Equivalence of Gradient Flows and Entropy solutions for singular nonlocal interaction equations in 1D

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The equations

Nonlocal interaction equation

$$\partial_t \mu = \partial_x\big(\mu(\partial_x W * \mu)\big)$$

$$\mu_0 \in P_2(\mathbb{R})$$

$$W(x) = \pm|x|$$

Conservation law

$$\partial_t F + \partial_x\, g(F) = 0$$

$$F_0 \in L^\infty(\mathbb{R})$$

$$g(F) = \pm(F^2 - F)$$

! In general solutions are not unique !

How are they connected?

$$F(x) = F_\mu(x) := \mu\big((-\infty, x]\big)$$

$$\partial_x |\cdot| * \mu = \int_{\mathbb{R}} \operatorname{sign}(x-y)\,d\mu(y)$$

$$= 2 \int_{-\infty}^{x} d\mu(y) - 1 = 2F - 1$$

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$$\Downarrow$$

$$\partial_t\mu = \partial_x\big(\mu(\pm\operatorname{sign} * \mu)\big) \underset{\partial_x}{\overset{\int_{-\infty}^{x}}{\rightleftarrows}} \partial_t F \pm \partial_x (F^2 - F) = 0$$

Well-posedness

- Nonlocal interaction equation ($\mu\ a.c.$)

$$\begin{cases} \partial_t \mu + \partial_x (v\mu) = 0 \\ v = -\partial_x W * \mu \end{cases}$$

Wasserstein Gradient Flow for

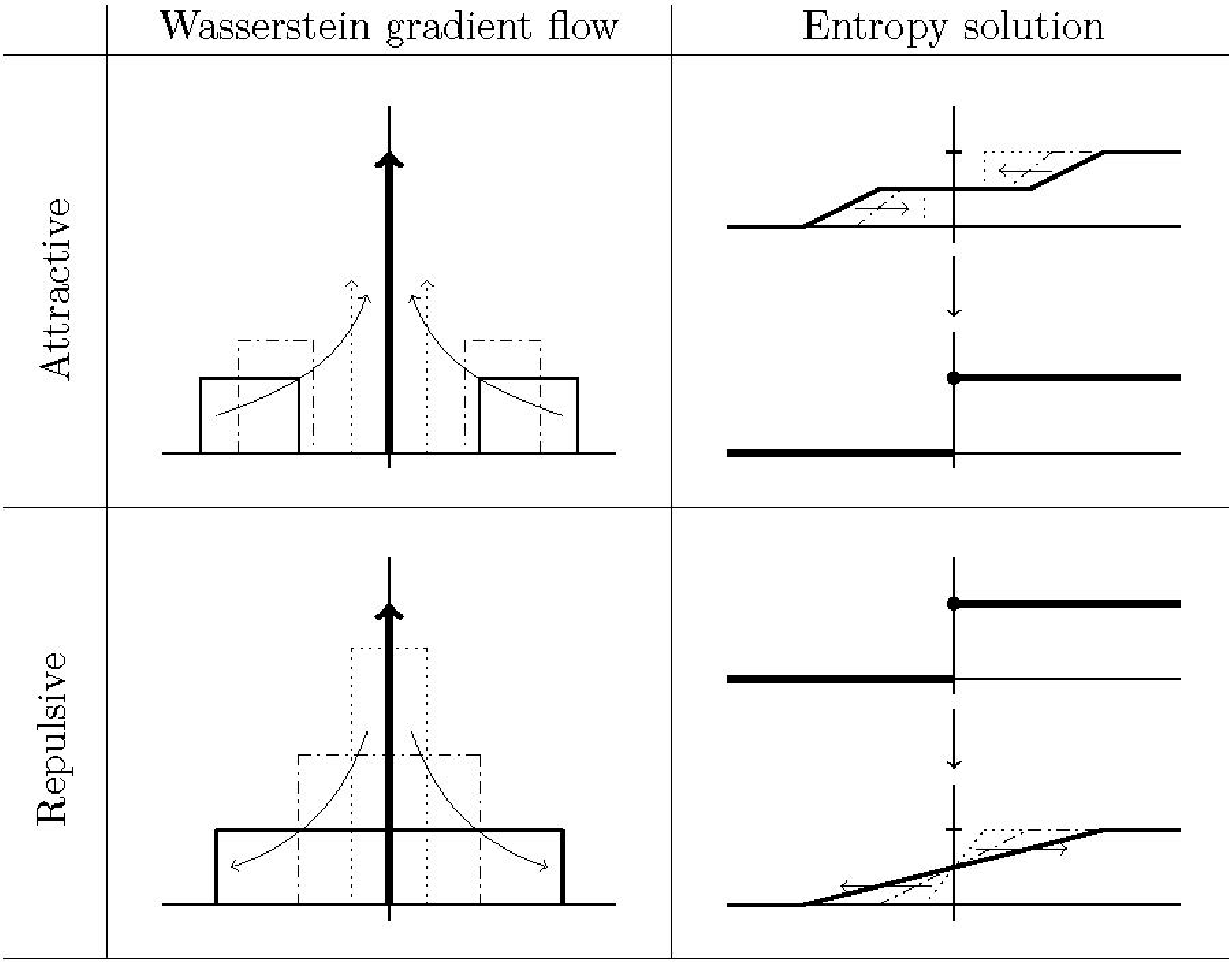
$$\mathcal{E}(\mu) = \frac{1}{2} \iint W(x-y) d\mu(x) d\mu(y)$$

- Conservation law (g convex)

$$\partial_t F + \partial_x g(F) = 0$$

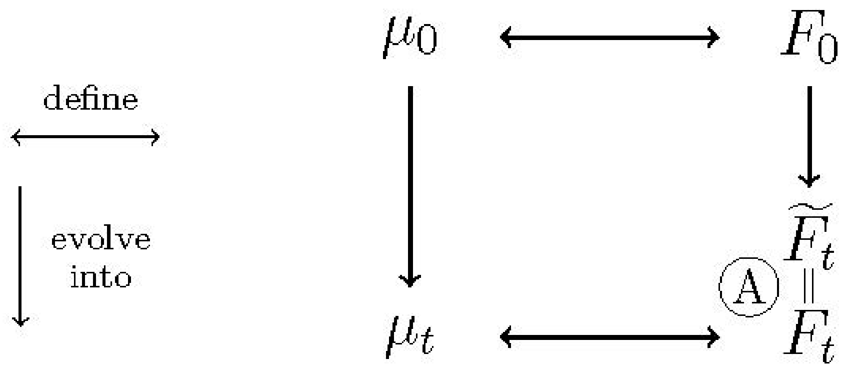
+Oleinik condition $\Longrightarrow$ Entropy Solution

Time reversal



Questions

- Solutions equivalent? (A)



- What do we gain if equivalence holds?
- Condition inside W-gradient flow?
- Why non-invariance under time reversal?

Answers

- Prove the equivalence
- Particle approximation
- Characterization of $\partial \mathcal{E}$

Outline

- Equivalence

1. Well-posedness
2. Pseudo-inverse
3. Sketch of the proof

- Particle approximation via wave-front-tracking
- Characterization of $\partial \mathcal{E}$

Entropy solution

$g(F) = \pm(F^2 - F\ )$ $F_0 \in L^\infty(\mathbb{R})$ non decreasing

$F \in L^\infty([0, +\infty), \mathbb{R})$ is *entropy solution* if

$$\begin{cases} \partial_t F + \partial_x g(F) = 0 \\ F = F_0 \qquad\qquad\qquad t = 0 \end{cases}$$

And, when g is convex, it satisfies the Oleinik condition

$$F(x+z, t) - F(x, t) \leq \frac{C}{t} z$$

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$$F(x+z, t) - F(x, t) \leq \frac{C}{t} z$$

Proposition

Given F_0, then there exists a unique entropy solution F_t starting from X_0. Moreover, $|F_{1,t} - F_{2,t}\ |_1 \ \leq |F_{1,0} - F_{2,0}|_1$.

Entropy solution

$$F_0(x) = \begin{cases} F^L & x < 0 \\ F^R & x > 0 \end{cases} \quad \text{with} \quad F^L < F^R$$

$$\boxed{g = F - F^2}$$

$$\text{R-H} \Rightarrow \dot{x}(t) = \frac{g(F^L) - g(F^R)}{F^L - F^R}$$

$$F(x,t) = \begin{cases} F^L & x < (1 - F^R - F^L)t \\ F^R & x > (1 - F^R - F^L)t \end{cases}$$

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$$\boxed{g = F^2 - F}$$

Oleinik condition

$$\mathrm{F}(x,t) = \begin{cases} F^L & x < (-1 + 2F^L)t \\ \dfrac{x+t}{2t} & \ldots < x < \ldots \\ F^R & x > (-1 + 2F^R)t \end{cases}$$

Wasserstein gradient flow

- $d_W^2(\rho, \mu) = \inf\left\{\int |x - y|^2 \mathrm{d}\boldsymbol{\gamma} \mid \boldsymbol{\gamma} \text{ transport plan } \rho \rightarrow \mu \right\}$
- Let $\mathcal{E}: P_2(\mathbb{R}) \rightarrow (-\infty, +\infty]$, $\mu \in D(\mathcal{E})$ and $k \in L^2(\mu)$, then $k \in \partial\mathcal{E}(\mu)$ if $\forall \rho$

$$\mathcal{E}(\rho) - \mathcal{E}(\mu) \geq \inf_{\boldsymbol{\gamma}} \int k(x)(y - x) d\boldsymbol{\gamma} + o(d_W(\rho, \mu))$$

Wasserstein gradient flow

- $d_W^2(\rho,\mu) = \inf\{\int |x-y|^2 \mathrm{d}\boldsymbol{\gamma} \mid \boldsymbol{\gamma} \text{ transport plan } \rho \to \mu\}$
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$$\mathcal{E}(\rho) - \mathcal{E}(\mu) \geq \inf_{\boldsymbol{\gamma}} \int k(x)(y-x)d\boldsymbol{\gamma} + o(d_W(\rho,\mu))$$

- Let $\mathcal{E}(\mu) = \frac{1}{2}\iint W(x-y)d\mu(x)d\mu(y)$ with $W(x) = \pm|x|$, then μ_t is a W-gradient flow for $\mathcal{E}$ if

$$\begin{cases} \partial_t\mu_t + \partial_x(v_t\mu_t) = 0 \\ v_t = -\partial^0\mathcal{E}(\mu_t) \end{cases}$$

Wasserstein gradient flow

Proposition (existence and uniqueness of W-gf)

Let $\mu_0 \in P_2(\mathbb{R})$, then there exists a unique gradient flow for the functional $\mathcal{E}$.

Moreover, for two given solutions μ_t and ν_t

$$d_W(\mu_t, \nu_t) \leq d_W(\mu_0, \nu_0).$$

Moreover, for $W = -|x|$ and for $t > 0$, $\mu_t \ll \mathcal{L}$.

Proposition

Let $\mu \ll \mathcal{L}$, then

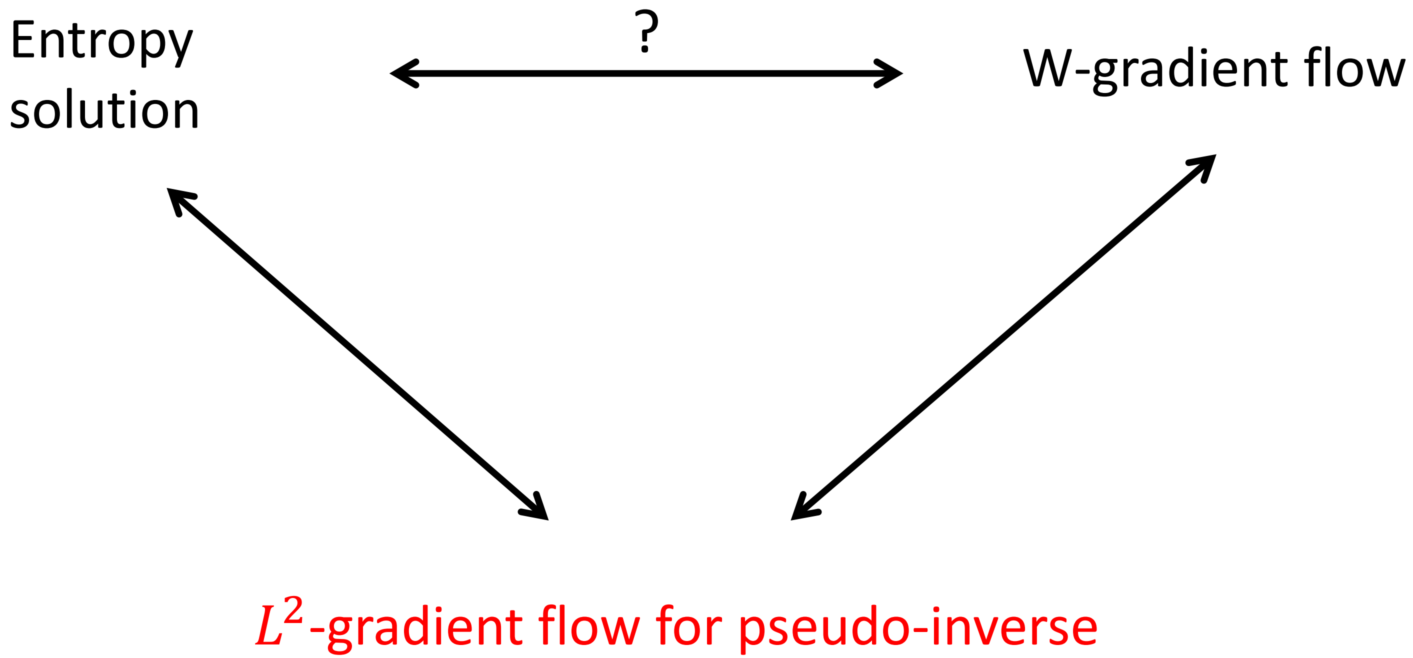
$$\partial^0 \mathcal{E}(\mu) = \int_{x \neq y} \pm \text{sign}\,(x-y) d\mu(y)$$

Carrillo, Di Francesco, Figalli, Slepcev, Laurent
GAB, Master thesis, supervisor G. Savarè

Equivalence

Entropy solution ⟷ (?) W-gradient flow

Equivalence



Pseudo-Inverse

- $\mathrm{K} := \{ f \in L^2((0,1)) \mid f \text{ is non-decreasing} \}$

$$\mu$$

$$F_\mu := \mu((-\infty, x])$$

$$X_\mu(s) := \inf\{x : F_\mu(x) > s\}$$

- Change of variable formula

$$\int_{\mathbb{R}} \phi(x)\, d\mu(x) = \int_0^1 \phi(X_\mu(s))\, ds$$

L^2-gradient flow

$$\mathcal{E}(X) := \frac{1}{2} \iint_{[0,1]^2} W\big(X(s) - X(z)\big) ds dz$$

$$\mathrm{I}_{\mathrm{K}}(X) := \begin{cases} 0 & \text{if } X \in \mathrm{K} \\ +\infty & \text{otherwise} \end{cases}$$

A curve X_t is a L^2-gradient flow if

$$\partial_t X_t \in -\partial(\mathcal{E} + \mathrm{I}_{\mathrm{K}})(\mathrm{X}_{\mathrm{t}})$$

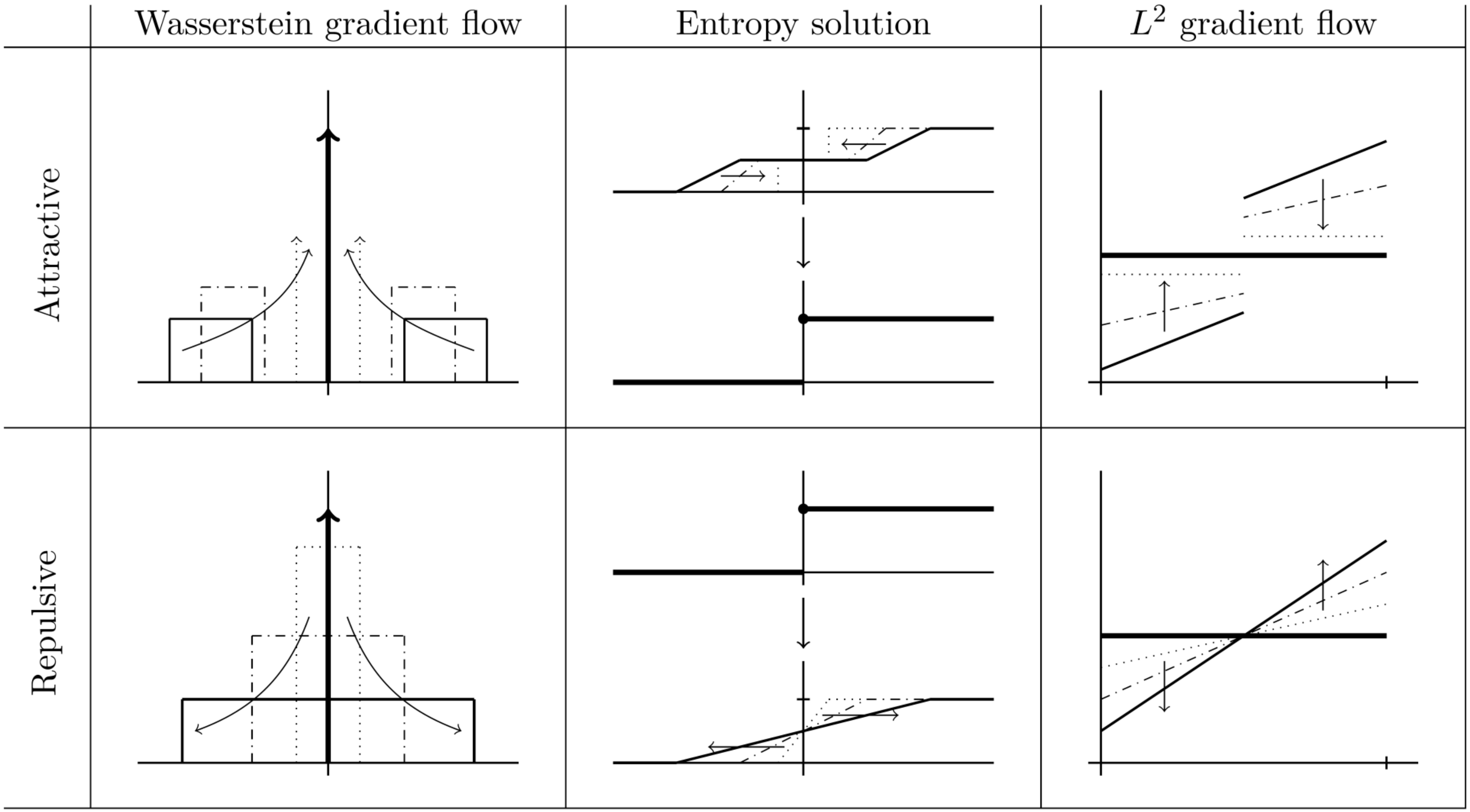
<u>Proposition</u>

Let $X_0 \in \mathrm{K}$, then ∃! gradient flow solution X_t starting from X_0.

Moreover, given X_1 and X_2, then $||X_{t,1} - X_{t,2}|| \leq ||X_1 - X_2||$.

Moreover for $W(x) = -|x|$ the solution X_t is strictly increasing.

Comparison


Wasserstein gradient flow
Entropy solution
L^2 gradient flow
Attractive
Repulsive

Sketch of the proof

- Finite combination of delta measures

$$\mu_0 = \sum_{j=1}^{N} m_j \delta_{x_j}$$

Attractive case

$$X(s,t) = \sum_{j=1}^{N} x_j(t)\, \chi_{[M_{j-1}, M_j)}(s)$$

$$\partial_t x_j(t) = -\sum_k m_k \text{sign}(x_j(t) - x_k(t))$$

$$\rightarrow F(x,t) = \mu_t\big((-\infty, x]\big) = \sum_{j=1}^{N} m_j \chi_{[x_j(t), +\infty)}$$

Sketch of the proof

Repulsive case

$$X(s,t) = X_0(s) + t(2s - 1) \rightarrow F(x,t) = \int_0^1 \chi_{(-\infty,x]}\big(X(s,t)\big)ds$$

Oleinik condition

$$F(x+z,t) - F(x,t) = \int_0^1 \chi_{(x,x+z]}\big(X(s,t)\big)ds = s^2 - s^1$$

$$\begin{cases} s^2 = \sup\{s \mid X(s,t) \in (x, x+z]\} \\ s^1 = \inf\{s \mid X(s,t) \in (x, x+z]\} \end{cases}$$

$$\Longrightarrow F(x+z,t) - F(x,t) = s^2 - s^1 \le \frac{X(s^2,t) - X(s^1,t)}{2t} \le \frac{z}{2t}$$

Sketch of the proof

- General initial measure

1- Approximate initial data μ_0 with μ_0^N

F_0 with F_0^N

X_0 with X_0^N

2- Contractivity properties

3- $\left|F_\mu - F_\nu\right|_1 = d_{W1}(\mu, \nu)$ & $||X_\nu - X_\mu|| = d_W(\mu, \nu)$

$\square$

Equivalence of the three solutions

Theorem

Let $W(x) = \pm|x|$, $g(F) = \pm(F^2 - F)$, $\mu_0 \in P_2(\mathbb{R})$, $F_0(x) = \mu_0\big((-\infty, x]\big)$ and X_0 the pseudo-inverse of F_0. Given μ_t absolutely continuous curve, the following are equivalent

1. μ_t is the unique Wasserstein gradient flow for $\mathcal{E}$ with initial condition μ_0
2. $F(x,t) = \mu_t((-\infty, x])$ is the unique entropy solution with initial condition F_0
3. $X_t(s) = \inf\{x \mid F(x,t) > s\}$ is the unique L^2-gradient flow for $\mathcal{E} + \mathrm{I}_{\mathrm{K}}$ with initial condition X_0

Particle approximation

- Attractive case $W(x) = |x|$

$$\mu_0 = \sum_{j=1}^{N} m_j \delta_{x_j}$$

$$\partial_t x_j(t) = -\sum_{k=1}^{N} m_k \operatorname{sign}(x_j(t) - x_k(t))$$

... done ...

Particle approximation

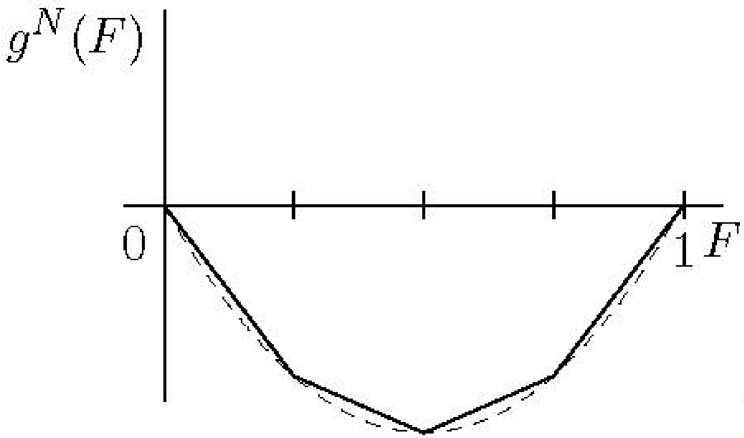
- Repulsive case $W(x) = -|x|$

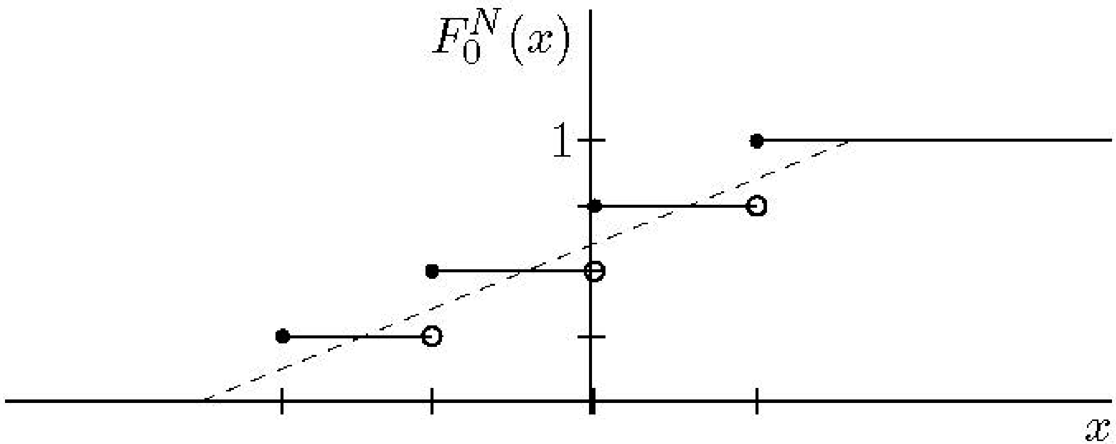
Immediate regularization!

How to avoid it?

Wave front tracking

1. Approximate initial data $F_0 \to F_0^N$
2. Approximate the flux $g \to g^N$
3. Solve the equation $\partial_t F^N + \partial_x g^N(F^N) = 0$





Wave front tracking

Theorem

$$\mu_t^N = \frac{1}{N}\sum \delta_{x_j^N(t)} \to \mu_t$$

$$\partial_t x_j^N(t) = \frac{1}{N}\sum_j \operatorname{sign}(x_j^N(t) - x_k^N(t))$$

Remark: avoid regularization + split delta

Extended subdifferential

<u>Proposition</u>

Let $\mu \in P_2(\mathbb{R})$ and $W(x) = -|x|$. If exists y such that $\mu(\{y\}) > 0$ then $\partial \mathcal{E} = \emptyset$.

Extended subdifferential

<u>Proposition</u>

Let $\mu \in P_2(\mathbb{R})$ and $W(x) = -|x|$. If exists y such that $\mu(\{y\}) > 0$ then $\partial\mathcal{E} = \emptyset$.

<u>Definition</u> (Extended sub-differential)

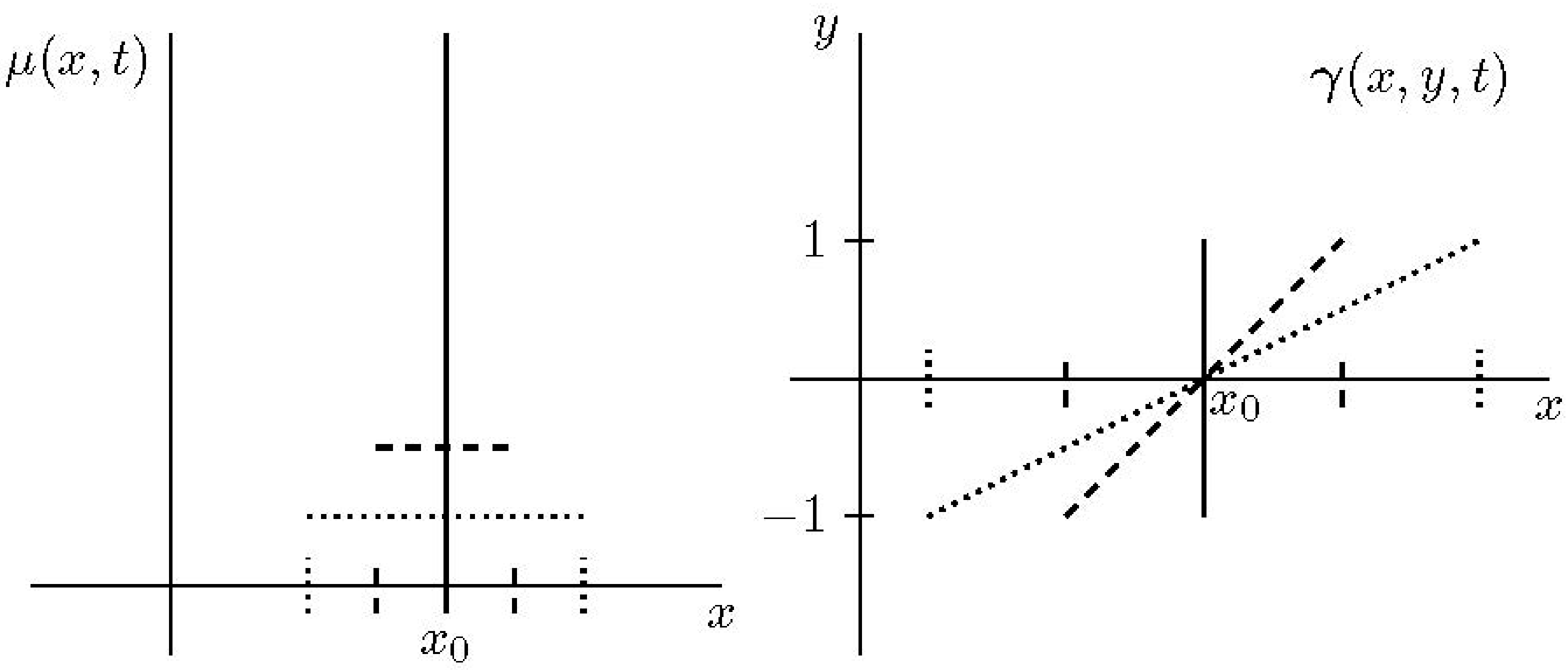
$\boldsymbol{\gamma} \in \boldsymbol{\partial}\mathcal{E}(\mu)$ if $(\pi_1)_{\#}\boldsymbol{\gamma} = \mu$ and

$$\mathcal{E}(\nu) - \mathcal{E}(\mu) \geq \inf \int_{R^3} x_2(x_3 - x_1)d\boldsymbol{\rho} + o(d_W)$$

Extended subdifferential

Given a measure $\mu = \nu + \sum m_i \delta_{x_i}$, then $\boldsymbol{\gamma} \in \boldsymbol{\partial}\mathcal{E}(\mu)$

$$\boldsymbol{\gamma}(x,y) = \sum \frac{1}{2} \delta_{x_i} \otimes \chi_{\Delta_i} + \big(i \otimes (2F-1)\big)_{\#}\nu$$



Open questions

- Equivalence for (more) general W
- W-gf for non λ-convex interaction in nD
- Wave-front-tracking approach for other cases
- …

Thank you for the attention