rigorous method, since all $g_k(x)$ are rational
- differential equations can be mechanically obtained
- corrections-to-scaling to arbitrary order

method applies to more general $q$-functional equations:

- e.g. for *algebraic* perimeter generating function $P(x, 1)$ and area moment generating functions $g_k(x)$
- Newton-Puiseaux expansion of $g_k(x)$ about dominant singularity
- limit law via asymptotic expansion of $F(s)$, transfer theorem and inverse Laplace trafo
- alternatively via double inverse Laplace trafo

## transfer theorem

**Theorem (Flajolet–Odlyzko 90)**

*For $\alpha \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ let $f(z)$ be analytic in the open indented disc*

$$D(\rho, \sigma, \phi) = \{z \in \mathbb{C} \mid |z| < \sigma, |\arg(z - \rho)| < \phi\},$$

*where $0 < \rho < \sigma$ and $0 < \phi < \pi/2$. If in the intersection of a small neighborhood of $\rho$ with $D(\rho, \sigma, \phi)$ we have*

$$f(z) \sim (1 - z/\rho)^{-\alpha} \qquad (z \to \rho),$$

*then $[z^n]f(z) \sim \rho^{-n} \frac{n^{\alpha-1}}{\Gamma(\alpha)}$ for $n \to \infty$.*

similar results for logarithmic singularities (see Flajolet 09)

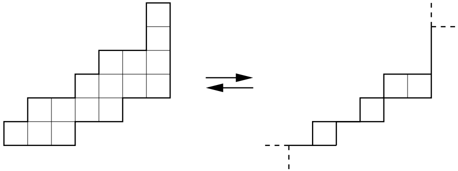
## examples

area limit law determined by singularity of perimeter generating function

- double pole: $\beta_{1,1/2}$
- simple pole: concentrated
- square root: area under Brownian excursion
- inverse square root: area under Brownian meander

# square-root-singularity: staircase polygons

decomposition of staircase polygons



quadratic $q$-difference equation

$$P(x,q) = \frac{qx^2}{1-(2qx+P(qx,q))}$$

area random variable in uniform fixed perimeter ensemble

$$\mu_m \sim \frac{\sqrt{\pi}}{4} m^{3/2} \qquad \sigma_m^2 \sim \frac{10-3\pi}{48} m^3$$

# square-root singularity: staircase polygons

**Theorem (cf Duchon 99, Takács 91)**

*The area random variables $X_m$ of staircase polygons satisfy*

$$\frac{X_m}{\mu_m} \xrightarrow{d} \frac{X}{\sqrt{\pi}} \qquad (m \to \infty)$$

*where $X$ is Airy distributed, i.e.*

$$\frac{\mathbb{E}[X^k]}{k!} = \frac{\Gamma(\gamma_0)}{\Gamma(\gamma_k)} \frac{\phi_k}{\phi_0},$$

*where $\gamma_k = (3k-1)/2$, and $\phi_k$ satisfies the quadratic recurrence*

$$\gamma_{k-1}\phi_{k-1} + \frac{1}{2}\sum_{l=0}^{k} \phi_l \phi_{k-l} = 0, \qquad \phi_0 = -1.$$

same result for path length in simply generated trees, discrete Bernoulli excursion area, construction cost for hash table, ...

# $q$-difference equations and universality

**Theorem (Takács 91, Duchon 99, R 05)**

*Airy limit law appears for solutions of*

$$P(x, q) = xF(x, q, P(qx, q))$$

*under sufficient assumptions on $F(x, q, y)$, e.g.*

- *$F$ polynomial, at least quadratic in $y$*
- *$F$ has no negative coefficients*

$\square$

interpretation

- thus square-root singularity generic
- similar statements for other types of singularity

# inverse square-root singularity

## Theorem (R 07, cf Takács 95)

*The area random variables $X_m$ of directed convex polygons satisfy*

$$\frac{X_m}{\mu_m} \xrightarrow{d} \frac{Z}{\mathbb{E}[Z]} \qquad (m \to \infty),$$

*with $Z$ the area random variable of the Brownian meander, i.e.,*

$$\frac{\mathbb{E}[Z^k]}{k!} = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_k)} \frac{\omega_k}{\omega_0} \frac{1}{2^{k/2}},$$

*where $\alpha_k = (3k+1)/2$, and $\omega_k$ satisfies the linear recurrence*

$$\alpha_{k-1}\omega_{k-1} + \sum_{l=0}^{k} \phi_l 2^{-l} \omega_{k-l} = 0, \qquad \omega_0 = 1.$$

# higher rank parameters for models of Bernoulli walks

- Bernoulli random walks, meanders, bilateral Dyck paths, Dyck paths
- discrete counterparts of Brownian motion, meanders, excursions and bridges (Kaigh 76, Aldous 92, Drmota–Marckert 05)
- higher rank parameters can be treated
- correspond to $k$-th area moments of Brownian motion, meanders, excursion, and bridges
- recursions for moments of joint distribution can be mechanically derived

# higher rank parameters for models of walks

Bernoulli random walk $b$

- $n(b)$ number of steps
- $k$-th moment of absolute height

$$n_k(b) = \sum_{s \in b} |h(s)|^k$$

  $h(s)$ height of walk at $s$, with $s = 0, 1, \ldots, n(b)$

- weight $w_b(\mathbf{u})$ of $b$

$$w_b(\mathbf{u}) = u_0^{n(b)} \cdot u_1^{n_1(b)} \cdot \ldots \cdot u_M^{n_M(b)}$$

- generating function $G^{(r)}(\mathbf{u}) = \sum_b w_b(\mathbf{u})$

# higher rank parameters for models of walks

## Theorem (Nguyễn Thế 03)

*Let $G^{(d)}(\mathbf{u})$, $G^{(b)}(\mathbf{u})$, $G^{(m)}(\mathbf{u})$ and $G^{(r)}(\mathbf{u})$ denote the generating functions of Dyck paths, bilateral Dyck paths, meanders, and Bernoulli random walks. Then*

$$\begin{aligned}
G^{(d)}(\mathbf{u}) &= \frac{1}{1 - u_0^2 u_1 \cdot \ldots \cdot u_M G^{(d)}(\mathbf{v}(\mathbf{u}))} \\
G^{(b)}(\mathbf{u}) &= \frac{1}{1 - 2u_0^2 u_1 \cdot \ldots \cdot u_M G^{(d)}(\mathbf{v}(\mathbf{u}))} \\
G^{(m)}(\mathbf{u}) &= G^{(d)}(\mathbf{u})(1 + u_0 \cdot \ldots \cdot u_M G^{(m)}(\mathbf{v}(\mathbf{u})) \\
G^{(r)}(\mathbf{u}) &= G^{(b)}(\mathbf{u})(1 + 2u_0 \cdot \ldots \cdot u_M G^{(m)}(\mathbf{v}(\mathbf{u}))
\end{aligned}$$

*with $v_k(\mathbf{u})$ given by*

$$v_k(\mathbf{u}) = \prod_{l=k}^{M} u_l^{\binom{l}{k}} \qquad (k = 0, 1, \ldots, M)$$

factorial moment generating functions

$$g_{\mathbf{k}}(u_0) := \frac{1}{\mathbf{k}!} \frac{\partial^{k_1}}{\partial u_1^{k_1}} \cdots \frac{\partial^{k_M}}{\partial u_M^{k_M}} G(\mathbf{u}) \Big|_{\mathbf{u}=\mathbf{u}_0}$$

- $\mathbf{k} = (k_1, \ldots, k_M) \in \mathbb{N}_0^M$, $\mathbf{u}_0 = (u_0, 1, \ldots, 1)$
- multi-index notation: $\mathbf{k}! = k_1! \cdot \ldots \cdot k_M!$, $|\mathbf{k}| = k_1 + \ldots + k_M$
- $\mathbf{k} \leq \mathbf{l}$ if $k_i \leq l_i$ for $i = 1, \ldots, M$.
- unit vectors $\mathbf{e}_k$, where $(\mathbf{e}_k)_i = \delta_{i,k}$ for $i = 1, \ldots, M$

**Theorem (R 05)**

*All generating functions $g_{\mathbf{k}}^{(\cdot)}(u_0)$ are algebraic, where $(\cdot) \in \{(d), (b), (m), (r)\}$. They are analytic for $|u_0| \leq u_c = 1/2$, except at $u_0 = \pm u_c$, with Puiseux expansions about $u_0 = u_c$ of the form*

$$g_{\mathbf{k}}^{(\cdot)}(u_0) = \sum_{l=0}^{\infty} f_{\mathbf{k},l}^{(\cdot)} (u_c - u_0)^{l/2 - \gamma_{\mathbf{k}}^{(\cdot)}}$$

**Theorem (R 05, ctnd)**

*The exponents* $\gamma_{\mathbf{k}}^{(\cdot)}$ *are given by*

$$\gamma_{\mathbf{k}}^{(d)} = -\frac{1}{2} + \sum_{i=1}^{M} \left(1 + \frac{i}{2}\right) k_i\,, \qquad \gamma_{\mathbf{k}}^{(b)} = \gamma_{\mathbf{k}}^{(m)} = \gamma_{\mathbf{k}}^{(d)} + 1, \qquad \gamma_{\mathbf{k}}^{(r)} = \gamma_{\mathbf{k}}^{(d)} + \frac{3}{2}.$$

*The leading coefficients* $f_{\mathbf{k},0}^{(\cdot)} = f_{\mathbf{k}}^{(\cdot)}$ *satisfy, for* $\mathbf{k} \neq \mathbf{0}$*, the recursions*

$$\gamma_{\mathbf{k}-\mathbf{e}_1}^{(d)} f_{\mathbf{k}-\mathbf{e}_1}^{(d)} + 2 \sum_{i=1}^{M-1} (i+1)(k_i+1) f_{\mathbf{k}-\mathbf{e}_{i+1}+\mathbf{e}_i}^{(d)} + \sum_{\mathbf{0}\leq \mathbf{l} \leq \mathbf{k}} f_{\mathbf{l}}^{(d)} f_{\mathbf{k}-\mathbf{l}}^{(d)} = 0,$$

$$\gamma_{\mathbf{k}-\mathbf{e}_1}^{(b)} f_{\mathbf{k}-\mathbf{e}_1}^{(b)} + 2 \sum_{i=1}^{M-1} (i+1)(k_i+1) f_{\mathbf{k}-\mathbf{e}_{i+1}+\mathbf{e}_i}^{(b)} - 8 \sum_{\mathbf{0}\leq \mathbf{l} \leq \mathbf{k}} f_{\mathbf{l}}^{(b)} f_{\mathbf{k}-\mathbf{l}}^{(b)} = 0,$$

$$\gamma_{\mathbf{k}-\mathbf{e}_1}^{(m)} f_{\mathbf{k}-\mathbf{e}_1}^{(m)} + 2 \sum_{i=1}^{M-1} (i+1)(k_i+1) f_{\mathbf{k}-\mathbf{e}_{i+1}+\mathbf{e}_i}^{(m)} + \sum_{\mathbf{0}\leq \mathbf{l} \leq \mathbf{k}} f_{\mathbf{l}}^{(m)} f_{\mathbf{k}-\mathbf{l}}^{(d)} = 0,$$

$$f_{\mathbf{k}}^{(r)} = \sum_{\mathbf{0}\leq \mathbf{l} \leq \mathbf{k}} f_{\mathbf{l}}^{(b)} f_{\mathbf{k}-\mathbf{l}}^{(m)},$$

*with boundary conditions* $f_{\mathbf{0}}^{(d)} = -4$, $f_{\mathbf{0}}^{(b)} = 1/2$, $f_{\mathbf{0}}^{(m)} = 1$, $f_{\mathbf{0}}^{(r)} = 1/2$, *and* $f_{\mathbf{k}}^{(\cdot)} = 0$ *if* $k_j < 0$ *for some* $1 \leq j \leq M$*. The coefficients* $f_{\mathbf{k}}^{(\cdot)}$ *are strictly positive for* $\mathbf{k} \neq \mathbf{0}$*.*

# area of discrete meanders with bounded step set

discrete meanders

- step set $\mathcal{S} \subseteq \{-c, -c+1, \ldots, d-1, d\} \subset \mathbb{Z}$
- $c, d$ positive, $-c, d \in \mathcal{S}$

generating functions

- step polynomial $S(u) = \sum_{s_i \in \mathcal{S}} s_i u^i$ with weights $s_i > 0$
- aperiodic: $u^c S(u) = H(u^p)$ for polynomial $H$ only for $p = 1$
- $F(z, q, u)$ perimeter, area, and final height gf

$q$-shift of length and final height functional equation

$$F(z,q,u) = 1 + zS(uq)F(z,q,uq) - z\sum_{i=0}^{c-1} r_i(uq)G_i(z,q)$$

$r_i(u) = u^i \left(s_{-c}u^{-c} + \ldots + s_{-(i+1)}u^{-(i+1)}\right)$, $G_i(z,q) = [u^i]F(z,q,u)$

asymptotic analysis via moment pumping and kernel method possible!

# area of discrete meanders with bounded step set

**Theorem (Schwerdtfeger 14)**

*Fix drift $\gamma = S'(1)/S(1)$ and consider the area random variables $Z_m$ for meanders of length $m$. After proper rescaling:*

- *negative drift: convergence to Brownian excursion area*
- *zero drift: convergence to Brownian meander area*
- *positive drift: concentration*

remarks

- $\gamma = 0$ also via FCLT (Iglehardt 74)
- $\gamma > 0$ with Gaussian CLL (Iglehardt 74)
- $\gamma < 0$ for *non-lattice* step sets (Kao 78, Durrett 80)

# Wiener index for simply generated trees (Janson 03)

simply generated trees

- realised as conditioned Galton-Watson trees of size $n$ defined by offspring distribution $X$
- Wiener index

$$W(T) = \frac{1}{2} \sum_{v,w \in T} d(v, w)$$

- simpler quantity

$$Q(T) = \sum_{v,w \in T} d(v \wedge w, o) = n(T)m(T) - W(T)$$

$v \wedge w$ last common ancestor, $n(T)$ # nodes, $m(T)$ pathlength

## Theorem (Janson 03)

*Assume* $\mathbb{E}[X] = 1$ *and* $0 < \sigma^2 := \mathbb{V}[X] < \infty$. *With* $e(u)$ *the standard Brownian excursion, we have*

$$\left( \frac{m(T_n)}{n^{3/2}/\sigma}, \frac{Q(T_n)}{n^{5/2}/\sigma} \right) \xrightarrow{d} \left( 2 \int_0^1 e(t)\mathrm{d}t, 4 \int \int_{0<s<t<1} \min(e(u))\, \mathrm{d}s\mathrm{d}t \right)$$

*The joint moments of the rhs* $(\xi, \eta)$ *are given by*

$$\mathbb{E}[\xi^k \eta^\ell] = \frac{k!\ell!\sqrt{\pi}}{2^{(5k+7\ell-4)/2}\Gamma((3k+5\ell-1)/2)} a_{k\ell},$$

*where the numbers* $a_{k\ell}$ *with* $a_{10} = a_{01} = 1$ *satisfy*

$$a_{k,\ell} = 2(3k+5\ell-4)a_{k-1,\ell} + 2(3k+5\ell-6)(3k+5\ell-4)a_{k,\ell-1} + \frac{1}{2}\sum\sum_{0<i+j<k+\ell} a_{i,j}a_{k-i,\ell-j},$$

*with* $a_{k\ell} = 0$ *if* $k < 0$ *or* $\ell < 0$.

## singularities of $q$-functional equations

- $q$-functional equation for polynomial $F$

$$F(x, q, P(x, q), P(qx, q)) = 0$$

- typical case: algebraic equation for $P(x, 1)$

$$F(x, 1, P(x, 1), P(x, 1)) = 0$$

- degenerate case: algebraic differential equation for $P(x, 1)$

$$G(x, P(x, 1), P'(x, 1)) = 0$$

  (note $(f(qx) - f(x))/(q - 1) \to f'(x)$)

- singularities of $D$-finite functions (linear DE with polynomial coefficients) have been classified
- prudent polygons satisfy $q$-functional equation and have non-$D$-finite perimeter generating function

# singularities of prudent polygons

three-sided prudent polygons (Schwerdtfeger 10)

- $P_3(x)$ radius of convergence $\sigma = \tau^2$ with square root singularity $1/2$

$$\sigma = 0.24412\ldots, \text{ where } \tau^5 + 2\tau^2 + 3\tau - 2 = 0$$

- meromorphic in slit disc $\{|x| < \rho\} \setminus [\sigma, \rho\}$

$$\rho = 0.29559\ldots, \text{ where } 1 - 3\rho - \rho^2 - \rho^3 = 0$$

- infinitely many singularities in $[\sigma, \rho)$ accumulating in $\rho$

prudent polygons

- numerical analysis (Guttmann et al 11)
- 500 terms from functional equation
- radius of convergence $x_c \approx 0.22647\ldots$, exponent $\approx 5/2$

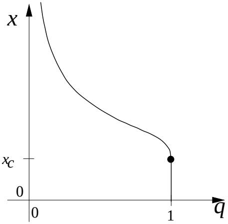
# singularities of three-choice polygons

- $P(x)$ half-perimeter generating function
- analysis of $8^{\text{th}}$ order ODE (Guttmann, Jensen 05)
- dominant singularity $x_c = 1/4$ with

$$P(x) \sim A(1-4x)^{-1/2} + B(1-4x)^{-1/2} \log(1-4x) \qquad (x \to 1/4^-)$$

# phase diagrams

behaviour of SAP area for weights $p_{m,n}q^n$, fixed large perimeter $m$?




- radius of convergence $x_c(q)$ of $x \mapsto P(x, q)$
- type of singularity does not change on critical line $q < 1$ resp. $q = 1$
- $q < 1$ deflated phase (branched polymers)
- $q > 1$ inflated phase (ball-shaped ring polymers)
- extended phase $q = 1$ (collapse phase transition)
- concentration for $q \neq 1$

# phase diagrams

polygon models

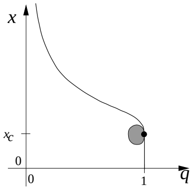
- proved/provable for solvable models ($q$-functional equation)
- partly proved for SAPs

similarly for

- solvable models of walks and trees
- behaviour of perimeter for weights $p_{m,n}x^m$, fixed large area $n$

# singular behaviour of staircase polygons (Prellberg 95)

singular behaviour of $P(x,q)$ in domain $\mathcal{D}$ about $(x,q)=(x_c,1)$




$$P(x,q)-\frac{1}{4}\sim(1-q)^{1/3}F\left(\frac{1-\frac{x}{x_c}}{(1-q)^{2/3}}\right)\quad(x,q)\to(x_c,1)$$

- $F:(4^{-1/3}a_0,\infty)\to\mathbb{R}$ scaling function $F(s)=\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}s}\log\mathrm{Ai}(4^{1/3}s)$
- $a_0=-2.338\ldots$ location of Airy function zero of smallest modulus
- proof with explicit expression for $P(x,q)$ by delicate saddle point analysis

## scaling function for staircase polygons

$$P(x,q) - \frac{1}{4} \sim (1-q)^{1/3} F\left(\frac{1-\frac{x}{x_c}}{(1-q)^{2/3}}\right) \quad (x,q) \to (x_c, 1) \text{ in } \mathcal{D}!!$$

- scaling function at 0 determines agf at criticality

$$P^{(sing)}(x_c, q) \sim (1-q)^{\frac{1}{3}} F(0) \qquad (q \to 1)$$

- scaling function at $\infty$ determines pgf at criticality

$$\begin{aligned} P^{(sing)}(x,1) &\sim (1-q)^{\frac{1}{3}} F\left(\frac{1-\frac{x}{x_c}}{(1-q)^{2/3}}\right) \qquad (x,q) \to (x_c,1) \\ &\sim (1-q)^{\frac{1}{3}} f_0 \left(\frac{1-\frac{x}{x_c}}{(1-q)^{2/3}}\right)^{1/2} \qquad (x,q) \to (x_c,1) \\ &\sim f_0 \left(1 - \frac{x}{x_c}\right)^{1/2} \qquad (x \to x_c) \end{aligned}$$

# scaling function for staircase polygons

scaling relation for $P(x, q)$ remains valid under arbitrary differentiation!

- area limit law from asymptotic expansion about $s = \infty$

$$F(s) \sim \sum_{k \geq 0} (-1)^k \frac{\phi_k}{2^{2k+1}} s^{-\gamma_k} \qquad (s \to \infty)$$

- critical perimeter limit law from Taylor expansion about $s = 0$

$$F(s) = \sum_{k \geq 0} \frac{b_k}{4^{k/3+1}} s^k$$

- scaling function describes "crossover from extended to deflated phase"

# self-avoiding polygons: area limit law

## Conjecture (cf R Guttmann Jensen 01)

*$X_m$ area RV for SAPs in uniform fixed perimeter ensemble*

$$\mathbb{P}(X_m = n) = \frac{p_{m,n}}{\sum_n p_{m,n}}$$

*$p_{m,n}$ # square lattice SAPs of half-perimeter $m$, area $n$*

$$\frac{X_m}{\mathbb{E}[X_m]} \xrightarrow{d} \frac{X}{\sqrt{\pi}},$$

*where $X$ is Airy distributed.*

# self-avoiding polygons: critical perimeter limit law

## Conjecture (cf R Guttmann Jensen 04)

*$Y_n$ perimeter RV for SAPs in non-uniform fixed area ensemble*

$$\mathbb{P}[Y_n = m] = \frac{p_{m,n} x_c^m}{\sum_m p_{m,n} x_c^m}$$

*Then $Y_n/n^{2/3} \xrightarrow{d} Y$, where for some constant $C > 0$*

$$\frac{\mathbb{E}[Y^k]}{k!} = \frac{\Gamma(\beta_0)}{\Gamma(\beta_k)} \frac{b_k}{b_0} C^k, \qquad \beta_k = (2k-1)/3$$

*The numbers $b_k$ are given by $F(s) = -\frac{\mathrm{d}}{\mathrm{d}s} \log Ai(s) = \sum_{k \geq 0} b_k s^k$.*

# self-avoiding polygons: perimeter limit laws

## Conjecture

*$Y_n$ perimeter RV for SAPs in uniform fixed area ensemble*

$$\mathbb{P}(Y_n = m) = \frac{p_{m,n}}{\sum_m p_{m,n}}$$

*$p_{m,n}$ # square lattice SAPs of half-perimeter $m$, area $n$*
*The standardised perimeter RV is asymptotically normal,*

$$\frac{Y_n - \mathbb{E}[Y_n]}{\sqrt{\mathbb{V}[Y_n]}} \xrightarrow{d} \mathcal{N}(0,1)$$

# scaling function: definition

- assumptions on coefficients $p_{m,n}$
  - $p_{m,n}$ non-negative, positive only if $Am \leq n \leq Bm^2$
  - radius of convergence $x_c$ of $P(x, 1)$ satisfies $0 < x_c \leq 1$
- domain of scaling function approximation $(s_0 < 0, \varphi > 0)$

$$\mathcal{D}(s_0) = \left\{(x, q) \in (0, \infty) \times (0, 1) : s_0 < \frac{1 - x/x_c}{(1-q)^\varphi}\right\}$$

- $F : (s_0, \infty) \to \mathbb{R}$ and $c > 0$, $\theta > 0$ s.t. for $P^{(sg)}(x, q) = P(x, q) - c$

$$P^{(sg)}(x, q) \sim (1-q)^\theta F\left(\frac{1 - x/x_c}{(1-q)^\varphi}\right) \qquad (x, q) \to (x_c^-, 1^-) \text{ in } \mathcal{D}(s_0)$$

- then $F$ is called scaling function, $\varphi, \theta$ critical exponents

# scaling function and one-variable generating functions

The scaling function determines the leading singular behaviour of the perimeter generating function and of the critical area generating function.

## Proposition (Eisner 10)

- *Assume* $F(s) \sim f_0 s^{-\gamma_0}$ *as* $s \to \infty$ *and* $\frac{\partial^r}{\partial x^r} P(x,1) \to \infty$ *as* $x \to x_c^-$ *for some* $r \in \mathbb{N}_0$*. Then* $\gamma_0 = -\theta/\varphi$ *and*

$$P^{(sg)}(x,1) \sim f_0 (1 - x/x_c)^{-\gamma_0} \qquad (x \to x_c^-)$$

- *Assume* $F(s) \sim h_0 s^{\alpha_0}$ *as* $s \to 0$ *and* $\frac{\partial^r}{\partial q^r} P(x_c,q) \to \infty$ *as* $q \to 1^-$ *for some* $r \in \mathbb{N}_0$*. Then* $\alpha_0 = 0$ *and*

$$P^{(sg)}(x_c,q) \sim h_0 (1-q)^{\theta} \qquad (q \to 1^-)$$

# scaling functions: differentiability

asymptotic expansions integrable, but generally not differentiable:

- $f(x) = x + \sin(x)$, $f'(x) = 1 + \cos(x)$
- $f(x) \sim x$ as $x \to \infty$ and $f'(x) = 1 + O(1)$, but $f'(x) \not\sim 1$

monotonicity assumptions:

**Theorem (Olver 74)**

*Let $f : \mathbb{R} \to \mathbb{R}$ be continously differentiable with $f'(x)$ eventually monotonically increasing. Assume that there is $p \geq 1$ such that*

$$f(x) \sim x^p \qquad (x \to \infty)$$

*Then we have*

$$f'(x) \sim p x^{p-1} \qquad (x \to \infty)$$

combinatorial generating functions are monotonic!

# scaling functions: differentiability

## Theorem (Eisner 10)

*Let $F(s)$ be analytic in some domain containing $(s_0, \infty)$ with*

$$F(s) \sim \sum_{k\geq 0} (-1)^k f_k s^{-\gamma_k} \qquad (s \to \infty)$$

*Fix $k \in \mathbb{N}_0$ and assume $F^{(k+1)}$ decreasing for even $k+1$, resp. increasing for odd $k+1$. Assume that for some $r \in \mathbb{N}_0$*

$$\frac{\partial^r}{\partial x^r} P(x,1) \to \infty \qquad (x \to x_c^-), \qquad \frac{\partial}{\partial q} P(x_c, q) \to \infty \qquad (q \to 1^-)$$

*Then $\gamma_k = (k-\theta)/\varphi$ and*

$$\frac{1}{k!} \frac{\partial^k}{\partial q^k} P^{(sg)}(x,q)\Big|_{q=1} \sim \frac{f_k}{\left(1 - \frac{x}{x_c}\right)^{\gamma_k}} \qquad (x \to x_c^-)$$

analogous result for perimeter moment series

# limit distributions and scaling functions

two-parameter tree-like structures: recipe for scaling functions

- $P(x,q)$ generating function (e.g. perimeter and area)
- phase diagram has critical point $(x_c, 1)$
- extract critical exponents $\varphi$, $\theta$ as $x \to x_c^-$

$$P^{(sg)}(x,1) \sim f_0 \left(1 - \frac{x}{x_c}\right)^{\frac{\theta}{\varphi}}, \qquad \frac{\partial}{\partial q} P^{(sg)}(x,q)\Big|_{q=1} \sim f_1 \left(1 - \frac{x}{x_c}\right)^{\frac{\theta-1}{\varphi}}$$

- extract limit distribution $X$ of area
- candidate for scaling function: double Laplace trafo of $X$

$$F(s) = \int_0^\infty e^{-st} \mathbb{E}[e^{-t^{\frac{1}{\varphi}} X}] \frac{1}{t^{1+\frac{\theta}{\varphi}}} \mathrm{d}t$$

(here $\theta < 0$, similarly for $\theta > 0$)

- check continuity assumptions (!)