Mass transport via Current Reservoirs:

a microscopic model for a Free Boundary Problem

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–
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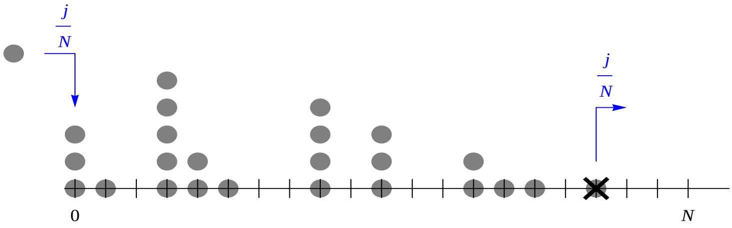
Transport via Current Reservoirs

Continuous time Independent Random Walkers in $\{0,1,\dots,N\}$ $\rightarrow$ *jumps outside suppressed*



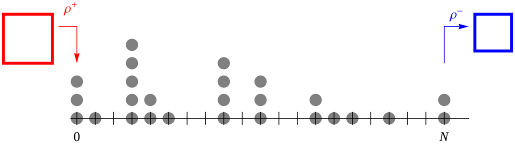
- particle created in 0 $\rightarrow$ at rate jN^{-1}
- rightmost particle deleted $\rightarrow$ at rate jN^{-1}

$\Big\}$ $\longrightarrow$ *to produce a* current $-2j$

References:

- IRW $\rightarrow$ [CDGP]: G.Carinci, A.De Masi, C.Giardina, E.Presutti
- SEP $\rightarrow$ [DPTV], [DFP]: A.De Masi, P.Ferrari, D.Tsagkarogiannis, E.Presutti, M.E.Vares

Density Reservoirs

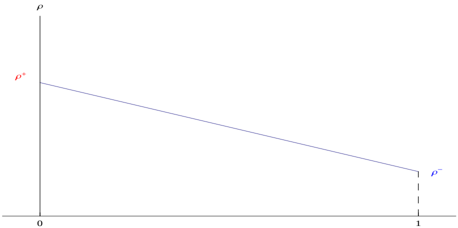


Hydrodynamic Limit $\longrightarrow \begin{cases} \dfrac{\partial \rho}{\partial t} = \dfrac{\partial^2 \rho}{\partial r^2} \\ \rho(0,t) = \rho^+ \quad \rho(1,t) = \rho^- \end{cases}$

Fick's Law

↓

linear stationary profiles



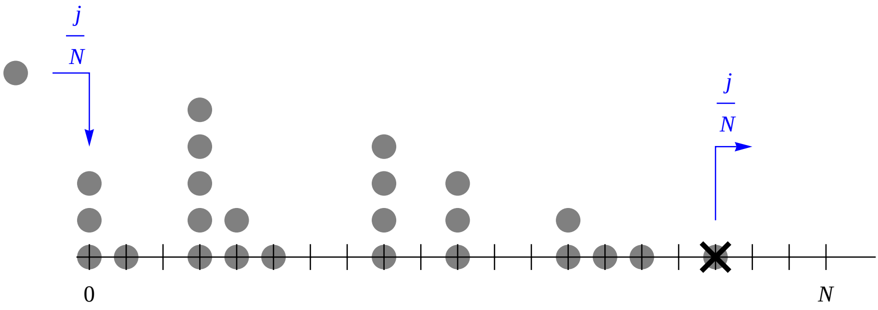
Plan of the talk

Motivations

1. Topological Interactions
2. Multiscale Phenomena
3. Microscopic Models for Free Boundary Problems
4. Beyond the classical Existence and Uniqueness results for the FBP

Independent particles with current reservoirs: [CDGP]

Independent Random Walk in $\{0, 1, \ldots, N\}$ $\rightarrow$ *jumps outside suppressed*



- particle created in 0 $\rightarrow$ at rate $\frac{j}{N}$
- rightmost particle deleted $\rightarrow$ at rate $\frac{j}{N}$

$\longrightarrow$ *to produce a* current $-2j$

The Generator

Configurations $\longrightarrow$ $\xi(x) =$ number of particles at x, $x \in \{0, 1, \dots, N\}$

$\xi \in \{0, 1, \dots, N\}^{\mathbb{N}}$ $N \in \mathbb{N}$

Generator: $L = \frac{j}{N} L_a + L_0 + \frac{j}{N} L_d$:

$L_0 =$ generator of independent symmetric random walks in $\{0, 1, \dots, N\}$ with reflecting boundaries

$$L_0 f(\xi) = \frac{1}{2} \sum_{x=0}^{N} \xi(x) \left(f(\xi^{x,x+1}) - f(\xi) \right) + \xi(x+1) \left(f(\xi^{x+1,x}) - f(\xi) \right)$$

where $\xi^{x,y}$ is the configuration obtained from ξ moving a particle from x to y.

L_a = add a particle at the origin

$$L_a f(\xi) = f(\xi + \mathbf{1}_0) - f(\xi)$$

L_d = remove a particle at the rightmost occupied site

$$L_b f(\xi) = f(\xi - \mathbf{1}_{X_\xi}) - f(\xi)$$

$$X_\xi := \min \left\{ y \in \{0, 1, \dots, N\} \, : \, \xi(y) > 0 \right\}$$

The Hydrodynamic Limit

$\exists\ \rho_t = \rho_t(r), r \in [0, 1]$ such that

$$\frac{1}{N}\xi_{N^2 t} \to \rho_t \qquad \text{as} \qquad N \to \infty$$

The Hydrodynamic Limit

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$$\frac{1}{N}\xi_{N^2 t} \to \rho_t \qquad \text{as} \qquad N \to \infty$$

Theorem

$\exists\ \rho_t = \rho_t(r), r \in [0,1], t \geq 0$, non negative and $\mathbf{L^1}$ such that "$\xi_{N^2 t}$ converges to ρ_t weakly" which means that for any $\zeta > 0$

$$\lim_{N\to\infty} P_\xi^{(N)}\left[\max_{x\in\{0,\dots,N\}}\left|\frac{1}{N}F_N(x;\xi_{N^2 t}) - F(N^{-1}x;\rho_t)\right| > \zeta\right] = 0$$

where

$$F_N(x;\xi) := \sum_{y=x}^{N}\xi(y); \qquad F(r;\rho) := \int_r^1 \rho(r')dr'$$

proved in [CDGP] under suitable assumptions on the initial datum.

Strategy

Barriers

Identification of the limit (heuristics): the FBP

Let $\rho_t(\cdot)$ is the hydrodynamic limit of ξ_t and R_t its *"boundary"*:

$$R_t := \inf \left\{ r \in [0,1] : \ \rho(z,t) = 0 \, \forall z \geq r \right\}$$

then $(R_t, \rho_t(\cdot))$ *is a "Generalized Solution" of the above defined FBP.*

- $j = 0$: *no births and deaths*

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial^2 \rho}{\partial r^2}, \qquad \left.\frac{\partial \rho}{\partial r}\right|_0 = \left.\frac{\partial \rho}{\partial r}\right|_1 = 0$$

  The heat equation with Neumann boundary conditions.

- $j \neq 0$: *adding births and deaths*

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial^2 \rho}{\partial r^2} + jD_0 - jD_{R_t}, \qquad r \in [0, R_t]$$

  - D_r = Dirac delta at r

The Free Boundary Problem

$\longrightarrow$ *If $\rho_t(r)$ is smooth in $(0, R_t)$, integrating by parts we obtain the FBP in its calssical formulation*

The pair $(R_t, \rho(\cdot, t))$ is a **Classical Solution** of the FBP with initial datum (R_0, ρ_0) if it is *"smooth enough"* and satisfies

$$\begin{cases} \frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2} & r \in (0, R_t) \\ \rho(R_t, t) = 0 & \\ \frac{\partial \rho}{\partial r}\Big|_{r=0^+} = \frac{\partial \rho}{\partial r}\Big|_{r=R_t^-} = -2j & \\ \rho(r, 0) = \rho_0(r) & r \in (0, R_0) \end{cases}$$

$\longrightarrow$ *The total mass is conserved:*

$$\int_0^{R_t} \rho(r,t)\, dr = \int_0^{R_0} \rho_0(r)\, dr$$

The Stefan problem

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial r^2}, \qquad v(r,t)\Big|_{r=0,X_t} = 0 \\ \frac{dX_t}{dt} = -(2j)^{-1}\frac{\partial v(r,t)}{\partial r}\Big|_{r=X_t} \end{cases}$$

is obtained from the FBP *by setting*

$$v(r,t) := -\frac{1}{2}\frac{\partial \rho}{\partial r}(r,t) - j$$

then $$\rho(r,t) = 2\int_r^{X_t} \Big(v(r',t) + j\Big)dr'$$

the equation for X_t *is obtained by differentiating the identity* $\rho(X_t,t) = 0$.

$\longrightarrow$ *Local existence and uniqueness of classical solutions for the Stephan problem are known.*

Stationary macroscopic profiles:

$\longrightarrow$ *Linear Profiles with slope $-2j$ are stationary:*

$$(R^{(M)}, \rho^{(M)}), \qquad \rho^{(M)}(r) := a_M - 2jr, \qquad 0 \le r \le R^{(M)} := \min\left\{\frac{a_M}{2j}, 1\right\}$$

The linear profiles are parametrized by $M :=$ *Total Mass* $\longrightarrow$ $\int_0^1 \rho^{(M)}(r)\, dr = M$

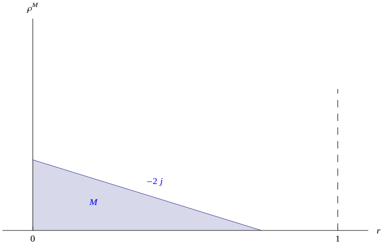

Figure : *Stationary solution for $M < j$*

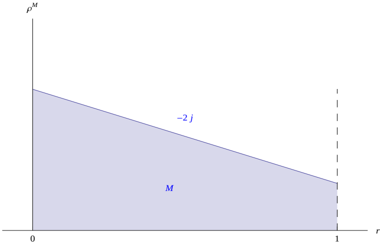

Figure : *Stationary solution for $M > j$*

$\mathcal{M} := \left\{\rho^{(M)},\ M > 0\right\}$ $\longrightarrow$ *one-dimensional Manifold of Classical Stationary Solutions*

Hydrodynamic Limit: Strategy of the Proof

We prove that

"$N^{-1}\xi^{(N)}_{N^2 t} \longrightarrow \rho_t$" as $N \to \infty$ Hydrodynamic Limit Theorem

where $(\rho_t(\cdot), R_t)$, (R_t boundary of ρ_t) is a "generalized solution" of the FBP

$(u_t(\cdot), X_t)$ Generalized Solution of the FBP:= Limit of Quasi-Solutions

where

a Quasi-Solution is obtained by relaxing the mass conservation costraint in the FBP

Strategy of the Proof

1. Characterization of ρ_t as the unique separating element of the *"Barriers"* through:
   - approximating microscopic processes
   - mass transport inequalities
2. Characterization of u_t as the unique separating element of the *Barriers*

Key Idea: Monotonicity

Compare the *original process* ξ_t with the *auxiliary process* ξ_t^-

Fix a time $T > 0$

$$\xi_t^- \longrightarrow \begin{cases} \text{in } [0,T) \longrightarrow \text{ evolution with Independent Random Walk} \\ \text{at time } T \longrightarrow \begin{cases} N^+(T) \text{ particles are added at site 0} \\ \text{the } N^-(T) \text{ rightmost particles are removed} \end{cases} \end{cases}$$

where

$$\left.\begin{array}{lcl} N^+(T) & := & \text{number of particles added up to time } T \\ N^-(T) & := & \text{number of particles removed up to time } T \end{array}\right\} \longrightarrow \text{ in the } \underline{\text{original process}} \ \xi_t$$

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then

"ξ_T^- is obtained from ξ_T by moving mass to the left"

Key Idea: Monotonicity

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then

"ξ_T^- is obtained from ξ_T by moving mass to the left"

$\xi_t^+ \longrightarrow$ is defined analogously, but the addition/removal mechanism is performed at time 0

"ξ_T^+ is obtained from ξ_T by moving mass to the right"

Mass Transport Inequalities

Definition (Partial order)

For $\xi, \xi' \in \{0, \ldots, N\}^{\mathbb{N}}$ we say that

$$\xi \leq \xi'$$

iff ξ' is obtained from ξ my moving mass to the right, e.g.

$$F_N(x;\xi) \leq F_N(x;\xi') \qquad \text{for all} \quad x \in \{0, \ldots, N-1\}$$

where

$$F_N(x;\xi) = \sum_{y \geq x} \xi(y)$$

THEN

$$\xi_t^- \leq \xi_t \leq \xi_t^+$$

"stochastically": the two processes can be both realized on a same space where the inequality holds pointwise almost surely.

Approximating processes

IDEA $\longrightarrow$ divide the time interval $[0, N^2 t]$ (Hydrodynamic Time Scale) into small intervalls of lenght $N^2\delta$, δ small

$$\xi_t^{(\delta,\pm)} \longrightarrow \begin{cases} \text{evolution with Independent Random Walk} \longrightarrow \text{in } (kN^2\delta, (k+1)N^2\delta) \\ \text{addition/removal mechanism} \longrightarrow \begin{cases} \text{at the beginning of the intervals for } \xi^{(\delta,+)} \\ \text{at the end of the intervals for } \xi^{(\delta,-)} \end{cases} \end{cases}$$

THEN

$$\xi_{kN^2\delta}^{(\delta,-)} \leq \xi_{kN^2\delta} \leq \xi_{kN^2\delta}^{(\delta,+)} \qquad \text{for all} \quad k \in \mathbb{N}$$

Idea of the Proof: Barriers

IDEA $\longrightarrow$ $\xi_t^{(\delta,\pm)}$ evolve as Independent Random Walk into the intervals, then they can be treated with traditional techniques to get the Hydrodynamic Limit.

$$\begin{array}{ccccc} \xi_{kN^2\delta}^{(\delta,-)} & \leq & \xi_{kN^2\delta} & \leq & \xi_{kN^2\delta}^{(\delta,+)} \\ \downarrow & & \downarrow & & \downarrow \\ S_{k\delta}^{(\delta,-)} & \leq & ? & \leq & S_{k\delta}^{(\delta,+)} \end{array}$$

as $N \to \infty$ Hydrodynamic Limit

in the sense of Mass Transport!

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in the sense of Mass Transport!

We expect that

- the mass-transport order is preserved in the limit
- $\left|S_{k\delta}^{(\delta,+)} - S_{k\delta}^{(\delta,-)}\right| \to 0$ as $\delta \to 0$ in some sense

this would characterize the hydrodynamic limit ρ_t as the limit as $\delta \to 0$ of $S_t^{(\delta,\pm)}$ in some sense

Hydrodynamic Limit for the Approximating Processes

We prove that

$$\xi^{(\delta,\pm)}_{kN^2\delta} \longrightarrow S^{(\delta,\pm)}_{k\delta} \qquad \text{as} \qquad N \to \infty$$

in the following sense:

Theorem

Given any $T > 0$ for any $\delta > 0$ small enough, any $k : k\delta \leq T$ and any $\zeta > 0$

$$\lim_{N\to\infty} P^{(N)}_{\xi_0}\left[\max_{x\in\{0,\dots,N\}} \left|N^{-1}F_N(x;\xi^{(\delta,\pm)}_{kN^2\delta}) - F(N^{-1}x; S^{(\delta,\pm)}_{k\delta}(\rho_0))\right| \leq \zeta\right] = 1$$

where

$$F_N(x;\xi) := \sum_{y=x}^{N} \xi(y), \qquad F(r;\rho) := \int_r^1 \rho(r')dr'$$

and ρ_0 and ξ_0 are "close" in some sense.

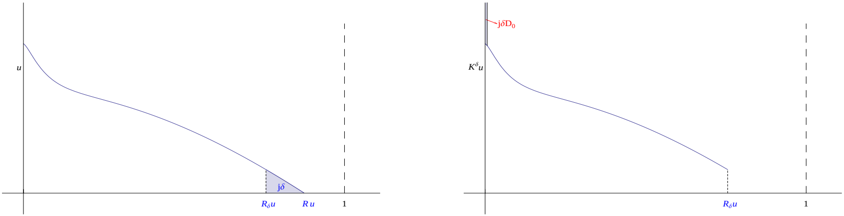
Barriers:

$$\mathbf{S}_{\mathbf{k}\delta}^{(\delta,+)}(\rho) := G_\delta^{\text{neum}} * K^{(\delta)} \cdots\cdots G_\delta^{\text{neum}} * K^{(\delta)} \rho \qquad (k \text{ times})$$

$$\mathbf{S}_{\mathbf{k}\delta}^{(\delta,-)}(\rho) := K^{(\delta)} G_\delta^{\text{neum}} * \cdots\cdots K^{(\delta)} G_\delta^{\text{neum}} * \rho \qquad (k \text{ times})$$

where

- $G_\delta^{\text{neum}}(r, r')$ = Green function of the heat equation in $[0, 1]$ with *Neumann b. c.*
- $K^{(\delta)}$ = *"the cut and paste map"*



$$\mathbf{K^{(\delta)} u = j\delta D_0 + u\, 1_{r \in [0, R_\delta(u)]}}$$

with $\quad R_\delta(u) \quad$ such that $\quad F(R_\delta(u), u) = \int_{R_\delta}^1 u(r)dr = j\delta$

Macroscopic Mass trasport inequalities

Call $F(r;u) := \int_r^1 u(r)dr, \quad u \geq 0$

Definition

For any integrable u and v

$$u \leq v \quad \text{iff} \quad F(r;u) \leq F(r;v), \qquad \forall r \in [0,1]$$

- $F(r;u)$ is a non increasing function of r which starts at 0 from the total mass of u, $F(0;u)$

Lemma

For any $\delta > 0$ and any integer k

$$S_{k\delta}^{(\delta,-)}(u) \leq S_{k\delta}^{(\delta,+)}(u)$$

Hydrodynamic Limit: Barriers separating element

Definition

We say that a function $u(\cdot, t)$, $u \in L^\infty([0,1], \mathbb{R}_+)$, separates the barriers $\{S^{(\delta,\pm)}_{k\delta}(u)(\cdot)\}$ iff

$$S^{(\delta,-)}_t(u)(\cdot) \le u(\cdot, t) \le S^{(\delta,+)}_t(u)(\cdot) \qquad \text{for all } \delta > 0 \text{ and } t \text{ such that } t = k\delta,\ k \in \mathbb{N}$$

Theorem (Existence and uniqueness of separating elements)

Let $u \in L^\infty([0,1], \mathbb{R}_+)$ and $F(0; u) > 0$. Then there exists a unique function $u(r,t)$ which separates the barriers $\{S^{(\delta,\pm)}_{k\delta}(u)\}$. $u(r,t)$ is continuous on the compacts of $[0,1] \times (0, \infty)$ and $u(\cdot, t)$ converges weakly to $u(\cdot)$ as $t \to 0$.

Theorem (Characterization of hydrodynamic limit)

The hydrodynamic limit $\rho(r,t)$ of ξ_t separates the barriers $\{S^{(\delta,\pm)}_{k\delta}(\rho_0)\}$.

Theorem

The Free Boundary Problem

The pair $(X_t, u(\cdot,t))$ is a **Classical Solution** of the FBP with initial datum (X_0, u_0) in the time interval $[0,T)$ if it satisfies

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial r^2} & r \in (0, X_t), \quad t \in [0,T) \\ u(X_t, t) = 0 & t \in [0,T) \\ \frac{\partial u}{\partial r}\Big|_{r=0^+} = -2j & t \in [0,T) \\ \frac{\partial u}{\partial r}\Big|_{r=X_t^-} = -2j & t \in [0,T) \\ u(r,0) = u_0(r) & r \in (0, X_0), \quad X_{t=0} = X_0 \end{cases}$$

i) $X_t \in C^1([0,T), \mathbb{R}_+)$;

ii) $u(\cdot,t) \in C^2((0,R_t), \mathbb{R}_+)$ and it has limits with its derivatives at 0 and X_t, $\forall t \in [0,T)$; $u(r,\cdot)$ differentiable $\forall r \in [0, X_t]$.

$\longrightarrow$ *The total mass is conserved:*

$$\int_0^{X_t} u(r,t)\,dr = \int_0^{X_0} u_0(r)\,dr$$

The Free Boundary Problem: an equivalent formulation

The pair $(X_t, u(\cdot,t))$ is a **Classical Solution** of the FBP with initial datum (X_0, u_0) in the time interval $[0,T)$ if it satisfies

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial r^2} & r \in (0, X_t), \quad t \in [0,T) \\ u(X_t, t) = 0 & t \in [0,T) \\ \frac{\partial u}{\partial r}\Big|_{r=0+} = -2j & t \in [0,T) \\ \int_0^{X_t} u(r,t)\, dr = \int_0^{X_0} u_0(r)\, dr & t \in [0,T) \\ u(r,0) = u_0(r) & r \in (0, X_0), \quad X_{t=0} = X_0 \end{cases}$$

i) $X_t \in C^1([0,T), \mathbb{R}_+)$;

ii) $u(\cdot,t) \in C^2((0,X_t), \mathbb{R}_+)$ and it has limits with its derivatives at 0 and X_t, $\forall t \in [0,T)$; $u(r,\cdot)$ differentiable $\forall r \in [0, X_t]$.

Key Idea

For a given X_t consider the problem without the mass conservation constraint:

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial r^2} & r \in (0, X_t), \quad t \in [0,T) \\ v(R_t, t) = 0 & t \in [0,T) \\ \frac{\partial v}{\partial r}\Big|_{r=0^+} = -2j & t \in [0,T) \\ v(r,0) = v_0(r) & r \in (0, X_0), \quad X_{t=0} = X_0 \end{cases}$$

then

$$v(r,t) := \int G_{0,t}^{X,\,\mathrm{neum}}(r',r)v_0(r')\,dr' + \int_0^t jG_{s,t}^{X,\,\mathrm{neum}}(0,r)\,ds$$

where

$G_{s,t}^{X,\,\mathrm{neum}}(r,\cdot)$ = *law density* of $\longrightarrow$ Brownian motion B_t starting from r at time s, reflected at 0 and restricted to trajectories so that $B_{s'} < X_{s'}$, $\forall s' \in [s,t]$

$$\int_I G_{s,t}^{X,\,\mathrm{neum}}(r',r)dr = P_{r';s}[\tau_s^X > t;\, B_t \in I], \qquad \tau_s^X = \inf\{t \geq s : B_t \geq X_t\}, \quad I \subset \mathbb{R}_+$$

Quasi-Solutions and Generalized Solutions

Definition (Quasi-solutions)

$(X_t, u(\cdot,t), \epsilon)$ *is a* quasi-solution *of the FBP in the time interval* $[0,T)$ *with initial datum* u_0 *and* accuracy parameter ϵ *if:*

- $(X_t, u(\cdot,t))$ *satisfies the problem where the mass conservation constraint is replaced by*

$$\sup_{t\le T}\left|\int_0^{X_t} u(r,t)\,dr - \int_0^{X_0} u(r,0)\,dr\right| \le \epsilon, \qquad t\in[0,T]$$

FBP

- $X_t > 0$ *is Lipschitz and piecewise* C^1 *(with finitely many discontinuities of the derivative)*
- $u(r,t)$ *is "smooth".*

Definition (Generalized solutions)

$(X_t, u(r,t))$ *is a* generalized solution *of the FBP in* $[0,T)$ *with initial datum* u_0 *if there exists a sequence* $(X_t^{(n)}, u^{(n)}(\cdot,t), \epsilon_n)$*,* $t\in[0,T]$*, of quasi-solutions in* $[0,T)$ *with initial datum* u_0 *such that*

$$\lim_{n\to\infty}\epsilon_n = 0 \qquad \textit{and} \qquad \lim_{n\to\infty} u^{(n)} = u \quad \textit{weakly}$$

Strategy of the Proof

Definition (Partial order modulo m)

For any integrable u and v and $m > 0$, we define

$$u \le v \quad \textit{modulo } m \quad \textit{iff} \quad \textit{for all } r \ge 0\textit{:} \;\; F(r;u) \le F(r;v) + m$$

We use the probabilistic representation of the quasi-solution and the relaxed condition on the mass to prove that:

Proposition

If $(X_t, u^{(\epsilon)}(\cdot,t), \epsilon)$ is a quasi-solution of the FBP with accuracy ϵ then for any $\delta > 0$, there is c so that for all $k \in \mathbb{N}$ such that $k\delta \le T$

$$S_{k\delta}^{(\delta,-)}(u^{(\epsilon)}(\cdot,0)) \le u^{(\epsilon)}(\cdot,k\delta) \le S_{k\delta}^{(\delta,+)}(u^{(\epsilon)}(\cdot,0)) \qquad \textit{modulo} \quad ck\epsilon$$

THEN

The Generalized Solution $u = \lim_{\epsilon \to 0} u^{(\epsilon)}$ of the FBP is the unique separating element between barriers!

Existence and Uniqueness

Theorem (Existence and uniqueness)

For any $u_0 \in L^\infty(\mathbb{R}_+, \mathbb{R}_+) \cap L^1(\mathbb{R}_+, \mathbb{R}_+)$ *and any* $T > 0$ *the following holds.*

(a) *There exists a Generalized Solution* $(X_t, u(r,t))$ *of the FBP in* $[0, T)$ *with initial datum* u_0.

(b) *Let* $S_t(u_0)$ *be the Separating Element of the Barriers* $\{S_{k\delta}^{(\delta,\pm)}(u_0)\}$.
Then, if $u(\cdot, t)$ *is a generalized solution of the FBP in* $[0, T)$ *with initial datum* u_0 *then*

$$u(\cdot, t) = S_t(u_0) \qquad \text{for all} \qquad t \in [0, T)$$

Consequence:

"The Hydrodynamic Limit of ξ_t is equal to the Generalized Solution of the FBP"

$$\text{“}\lim_{N\to\infty}(N^{-1}\xi_{N^2 t}, R_{\xi_{N^2 t}}) = (u(\cdot, t), X_t)\text{”}$$

Stationary macroscopic profiles:

$\longrightarrow$ *Linear Profiles with slope $-2j$ are stationary:*

$$(R^{(M)}, \rho^{(M)}), \qquad \rho^{(M)}(r) := a_M - 2jr, \qquad 0 \le r \le R^{(M)} := \min\left\{\frac{a_M}{2j}, 1\right\}$$

The linear profiles are parametrized by $M :=$ *Total Mass* $\longrightarrow$ $\int_0^1 \rho^{(M)}(r)\,dr = M$

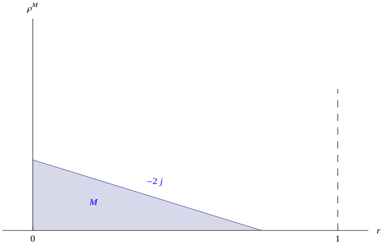


Figure : *Stationary solution for $M < j$*

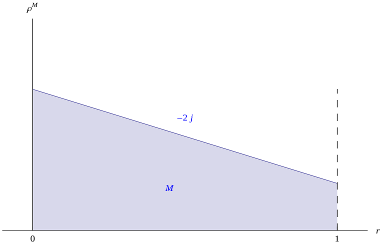


Figure : *Stationary solution for $M > j$*

$$\mathcal{M} := \left\{\rho^{(M)},\ M > 0\right\}$$ $\longrightarrow$ *one-dimensional Manifold of Classical Stationary Solutions*

Stability of the manifold of stationary profiles.

Theorem (Stability)

Let $\int_0^1 \rho_0(r)dr = M$ *and* ρ_t *the hydro-limit starting from* ρ_0*. Then, as* $t \to \infty$*,* ρ_t *converges weakly to* $\rho^{(M)}$ *in the sense that*

$$\lim_{t\to\infty} F(r;\rho_t) = F(r;\rho^{(M)}), \qquad \forall r \in [0,1]$$

where $F(r;u) = \int_r^1 u(r)dr$

FICK's LAW:

Agreement: The Stationary Profiles are Stable and Linear

Disagreement: The Stationary Profile is not Unique because the Desity is not fixed (unlike the case of Density Reservoirs)

Microscopic Stationary State

- *On the one hand:*

ξ_t is an *irreducible*, *aperiodic* Markov Process $\Rightarrow$ $\left\{ \begin{array}{l} \text{if it has a } \textit{stationary state} \text{ then} \\ \text{it is even a } \textit{limiting state} \\ \text{and it is } \textit{unique} \end{array} \right.$

- *On the other hand:*

"$N^{-1}\xi_{N^2 t} \longrightarrow \rho_t$" as $N \to \infty$ Hydrodynamic Limit (t fixed)

"$\rho_t \longrightarrow \rho^{(m)}$" as $t \to \infty$ with $\rho^{(m)} \in \mathcal{M}$, $m = \lim_{N\to\infty} N^{-1}|\xi_0|$

Interchange of limits in not allowed!

THEN

There is a <u>second time scale</u> beyond the hydrodynamic one

where we expect to observe one of the two following scenarios

- either there is a preferential profile
- or none of such profiles is stationary

The total number of particles

$|\xi_t| =$ Total Number of particles at time t

- particle added: $|\xi| \to |\xi| + 1 \quad \to \quad$ at rate $\frac{j}{N}$
- particle deleted: $|\xi| \to |\xi| - 1 \quad \to \quad$ at rate $\frac{j}{N}$

$\Big\} \longrightarrow$ $|\xi_t|$ *performs a* symmetric random walk *with jumps* ± 1 *at rate* $\frac{j}{N}$

The density $\frac{|\xi_t|}{N}$ changes after times of the order N^3:

$$M_t^N := \frac{|\xi_{N^3 t}|}{N} \longrightarrow B_t \quad \text{as} \quad N \to \infty$$

where $B_t :=$ Brownian Motion *on* $\mathbb{R}^+$ *with reflecting boundary conditions at* 0

Superhydrodynamic Limit

Brownian motion on the manifold of stationary profiles

Theorem (Super-hydrodynamic limit)

Let $\xi^{(N)}$ be a sequence such that $|\xi^{(N)}|N^{-1} \to m > 0$ as $N \to \infty$. Let t_N be an increasing, divergent sequence, then the process $\xi_{N^2 t_N}$ has two regimes:

- *Subcritical. Suppose $N^{-1}t_N \to 0$, then*

$$\lim_{N\to\infty} P^{(N)}_{\xi^{(N)}}\left[\max_{x\in\{0,\dots,N\}}\left|\frac{1}{N}F_N(x;\xi_{N^2 t_N}) - F(N^{-1}x;\rho^{(m)})\right| \le \zeta\right] = 1 \tag{1}$$

- *Critical. Let $t_N = Nt$ then*

$$\lim_{N\to\infty} P^{(N)}_{\xi^{(N)}}\left[\max_{x\in\{0,\dots,N\}}\left|\frac{1}{N}F_N(x;\xi_{N^3 t}) - F(N^{-1}x;\rho^{(M_t^{(N)})})\right| \le \zeta\right] = 1 \tag{2}$$

where $M_t^{(N)} := \epsilon|\xi^{(N)}|_{N^3 t}|$ converges in law as $N \to \infty$ to B_{jt}, where $(B_t)_{t\ge 0}$, $B_0 = m$, is the Brownian motion on $\mathbb{R}_+$ reflected at the origin.

References

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2. Super-hydrodynamic limit in interacting particle systems, (2014) **arxiv:1312.0640**
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