On rebellious voter models

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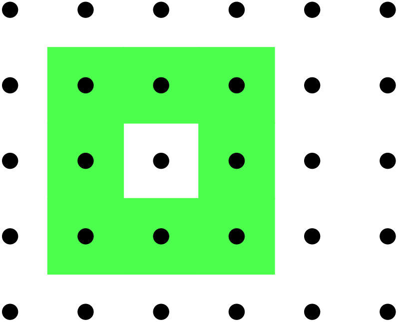
joint with Anja Sturm and Karel Vrbenský

The Neuhauser-Pacala model

Denote a point in $\mathbb{Z}^d$ by $i = (i_1, \ldots, i_d)$.

Def *neighborhood* of a site $\mathcal{N}_i := \{j \in \mathbb{Z}^d : 0 < \|i - j\|_\infty \le R\}$.



(Here $R = 1$, $d = 2$).

The Neuhauser-Pacala model

Def *local frequency* $f_\tau(i) := |\mathcal{N}_i|^{-1}|\{j \in \mathcal{N}_i : x(j) = \tau\}|$.

1	0	1	1	0	0
1	1	0	1	1	1
1	1	0	0	1	1
0	0	1	1	1	0
1	0	1	1	1	0

Here $f_0(i) = 3/8$, $f_1(i) = 5/8$.

The Neuhauser-Pacala model

Fix rates $\alpha_{01}, \alpha_{10} \geq 0$.

$$\begin{array}{cccccc}
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & \dagger & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0
\end{array}$$

With rate $f_0 + \alpha_{01} f_1$ an organism of type 0 dies...

The Neuhauser-Pacala model

1	0	1	1	0	0
1	1	0	1	1	1
1	1	1	0	1	1
0	0	1	1	1	0
1	0	1	1	1	0

. . . and is replaced by a random type from the neighborhood.

The Neuhauser-Pacala model

Neuhauser & Pacala (1999): Markov process in the space $\{0,1\}^{\mathbb{Z}^d}$ of spin configurations $x = (x(i))_{i\in\mathbb{Z}^d}$, where spin $x(i)$ flips:

$$0 \mapsto 1 \text{ with rate } f_1(f_0 + \alpha_{01} f_1),$$
$$1 \mapsto 0 \text{ with rate } f_0(f_1 + \alpha_{10} f_0),$$

with

$$f_\tau(i) := \frac{|\{j \in \mathcal{N}_i : x(j) = \tau\}|}{|\mathcal{N}_i|} \quad \mathcal{N}_i := \{j : 0 < \|i - j\|_\infty \le R\}.$$

the local frequency of type $\tau = 0, 1$.

Interpretation: *Interspecific competition rates* α_{01}, α_{10}. Organism of type 0 dies with rate $f_0 + \alpha_{01} f_1$ and is replaced by type sampled at random from distance $\le R$.

The Neuhauser-Pacala model

Parameter α_{01} measures the strength of competition felt by type 0 from type 1 (compared to strength 1 from its own type).
If $\alpha_{01} < 1$, then type 0 dies *less* often due to competition from type 1 than from competition with its own type: *balancing selection*.
If $\alpha_{01} > 1$, then type 0 dies *more* often due to competition from type 1 than from competition with its own type, i.e., type 1 is an *agressive species*.

By definition, type 0 *survives* if starting from a single organism of type 0 and all other organisms of type 1, there is a positive probability that the organisms of type 0 never die out.

By definition, one has *coexistence* if there exists an invariant law concentrated on states where both types are present.

Mean field model

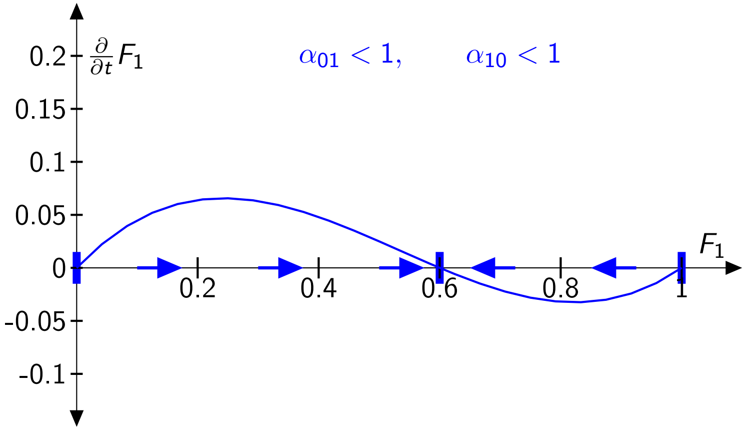
In the *mean field model*, the lattice $\mathbb{Z}^d$ is replaced by a complete graph with N vertices. In this case, the neighborhood $\mathcal{N}_i$ of a vertex i is simply all sites except i.

In the limit $N \to \infty$, the frequencies $F_\tau(t)$ of type $\tau = 0, 1$ satisfy a differential equation:

$$\begin{aligned} \tfrac{\partial}{\partial t} F_1(t) = {} & F_1(t)\big(F_0(t) + \alpha_{01} F_1(t)\big) F_0(t) \\ & - F_0(t)\big(F_1(t) + \alpha_{10} F_0(t)\big) F_1(t). \end{aligned}$$

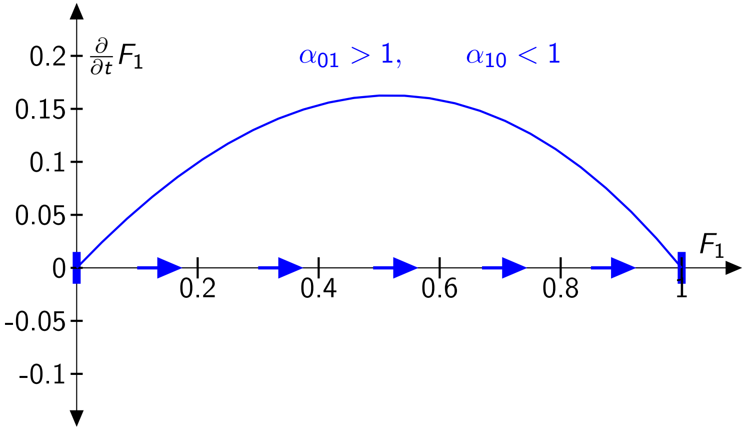
with $F_0 = 1 - F_1$.

Mean field model



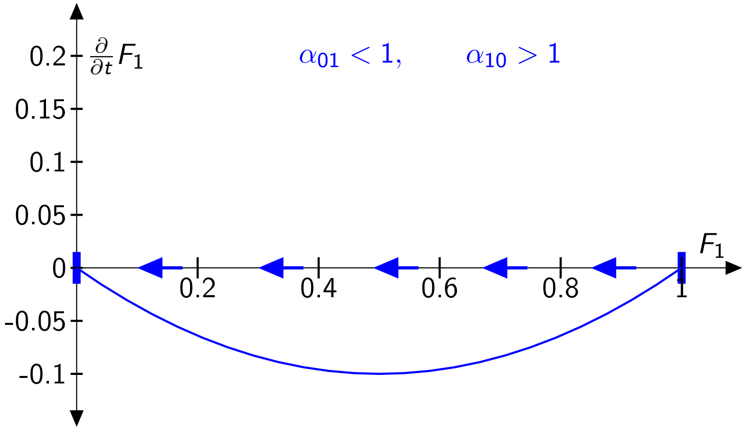
Balancing selection ($\alpha_{01} = 0.6, \alpha_{10} = 0.4$).

Mean field model



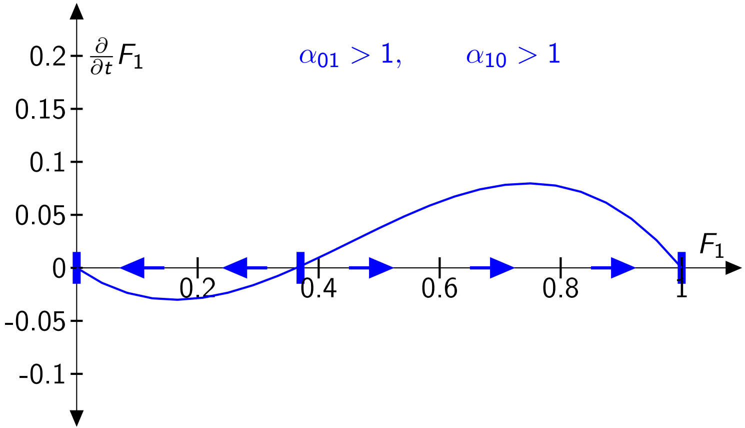
Type 1 is an agressive species ($\alpha_{01} = 1.7, \alpha_{10} = 0.4$).

Mean field model



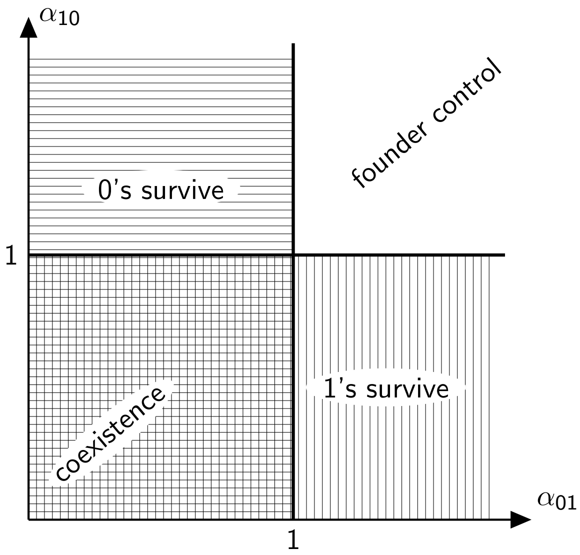
Type 0 is an agressive species ($\alpha_{01} = 0.6, \alpha_{10} = 1.4$).

Mean field model

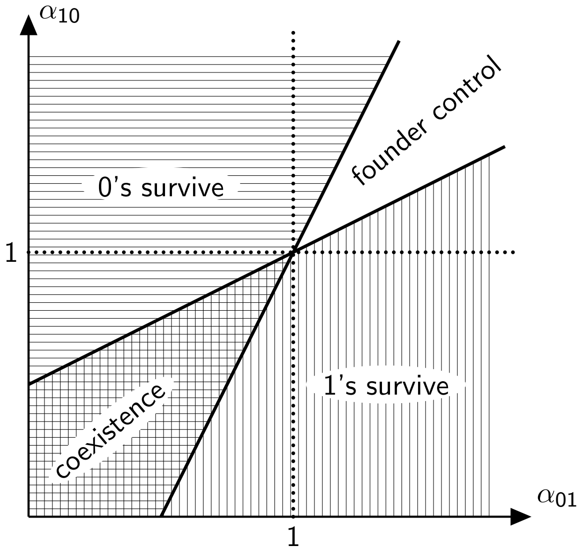


Both types are agressive species ($\alpha_{01} = 1.7, \alpha_{10} = 1.4$).

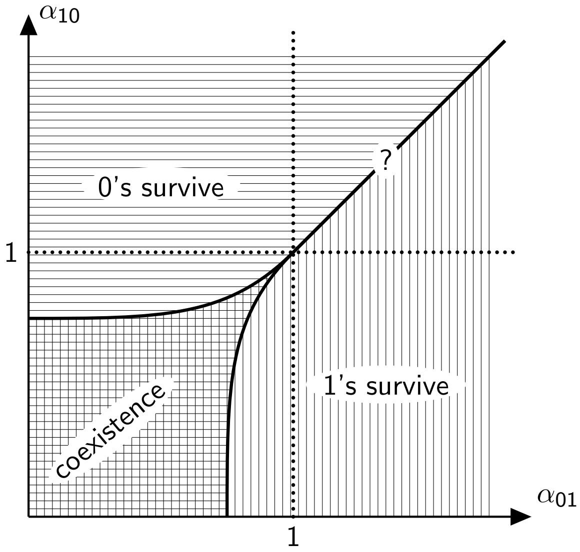
Mean field model



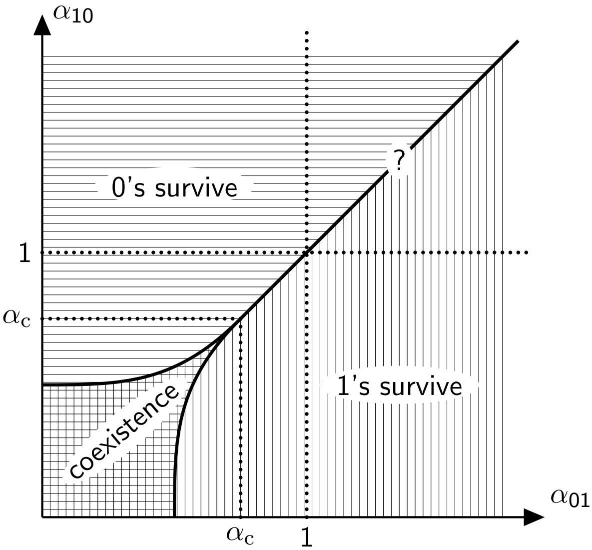
Dimension $d \geq 3$



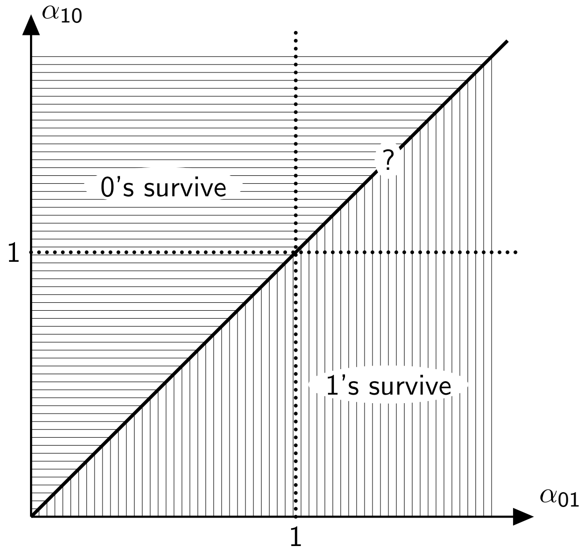
Dimension $d = 2$



Dimension $d = 1$, range $R \geq 2$



Dimension $d = 1$, range $R = 1$



Rigorous results

Sudbury, AOP, 1990

Neuhauser & Pacala, AAP, 1999

Cox & Perkins, AOP, 2005

Cox & Perkins, PTRF, 2007

Cox & Perkins, AAP, 2008

Sturm & S., AAP, 2008

Sturm & S., ECP, 2008

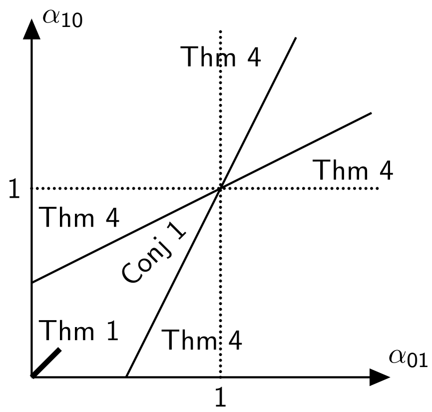
Cox, Merle, & Perkins, EJP, 2010

S., ECP, 2013

Cox, Durrett, & Perkins, Astérisque, 2013

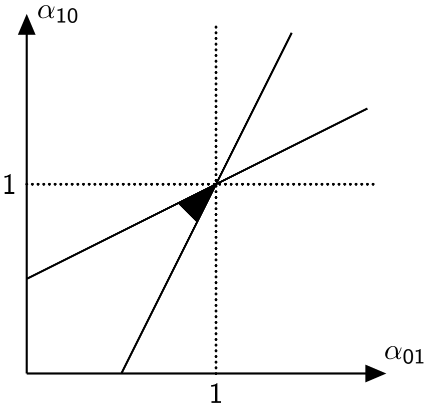
Cox & Perkins, AAP, 2014

Rigorous results



Neuhauser & Pacala (1999) have proved that in the spatial model, the regions of coexistence and founder control are reduced. Except when $d = 1 = R$, coexistence is possible for $\alpha_{01} = \alpha_{10} = \alpha$ small enough. They conjectured that this is true for all $\alpha < 1$.

Rigorous results



Cox & Perkins (2007) have proved coexistence in a cone near $(1, 1)$ for dimensions $d \geq 3$. Cox, Merle & Perkins (2010) have an analogue result for $d = 2$. The statement is believed to be false in dimension $d = 1$.

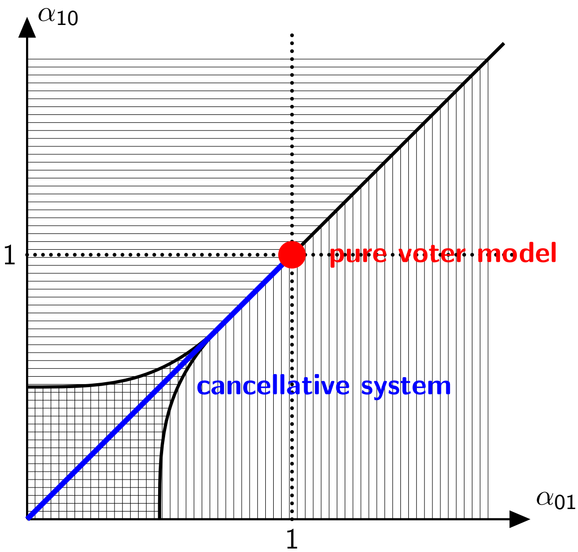
Voter model perturbations

For $(\alpha_{01}, \alpha_{10}) = (1, 1)$ we have a classical voter model.

In dimensions $d \geq 2$, Cox, Merle and Perkins prove that it is possible to send $\alpha_{01}, \alpha_{10} \to 1$ through a cone ($d \geq 3$) or cusp ($d = 2$) such that rescaled sparse models converge to supercritical *super Brownian motion*.

Using this, for $(\alpha_{01}, \alpha_{10})$ very close to $(1, 1)$, they can set up a comparison with oriented percolation and prove survival of the ones. By symmetry, the same holds for the zeros and one can conclude coexistence.

Special models



Cancellative systems

Equip $\{0,1\}$ with the usual product and with addition modulo 2, denoted as $\oplus$. Then $\{0,1\}$ is a *finite field*. We may view $\{0,1\}^{\mathbb{Z}^d}$ (equipped with $\oplus$) as a *linear space* over $\{0,1\}$.

Let $(A(i,j))_{i,j\in\mathbb{Z}^d}$ be a matrix with $0,1$-valued entries, such that $A(i,j)=1$ for finitely many i,j and $A(i,j)=0$ otherwise. Then we define

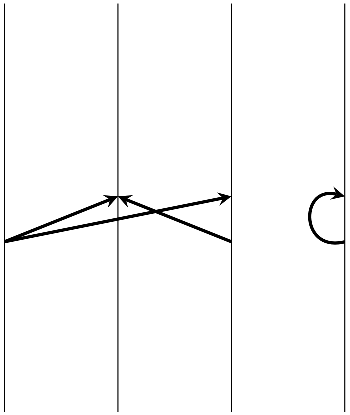
$$Ax(i) := \bigoplus_{j\in\mathbb{Z}^d} A(i,j)x(j).$$

A *cancellative system* $X=(X_t)_{t\geq 0}$ is a *linear system* w.r.t. to the finite field $\{0,1\}$. For certain A there is a nonnegative rate $r(A)$ such that the system makes the transition

$$x \mapsto x \oplus Ax$$

at Poisson times with rate $r(A)$.

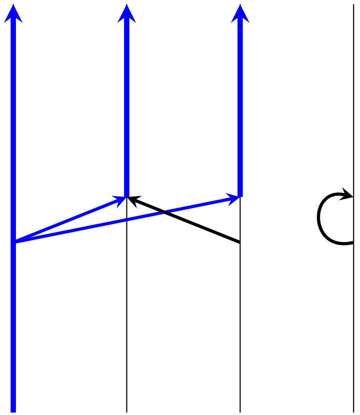
Graphical representation



Draw an arrow $i \to j$ whenever $A(j,i) = 1$.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Graphical representation



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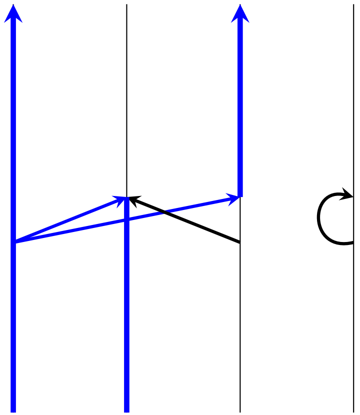

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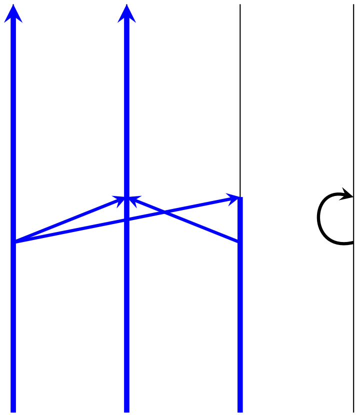

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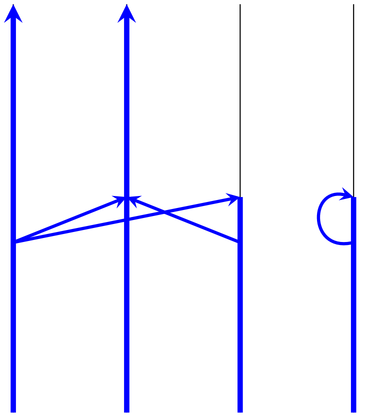


Draw an arrow $i \to j$ whenever $A(j,i) = 1$.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Cancellative systems duality

For $x, y \in \{0,1\}^{\mathbb{Z}^d}$, define

$$\langle x, y\rangle := \sum_i x(i)y(i) \quad \text{and} \quad \langle\!\langle x, y\rangle\!\rangle := \bigoplus_i x(i)y(i).$$

Then $\langle x, y\rangle$ is the number of sites i with $x(i) = 1 = y(i)$ and

$$\langle\!\langle x, y\rangle\!\rangle = 1_{\{\langle x, y\rangle \text{ is odd}\}}.$$

For any A,

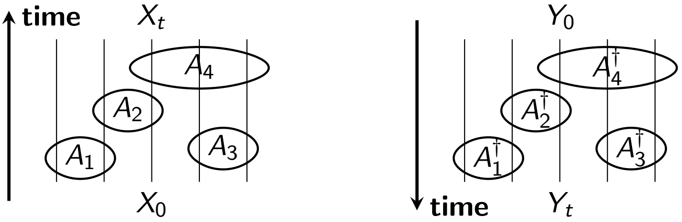
$$\langle\!\langle x, Ay\rangle\!\rangle = \langle\!\langle A^\dagger x, y\rangle\!\rangle,$$

where $A^\dagger(i,j) := A(j,i)$ is the *adjoint* of A.

Cancellative systems duality

Let X and Y be cancellative systems with rates satisfying

$$r_X(A) = r_Y(A^\dagger).$$



For each $t > 0$, we can *couple* such that for each $0 < u < t$, the processes $(X_s)_{0 \leq s \leq u}$ and $(Y_s)_{0 \leq s \leq t-u}$ are independent, and

$$\langle\langle X_t, Y_0 \rangle\rangle = \langle\langle X_u, Y_{t-u} \rangle\rangle = \langle\langle X_0, Y_t \rangle\rangle \qquad (0 \leq u \leq t).$$

Cancellative systems duality

Once again, if X and Y satisfy

$$r_X(A) = r_Y(A^\dagger).$$

Then X and Y are *pathwise dual* in the sense that for each $t > 0$ there exists a coupling such that

$$\langle\langle X_t, Y_0 \rangle\rangle = \langle\langle X_0, Y_t \rangle\rangle \qquad \text{a.s.}$$

In particular, they are dual in the sense that

$$\mathbb{P}\big[\langle X_t, Y_0 \rangle \text{ is odd}\big] = \mathbb{P}\big[\langle X_0, Y_t \rangle \text{ is odd}\big] \qquad (t \geq 0).$$

This formula holds also for random X_0 and Y_0 when we let X_t be independent of Y_0 and X_0 independent of Y_t.

Type symmetry and parity preservation

Def A cancellative system X is *type symmetric* if the transition $x \mapsto x'$ has the same rate as $(1-x) \mapsto (1-x')$.

Def A cancellative system X is *parity preserving* if a.s. $|X_t|$ is odd iff $|X_0|$ is odd $(t \geq 0)$.

- X type symmetric iff only jumps that involve A such that *each row contains an even number of ones*. (Even number of incoming arrows at each site.)
- X parity preserving iff only jumps that involve A such that *each column contains an even number of ones*. (Even number of outgoing arrows at each site.)

Consequence X type symmetric $\Leftrightarrow$ dual Y is parity preserving.

Interfaces

In the one-dimensional case, we have an extra tool available.

Let $\mathbb{Z} + \frac{1}{2} := \{k + \frac{1}{2} : k \in \mathbb{Z}\}$ and let $\mathbb{I} = \mathbb{Z}$ or $= \mathbb{Z} + \frac{1}{2}$.
Define a gradient operator $\nabla : \{0,1\}^{\mathbb{I}} \to \{0,1\}^{\mathbb{I}+\frac{1}{2}}$ by

$$\nabla x(i) := x(i - \tfrac{1}{2}) \oplus x(i + \tfrac{1}{2}).$$

If $(X_t)_{t\geq 0}$ is type symmetric, then $(\nabla X_t)_{t\geq 0}$ is a Markov process: the *interface model* of X.

$$\begin{array}{ccccccccccccccccc}
X & 0 & & 1 & & 1 & & 1 & & 0 & & 0 & & 1 & & 0 \\
\nabla X & & 1 & & 0 & & 0 & & 1 & & 0 & & 1 & & 1 &
\end{array}$$

Interface models are always parity preserving.

[S. '13] The interface model of a type symmetric cancellative spin system is a parity preserving cancellative spin system. Conversely, every parity preserving cancellative spin system is the interface model of a unique type symmetric cancellative spin system. Moreover, the following commutative diagram holds:

$$\begin{tikzcd} X \arrow[r, "\text{interface}"] \arrow[d, leftrightarrow, "\text{dual}"'] & Y' \arrow[d, leftrightarrow, "\text{dual}"] \\ Y & X' \arrow[l, "\text{interface}"] \end{tikzcd}$$

Here X, X' are type symmetric and Y, Y' are parity preserving. X and X' are dual with the non-local duality function $\langle\langle X, \nabla X' \rangle\rangle$.

Proof (sketch) Recall the duality function

$$\langle\langle x, y\rangle\rangle = \bigoplus_i x(i)y(i).$$

Then

$$\langle\langle x, \nabla y\rangle\rangle = \langle\langle \nabla x, y\rangle\rangle \qquad (x \in \{0,1\}^{\mathbb{I}},\ y \in \{0,1\}^{\mathbb{I}+\frac{1}{2}}).$$

If A is type symmetric, then $A^\dagger$ is the dual action and $\nabla A \nabla^{-1}$ is the corresponding action on interfaces. Now

$$(\nabla A \nabla^{-1})^\dagger = \nabla^{-1} A^\dagger \nabla$$

correspond to the dual of the interface model resp. the model whose interface model is the dual.

(Some care is needed to define ∇^{-1} but this is the basic idea.)

The symmetric Neuhauser-Pacala model

Claim The symmetric Neuhauser-Pacala model with $\alpha := \alpha_{01} = \alpha_{10} \leq 1$ is cancellative.

Proof For each i:

- With rate α, choose uniform $j \in \mathcal{N}_i$ and jump $x(i) \mapsto x(i) \oplus x(i) \oplus x(j)$ (voter dynamics).
- With rate $1 - \alpha$, choose uniform, independent $j, k \in \mathcal{N}_i$ and jump $x(i) \mapsto x(i) \oplus x(j) \oplus x(k)$ (rebellious dynamics).

Check that this yields the desired flip rates.

Dual of the Neuhauser-Pacala model

The dual Y of the symmetric Neuhauser-Pacala model is a *parity preserving* system of *branching* and *annihilating* random walks. Interpret $Y_t(i) = 1$ as a particle. For each i:

- With rate α, choose uniform $j \in \mathcal{N}_i$ and jump $x(i) \mapsto x(i) \oplus x(i)$ and $x(j) \mapsto x(j) \oplus x(i)$. *If there is a particle at i, then it jumps to j. If there already is a particle at j, then the two particles annihilate.*
- With rate $1 - \alpha$, choose uniform, independent $j, k \in \mathcal{N}_i$ and jump $x(j) \mapsto x(j) \oplus x(i)$ and $x(k) \mapsto x(k) \oplus x(i)$. *If there is a particle at i, then it produces particles at j and k that annihilate with any particles that may already be present.*

Classification of behavior

Let Y be parity preserving.

Def Y *persists* if there exists an invariant law that is concentrated on states other than $\underline{0}$ (all zero).

Def Y *survives* if $\mathbb{P}^y[Y_t \neq \underline{0}\ \forall t \geq 0] > 0$ for some *even* initial state y.

If $|Y_0|$ is finite and odd, then let $l_t := \inf\{i \in \mathbb{Z} + \frac{1}{2} : Y_t(i) = 1\}$ denote the left-most one and let

$$\hat{Y}_t(i) := Y(l_t + i) \qquad (t \geq 0,\ i \in \mathbb{N})$$

denote the process Y *viewed from the left-most one*.

Def Y is *stable* if $\hat{Y}$ is positively recurrent.

Def Y is *strongly stable* if $\hat{Y}$ is stable and $\mathbb{E}\big[|\hat{Y}_\infty|\big] < \infty$ in equilibrium.

Classification of behavior

Let X be type symmetric.

Def X exhibits *coexistence* if there exists an invariant law that is concentrated on states other than $\underline{0}$ and $\underline{1}$.

Def X *survives* if $\mathbb{P}^x[X_t \neq \underline{0}\ \forall t \geq 0] > 0$ for some *finite* initial state x.

Def X exhibits *(strong) interface tightness* if its interface model is (strongly) stable.

Interface tightness introduced for the contact process by Cox & Durrett (1995) and studied by Belhaouari, Mountford & Valle (2007) and & Sturm & S. (2008).

Abstract results

$$\begin{tikzcd} X \arrow[r, "\text{interface}"] \arrow[d, leftrightarrow, "\text{dual}"'] & Y' \arrow[d, leftrightarrow, "\text{dual}"] \\ Y & X' \arrow[l, "\text{interface}"] \end{tikzcd}$$

Claim

interface model Y' persists $\Leftrightarrow$ X coexists $\Leftrightarrow$ dual Y survives.

Proof of second claim

Start X in product measure with intensity $1/2$. Then
$\mathbb{P}\big[X_t(i) \neq X_t(j)\big] = \mathbb{P}\big[\langle X_t, \delta_i + \delta_j\rangle \text{ is odd}\big] =$
$\mathbb{P}^{\delta_i+\delta_j}\big[\langle X_0, Y_t\rangle \text{ is odd}\big] = \frac{1}{2}\mathbb{P}^{\delta_i+\delta_j}\big[Y_t \neq \underline{0}\big]$
$\underset{t\to\infty}{\longrightarrow} \frac{1}{2}\mathbb{P}^{\delta_i+\delta_j}\big[Y_t \neq \underline{0}\ \forall t \geq 0\big]$. *Odd upper invariant law.*

Claim X survives $\Leftrightarrow$ dual Y persists. (Similar.)

Thm [S. '13] Strong interface tightness implies noncoexistence.

Strong interface tightness implies noncoexistence

Lemma Assume that strong interface tightness holds for X. Let $\hat{Y}_\infty + i$ denote the configuration $\hat{Y}_\infty$ shifted by i. Then

$$h(x) := \sum_{i \in \mathbb{Z}+\frac{1}{2}} \mathbb{E}\big[\langle\!\langle x, \hat{Y}_\infty + i \rangle\!\rangle\big]$$

is a harmonic function for the process X' (dual of interface model of X). Moreover, there exist constants $0 < c \leq C < \infty$ s.t.

$$c|x| \leq h(x) \leq C|x|.$$

Proof of Thm (sketch) By martingale convergence, $h(X'_t)$ converges a.s., which implies that X' dies out a.s. The same holds for its interface model Y which is dual to X, so by duality X exhibits noncoexistence.

The rebellious voter model

The *rebellious voter model* is very similar to the one-dimensional symmetric Neuhauser-Pacala model.

For each i:

- With rate α, choose $j = i - 1$ or $j = i + 1$ with probab. $\frac{1}{2}$ each and jump $x(i) \mapsto x(i) \oplus x(i) \oplus x(j)$ (voter dynamics).
- With rate $1 - \alpha$, choose either $\{j, k\} = \{i - 2, i - 1\}$ or $\{j, k\} = \{i + 1, i + 2\}$ with probab. $\frac{1}{2}$ each and jump $x(i) \mapsto x(i) \oplus x(j) \oplus x(k)$ (rebellious dynamics).

The dual is a system of branching and annihilating nearest-neighbour random walks that always place offspring on the two sites immediately to their left or right.

The rebellious voter model

The rebellious voter model is *self-dual* in the sense that it is equal to the dual of its interface model, or more simply:

$$\begin{tikzcd} X \arrow[d, bend right, leftrightarrow, "\text{dual}"'] \arrow[d, bend left, "\text{interface}"] \\ Y \end{tikzcd}$$

Consequence Survival equivalent to coexistence.

The disagreement voter model

The $d = 1$ Neuhauser-Pacala model X with range $R = 1$ is up to reparametrization equal to the *disagreement voter model*, where for each i:

- With rate α, choose $j = i - 1$ or $j = i + 1$ with probab. $\frac{1}{2}$ each and jump $x(i) \mapsto x(i) \oplus x(i) \oplus x(j)$ (voter dynamics).
- With rate $1 - \alpha$, jump $x(i) \mapsto x(i) \oplus x(i-1) \oplus x(i+1)$ (disagreement dynamics).

In the *dual* model Y, a particle places offspring on the sites immediately to its left and right.

The *interface* model Y' is a mixture of annihilating random walk and exclusion dynamics.

Clearly Y' dies out for all $\alpha > 0$ hence X exhibits noncoexistence.

The exclusion process

Recall that in the symmetric, nearest-neighbor exclusion process, pairs of neighboring 0's and 1's make the transitions $01 \leftrightarrow 10$ at rate one. This model is both type symmetric and parity preserving. It is part of a commutative diagram where:

X = pure disagreement dynamics

Y = exclusion process

Z = double branching annihilating process

$$\begin{tikzcd} X \arrow[r, "\text{interface}"] \arrow[d, leftrightarrow, "\text{dual}"'] & Y \arrow[r, "\text{interface}"] \arrow[d, leftrightarrow, "\text{dual}"'] & Z \arrow[d, leftrightarrow, "\text{dual}"'] \\ Z & Y \arrow[l, "\text{interface}"] & X \arrow[l, "\text{interface}"] \end{tikzcd}$$

Ergodic results

[Sturm & S. '08] A symmetric Neuhauser-Pacala or rebellious voter model have at most one spatially homogeneous coexisting invariant law. If moreover $\alpha > 0$ and the dual model Y is not stable, then this is the long-time limit law started from any spatially homogeneous coexisting initial law.

[Sturm & S. '08] For the rebellious voter model with α sufficiently close to zero, there is a unique coexisting invariant law ν and one has *complete convergence*

$$\mathbb{P}[X_t \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \rho_0\delta_{\underline{0}} + \rho_1\delta_{\underline{1}} + (1-\rho_0-\rho_1)\nu,$$

where $\rho_\tau := \mathbb{P}[X_t = \underline{\tau} \text{ for some } t \geq 0]$.

[Cox & Perkins '14] There exists some $\alpha' < 1$ such that the symmetric Neuhauser-Pacala model in dimensions $d \geq 2$ exhibits complete convergence for $\alpha \in (\alpha', 1)$.

Ergodic results

Idea of proof Recall that if law of X_0 is product measure with intensity $1/2$, then

$$\mathbb{P}\big[\langle X_t, y\rangle \text{ is odd}\big] = \mathbb{P}^y\big[\langle X_0, Y_t\rangle \text{ is odd}\big] = \tfrac{1}{2}\mathbb{P}^y\big[Y_t \neq \underline{0}\big].$$

As a consequence, $\mathbb{P}[X_t \in \cdot\,]$ converges weakly to $\nu := \mathbb{P}[X_\infty \in \cdot\,]$ characterized by

$$\mathbb{P}\big[\langle X_\infty, y\rangle \text{ is odd}\big] = \tfrac{1}{2}\mathbb{P}^y\big[Y_t \neq \underline{0}\ \forall t \geq 0\big].$$

For more general initial laws, convergence will follow if

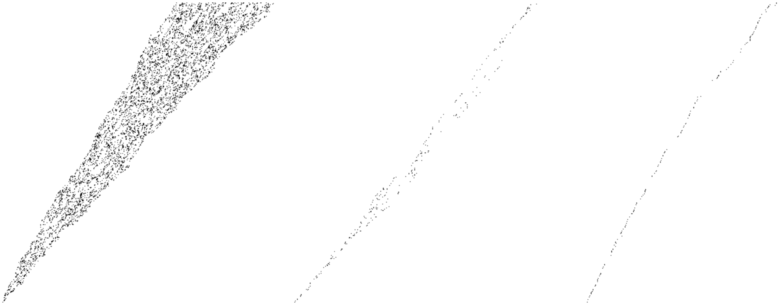
$$\mathbb{P}^y\big[\langle X_0, Y_t\rangle \text{ is odd}\big] \approx \tfrac{1}{2}\mathbb{P}^y\big[Y_t \neq \underline{0}\big] \quad \text{as } t \to \infty.$$

This requires one to show that conditional on survival, Y_t is large and sufficiently random so that $\langle X_0, Y_t\rangle$ is odd with probab. $\approx 1/2$.

Numerical simulation

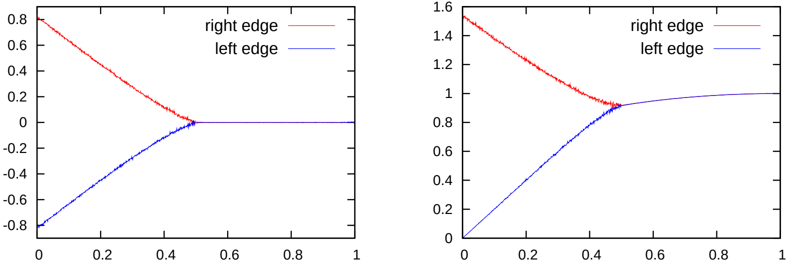
Interface process Y' of the two-sided rebellious voter model for $\alpha = 0.4, 0.5, 0.51, 0.6$.

One-sided rebellious interface model



Interface process Y' of the one-sided rebellious voter model for $\alpha = 0.3, 0.5, 0.6.$

Edge speeds



Edge speeds for the rebellious voter model (left) and its one-sided counterpart (right) [S. & Vrbenský '10].

Two functions of the process

Define the *survival probability*

$$\rho(\alpha) := \mathbb{P}^{\delta_0}[X_t \neq 0 \ \forall t \geq 0].$$

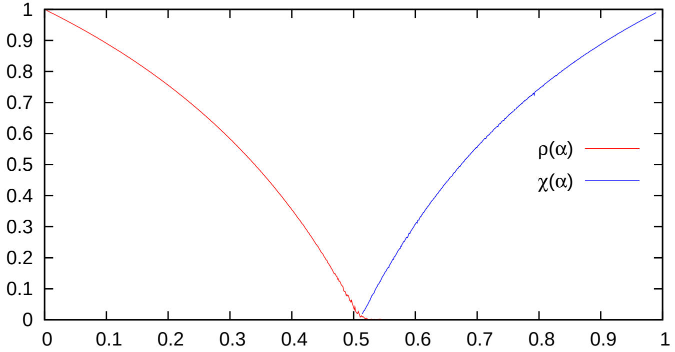
- coexistence $\Leftrightarrow \rho(\alpha) > 0$.

Define the *fraction of time spent with a single interface*

$$\chi(\alpha) := \mathbb{P}\big[|Y'_\infty| = 1\big].$$

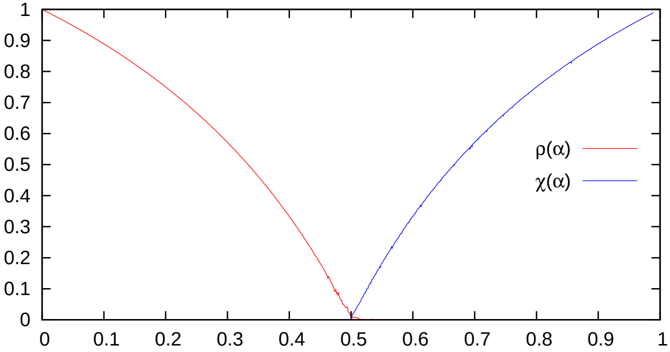
- interface tightness $\Leftrightarrow \chi(\alpha) > 0$.

Numerical data



The functions ρ and χ for the two-sided rebelious voter model.

Numerical data



The functions ρ and χ for the one-sided rebelious voter model.

Explicit formulas

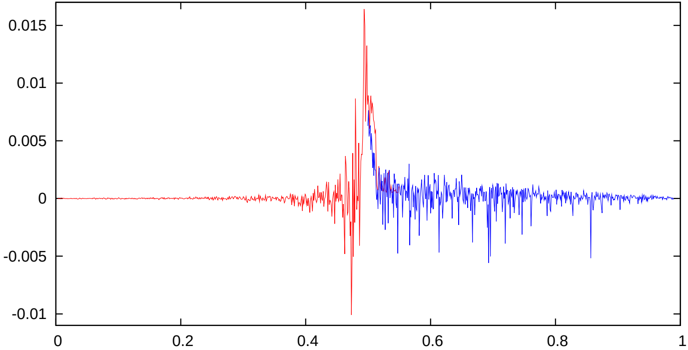
It seems that for the one-sided model, the functions ρ and χ are described by the explicit formulas:

$$\rho(\alpha) = 0 \vee \frac{1 - 2\alpha}{1 - \alpha} \quad \text{and} \quad \chi(\alpha) = 0 \vee \big(2 - \frac{1}{\alpha}\big).$$

In particular, one has the symmetry $\rho(1 - \alpha) = \chi(\alpha)$ and the critical parameter seems to be given by $\alpha_{\rm c} = 1/2$.

Explanation?

Numerical data



Differences of ρ and χ with presumed explicit formulas.

A critical exponent

Theoretical physicists believe that

$$\rho(\alpha) \sim (\alpha_c - \alpha)^\beta \quad \text{as} \quad \alpha \uparrow \alpha_c,$$

where β is a *critical exponent*.

It has been conjectured by I. Jensen (1994) that $\beta = 13/14$ and by Inui & Tretyakov (1998) that $\beta = 1$. More recent estimates are $\beta \approx 0.92$, $\beta \approx 0.95$ [Hinrichsen '00] [Ódor & Szolnoki '05]. Our formula would imply $\beta = 1$.

Stability

Let Y be a system of annihilating random walks where one particle can split into three. Recall that Y is stable if it spends a positive fraction of time with only one particle.

Start three n.n. random walks on $\mathbb{Z}$ on positions $i < j < k$ and let τ be the first time than any two of them meet. Then:

$$\mathbb{P}[\tau > t] \propto t^{-3/2} \quad \text{as } t \to \infty,$$
$$\mathbb{E}[\tau] = (k-j)(j-i) < \infty.$$

Conjecture The fact that $3/2 > 1$ and hence $\mathbb{E}[\tau] < \infty$ is essential for stable behavior.

Question What is the asymptotics of $\mathbb{P}[\tau > t]$ for other systems of recurrent walkers, e.g. in the domain of attraction of an α-stable law?