Wasserstein curvature of Markov processes

Bertrand Cloez

Université Paul Sabatier

Centre International de Mathématiques et d'Informatique TOULOUSE

Young European Probabilists

Markov processes

Let $(X_t)_{t\geq 0}$ be a Markov process on a polish space (E, d).

Markov processes

Let $(X_t)_{t\geq 0}$ be a Markov process on a polish space (E,d).

- Its semigroup $(P_t)_{t\geq 0}$ is defined by

$$\forall x\in E,\ \forall t\geq 0,\ P_tf(x)=\mathbb{E}\left[f(X_t)\mid X_0=x\right],$$

Markov processes

Let $(X_t)_{t\geq 0}$ be a Markov process on a polish space (E, d).

- Its semigroup $(P_t)_{t\geq 0}$ is defined by

$$\forall x \in E,\ \forall t \geq 0,\ P_t f(x) = \mathbb{E}\left[f(X_t) \mid X_0 = x\right],$$

- Its generator $\mathcal{L}$ is defined by

$$\mathcal{L}f(x) = \lim_{t\to 0+} \frac{P_t f(x) - f(x)}{t} = \partial_t P_t|_{t=0} f(x)$$

Markov processes

Let $(X_t)_{t\geq 0}$ be a Markov process on a polish space (E,d).

- Its semigroup $(P_t)_{t\geq 0}$ is defined by

$$\forall x \in E,\ \forall t \geq 0,\ P_t f(x) = \mathbb{E}\left[f(X_t) \mid X_0 = x\right],$$

- Its generator $\mathcal{L}$ is defined by

$$\mathcal{L}f(x) = \lim_{t\to 0+} \frac{P_t f(x) - f(x)}{t} = \partial_t P_t|_{t=0} f(x)$$

Natural questions :

- Convergence : $P_t f \to \pi$?
- Exponential convergence $\delta(\mu P_t, \pi) \leq Ce^{-\lambda t}$?

Wasserstein distance

- For any probability measures μ_1, μ_2 on (E, d) :

$$\begin{aligned}\mathcal{W}(\mu_1, \mu_2) &= \inf_{\Pi} \int_{E \times E} d(x, y) \Pi(dx, dy) \\ &= \inf_{X_1 \sim \mu_1, X_2 \sim \mu_2} \mathbb{E}\left[d(X_1, X_2)\right] \\ &= \sup_{\mathrm{Lip}(f) \leq 1} \int f d\mu_1 - \int f d\mu_2.\end{aligned}$$

Wasserstein distance

- For any probability measures μ_1, μ_2 on (E, d) :

$$\begin{aligned}\mathcal{W}(\mu_1, \mu_2) &= \inf_{\Pi} \int_{E\times E} d(x, y)\Pi(dx, dy) \\ &= \inf_{X_1\sim\mu_1, X_2\sim\mu_2} \mathbb{E}\left[d(X_1, X_2)\right] \\ &= \sup_{\mathrm{Lip}(f)\leq 1} \int f d\mu_1 - \int f d\mu_2.\end{aligned}$$

- if d is bounded then Convergence with $\mathcal{W}$ $\Leftrightarrow$ Convergence in law.

Wasserstein distance

- For any probability measures μ_1, μ_2 on (E, d) :

$$\begin{aligned}\mathcal{W}(\mu_1,\mu_2) &= \inf_{\Pi} \int_{E\times E} d(x,y)\Pi(dx,dy)\\ &= \inf_{X_1\sim\mu_1, X_2\sim\mu_2} \mathbb{E}\left[d(X_1,X_2)\right]\\ &= \sup_{\mathrm{Lip}(f)\leq 1} \int f d\mu_1 - \int f d\mu_2.\end{aligned}$$

- if d is bounded then Convergence with $\mathcal{W} \Leftrightarrow$ Convergence in law.
- Also called Kantorovich, Mallows, Monge, Fréchet, optimal transport, coupling, minimum-L^1...

Wasserstein curvature

Definition

The Wasserstein curvature of a Markov semigroup $(P_t)_{t\geq 0}$ is the largest constant ρ such that

$$\mathcal{W}(\mu P_t, \nu P_t) \leq e^{-\rho t}\mathcal{W}(\mu, \nu),$$

for any probability measure μ, ν and any $t \geq 0$.

Wasserstein curvature

Definition

The Wasserstein curvature of a Markov semigroup $(P_t)_{t\geq 0}$ is the largest constant ρ such that

$$\mathcal{W}(\mu P_t, \nu P_t) \leq e^{-\rho t}\mathcal{W}(\mu, \nu),$$

for any probability measure μ, ν and any $t \geq 0$.

- introduced independently by Joulin (2007), Ollivier (2007) and Sammer (2005).
- Motivated by generalizing Bakry-Emery curvature of diffusion processes or Ricci curvature of Riemannian Manifold.

Brownian motion and curvature

Let (M, g) be a smooth Riemannian manifold and $(P_t)_{t\geq 0}$ be solution to the heat equation

$$\forall t \geq 0,\ \partial_t P_t f = \Delta P_t f,$$

Brownian motion and curvature

Let (M, g) be a smooth Riemannian manifold and $(P_t)_{t\geq 0}$ be solution to the heat equation

$$\forall t \geq 0,\ \partial_t P_t f = \Delta P_t f,$$

Theorem (Sturm, Von Renesse, 05)

The two following assertions are equivalent :

$$\forall x \in M, \forall v \in \mathbb{R}^n, Ricci_x(v, v) \geq k\|v\|^2$$

$$\forall t \geq 0,\ \mathcal{W}_d(\mu P_t, \nu P_t) \leq e^{-kt} d(x, y).$$

Brownian motion and curvature

Let (M, g) be a smooth Riemannian manifold and $(P_t)_{t \geq 0}$ be solution to the heat equation

$$\forall t \geq 0,\ \partial_t P_t f = \Delta P_t f,$$

Theorem (Sturm, Von Renesse, 05)

The two following assertions are equivalent :

$$\forall x \in M, \forall v \in \mathbb{R}^n, Ricci_x(v, v) \geq k \|v\|^2$$

$$\forall t \geq 0,\ \mathcal{W}_d(\mu P_t, \nu P_t) \leq e^{-kt} d(x, y).$$

- if M is a sphere with radius r and dimension n then $\rho = (n-1)/r$.

Brownian motion and curvature

Let (M,g) be a smooth Riemannian manifold and $(P_t)_{t\geq 0}$ be solution to the heat equation

$$\forall t \geq 0,\ \partial_t P_t f = \Delta P_t f,$$

Theorem (Sturm, Von Renesse, 05)

The two following assertions are equivalent :

$$\forall x \in M, \forall v \in \mathbb{R}^n, Ricci_x(v,v) \geq k\|v\|^2$$

$$\forall t \geq 0,\ \mathcal{W}_d(\mu P_t, \nu P_t) \leq e^{-kt} d(x,y).$$

- if M is a sphere with radius r and dimension n then $\rho = (n-1)/r$.
- Also equivalent to the convexity of the entropy, Bakry-Emery curvature, $\mathcal{W}^2$ contraction . . .

Brownian motion and curvature

Let (M, g) be a smooth Riemannian manifold and $(P_t)_{t\geq 0}$ be solution to the heat equation

$$\forall t \geq 0,\ \partial_t P_t f = \Delta P_t f,$$

Theorem (Sturm, Von Renesse, 05)

The two following assertions are equivalent :

$$\forall x \in M, \forall v \in \mathbb{R}^n, Ricci_x(v, v) \geq k\|v\|^2$$

$$\forall t \geq 0,\ \mathcal{W}_d(\mu P_t, \nu P_t) \leq e^{-kt} d(x, y).$$

- if M is a sphere with radius r and dimension n then $\rho = (n-1)/r$.
- Also equivalent to the convexity of the entropy, Bakry-Emery curvature, $\mathcal{W}^2$ contraction ...
- Others definitions of curvature, see Sturm-Lott-Villani(06,09), Erbar-Maas (2011), Gozlan-Roberto-Samson-Tetali (2012)...

Convergence to equilibrium

Lemma (Convergence in Wasserstein distance)

If $\rho > 0$ *then there exists a unique invariant probability measure* π, *and*

$$\forall t \geq 0, \ \mathcal{W}(\mu P_t, \pi) \leq e^{-\rho t} \mathcal{W}(\mu, \pi).$$

Convergence to equilibrium

Lemma (Convergence in Wasserstein distance)

If $\rho > 0$ then there exists a unique invariant probability measure π, and

$$\forall t \geq 0,\ \mathcal{W}(\mu P_t, \pi) \leq e^{-\rho t} \mathcal{W}(\mu, \pi).$$

Theorem (Spectral gap / Poincaré inequality)

If $\rho > 0$ and π is reversible then $(P_t)_{t\geq 0}$ verifies a Poincaré inequality ; namely

$$\forall t \geq 0,\ Var_\pi(P_t f) \leq e^{-2\rho t} Var_\pi(f)$$

Convergence to equilibrium

Lemma (Convergence in Wasserstein distance)

If $\rho > 0$ then there exists a unique invariant probability measure π, and

$$\forall t \geq 0,\ \mathcal{W}(\mu P_t, \pi) \leq e^{-\rho t} \mathcal{W}(\mu, \pi).$$

Theorem (Spectral gap / Poincaré inequality)

If $\rho > 0$ and π is reversible then $(P_t)_{t\geq 0}$ verifies a Poincaré inequality ; namely

$$\forall t \geq 0,\ Var_\pi(P_t f) \leq e^{-2\rho t} Var_\pi(f) \Leftrightarrow Var_\pi(f) \leq \int \Gamma f d\pi$$

Convergence to equilibrium

Lemma (Convergence in Wasserstein distance)

If $\rho > 0$ then there exists a unique invariant probability measure π, and

$$\forall t \geq 0,\ \mathcal{W}(\mu P_t, \pi) \leq e^{-\rho t} \mathcal{W}(\mu, \pi).$$

Theorem (Spectral gap / Poincaré inequality)

If $\rho > 0$ and π is reversible then $(P_t)_{t\geq 0}$ verifies a Poincaré inequality ; namely

$$\forall t \geq 0,\ Var_\pi(P_t f) \leq e^{-2\rho t} Var_\pi(f) \Leftrightarrow Var_\pi(f) \leq \int \Gamma f d\pi$$

See Wang (2003), Chen (2004), Ollivier (2010), Hairer-Stuart-Vollmer (2011), Veysseire (2012).

Poissonian concentration

Theorem (Joulin 2007)

If $\rho > 0$ and there exist $A, B > 0$ such that

- $\sup_{t \geq 0} d(X_{t-}, X_t) \leq A$ *a.s.,*
- $\Gamma(f, f) \leq B \|f\|^2_{Lip}$,

then for all y and $t \geq 0$,

$$\mathbb{P}\left(|f(X_t) - \mathbb{E}[f(X_t)]| \geq y\right) \leq C e^{-ay \log(1+by))}$$

where $a, b > 0$ are some constants.

Poissonian concentration

Theorem (Joulin 2007)

If $\rho > 0$ and there exist $A, B > 0$ such that

- $\sup_{t\geq 0} d(X_{t-}, X_t) \leq A$ *a.s.,*
- $\Gamma(f,f) \leq B\|f\|^2_{Lip}$,

then for all y and $t \geq 0$,

$$\mathbb{P}\left(|f(X_t) - \mathbb{E}[f(X_t)]| \geq y\right) \leq Ce^{-ay\log(1+by))}$$

where $a, b > 0$ are some constants.

- Under the same assumptions, Joulin (2007) gives a concentration for the empirical measure.
- See also Ollivier, Joulin, ...

Deterministic Markov processes

Let $(x_t)_{t\geq 0}$ be the solution of

$$\forall t \geq 0,\ x_t' = F(x_t).$$

It is a Markov process.

Deterministic Markov processes

Let $(x_t)_{t \geq 0}$ be the solution of

$$\forall t \geq 0,\ x_t' = F(x_t).$$

It is a Markov process. What is its Wasserstein curvature with the usual distance ?

Deterministic Markov processes

Let $(x_t)_{t\geq 0}$ be the solution of

$$\forall t \geq 0,\ x_t' = F(x_t).$$

It is a Markov process. What is its Wasserstein curvature with the usual distance ?
It is the largest constant $\rho > 0$ such that

$$\forall x, y\ \langle x - y, F(x) - F(y)\rangle \leq -\rho \|x - y\|^2.$$

Deterministic Markov processes

Let $(x_t)_{t\geq 0}$ be the solution of

$$\forall t \geq 0,\ x_t' = F(x_t).$$

It is a Markov process. What is its Wasserstein curvature with the usual distance ?
It is the largest constant $\rho > 0$ such that

$$\forall x, y\ \langle x - y, F(x) - F(y)\rangle \leq -\rho \|x - y\|^2.$$

If $\rho > 0$ then $\exists x^*$ s.t.

$$\forall t \geq 0,\ \|x_t - x^*\|^2 \leq e^{-\rho t}\|x_0 - x^*\|^2.$$

Two point space

Markov processes generated by

$$\mathcal{L}f(n) = b\mathbf{1}_{n=0}(f(1) - f(0)) + d\mathbf{1}_{n=1}(f(0) - f(1)).$$

Two point space

Markov processes generated by

$$\mathcal{L}f(n) = b\mathbf{1}_{n=0}(f(1) - f(0)) + d\mathbf{1}_{n=1}(f(0) - f(1)).$$

Only one choice for the distance :

$$\forall x, y \in \{0, 1\} \; d(x, y) = c\mathbf{1}_{x \neq y}.$$

Two point space

Markov processes generated by

$$\mathcal{L}f(n) = b\mathbf{1}_{n=0}(f(1) - f(0)) + d\mathbf{1}_{n=1}(f(0) - f(1)).$$

Only one choice for the distance :

$$\forall x, y \in \{0, 1\}\ d(x, y) = c\mathbf{1}_{x \neq y}.$$

Wasserstein curvature is

$$\rho = b \wedge d$$

Birth and death processes

Markov processes generated by

$$\mathcal{L}f(n) = b(n)(f(n+1) - f(n)) + d(n)(f(n-1) - f(n)).$$

Birth and death processes

Markov processes generated by

$$\mathcal{L}f(n) = b(n)(f(n+1) - f(n)) + d(n)(f(n-1) - f(n)).$$

Theorem (Chafaï and Joulin, 2012)

The Wasserstein curvature associated to the distance d, defined by

$$\forall x, y \in \mathbb{N},\ d(x,y) = | \quad x \quad - \quad y \ |,$$

is given by

$$\rho \qquad = \inf_{n\geq 0} \left(d(n+1) - d(n) \qquad\qquad + b(n) - b(n+1) \qquad\qquad \right).$$

Birth and death processes

Markov processes generated by

$$\mathcal{L}f(n) = b(n)(f(n+1) - f(n)) + d(n)(f(n-1) - f(n)).$$

Theorem (Chafaï and Joulin, 2012)

The Wasserstein curvature associated to the distance d, defined by

$$\forall x, y \in \mathbb{N}, \ d(x,y) = |u(x) - u(y)|,$$

is given by

$$\rho(u) = \inf_{n \geq 0} \left(d(n+1) - d(n)\frac{u(n-1)}{u(n)} + b(n) - b(n+1)\frac{u(n+1)}{u(n)} \right).$$

Birth and death processes

Markov processes generated by

$$\mathcal{L}f(n) = b(n)(f(n+1) - f(n)) + d(n)(f(n-1) - f(n)).$$

Theorem (Chafaï and Joulin, 2012)

The Wasserstein curvature associated to the distance d, defined by

$$\forall x, y \in \mathbb{N},\ \ d(x,y) = |u(x) - u(y)|,$$

is given by

$$\rho(u) = \inf_{n\geq 0}\left(d(n+1) - d(n)\frac{u(n-1)}{u(n)} + b(n) - b(n+1)\frac{u(n+1)}{u(n)}\right).$$

By MChen, there exists u s.t. $\rho(u) = \lambda_0$ is the spectral gap

Ornstein-Uhlenbeck process

Let (X_t) be solution to

$$dX_t = -\lambda X_t dt + \sqrt{2} dB_t.$$

Ornstein-Uhlenbeck process

Let (X_t) be solution to

$$dX_t = -\lambda X_t dt + \sqrt{2} dB_t.$$

It is known that

$$P_t f(x) = \mathbb{E}_x\left[f(X_t)\right] = \mathbb{E}_x\left[f(xe^{-\lambda t} + N\sqrt{1 - e^{-2\lambda t}})\right],$$

Ornstein-Uhlenbeck process

Let (X_t) be solution to

$$dX_t = -\lambda X_t dt + \sqrt{2} dB_t.$$

It is known that

$$P_t f(x) = \mathbb{E}_x \left[f(X_t) \right] = \mathbb{E}_x \left[f(xe^{-\lambda t} + N\sqrt{1 - e^{-2\lambda t}}) \right],$$

and then

$$\begin{aligned} & P_t f(x) - P_t f(y) \\ & \quad = \mathbb{E}_x \left[f(xe^{-\lambda t} + N\sqrt{1 - e^{-2\lambda t}}) - f(ye^{-\lambda t} + N\sqrt{1 - e^{-2\lambda t}}) \right] \\ & \quad \leq e^{-\lambda t} \|f\|_{\mathrm{Lip}} \|x - y\|. \end{aligned}$$

Ornstein-Uhlenbeck process

Let (X_t) be solution to

$$dX_t = -\lambda X_t dt + \sqrt{2} dB_t.$$

It is known that

$$P_t f(x) = \mathbb{E}_x [f(X_t)] = \mathbb{E}_x \left[f(xe^{-\lambda t} + N\sqrt{1 - e^{-2\lambda t}}) \right],$$

and then

$$\begin{aligned} & P_t f(x) - P_t f(y) \\ & = \mathbb{E}_x \left[f(xe^{-\lambda t} + N\sqrt{1 - e^{-2\lambda t}}) - f(ye^{-\lambda t} + N\sqrt{1 - e^{-2\lambda t}}) \right] \\ & \leq e^{-\lambda t} \|f\|_{\mathrm{Lip}} \|x - y\|. \end{aligned}$$

We have $\rho = \lambda$.

Kolmogorov-Langevin diffusion

Let us consider that $E = \mathbb{R}^d$ and

$$\mathcal{L} = \Delta - \nabla V \cdot \nabla.$$

Kolmogorov-Langevin diffusion

Let us consider that $E = \mathbb{R}^d$ and

$$\mathcal{L} = \Delta - \nabla V \cdot \nabla.$$

Lemma

If $\mathrm{Hess}V \geq \kappa$ *then* $\rho \geq \kappa$.

Kolmogorov-Langevin diffusion

Let us consider that $E = \mathbb{R}^d$ and

$$\mathcal{L} = \Delta - \nabla V \cdot \nabla.$$

Lemma

If $\mathrm{Hess} V \geq \kappa$ *then* $\rho \geq \kappa$.

Theorem (Eberle 2011)

If there exist $K, L > 0$ *such that* $\mathrm{Hess} V \geq K$ *outside a ball and* $\mathrm{Hess} V \geq -L$ *then there exists a distance* d_f *such that its Wasserstein curvature, associated to* d_f*, is positive.*

Kolmogorov-Langevin diffusion

Let us consider that $E = \mathbb{R}^d$ and

$$\mathcal{L} = \Delta - \nabla V \cdot \nabla.$$

Lemma

If $\mathrm{Hess} V \geq \kappa$ *then* $\rho \geq \kappa$.

Theorem (Eberle 2011)

If there exist $K, L > 0$ *such that* $\mathrm{Hess} V \geq K$ *outside a ball and* $\mathrm{Hess} V \geq -L$ *then there exists a distance* d_f *such that its Wasserstein curvature, associated to* d_f*, is positive.*

- Proof by mirror (or reflexion) coupling + concave transformation of the usual distance.

Kolmogorov-Langevin diffusion

Let us consider that $E = \mathbb{R}^d$ and

$$\mathcal{L} = \Delta - \nabla V \cdot \nabla.$$

Lemma

If $\mathrm{Hess} V \geq \kappa$ *then* $\rho \geq \kappa$*.*

Theorem (Eberle 2011)

If there exist $K, L > 0$ *such that* $\mathrm{Hess} V \geq K$ *outside a ball and* $\mathrm{Hess} V \geq -L$ *then there exists a distance* d_f *such that its Wasserstein curvature, associated to* d_f*, is positive.*

- Proof by mirror (or reflexion) coupling + concave transformation of the usual distance.
- Almost optimal.

TCP process

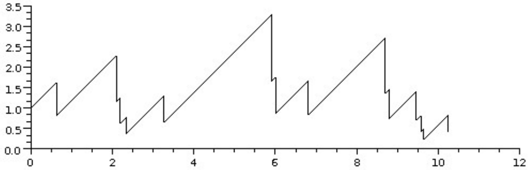
Theorem (C. 12)

If $(X_t)_{t\geq 0}$ *is generated by*

$$\forall x \geq 0,\ \mathcal{L}f(x) = f'(x) + r\left(f(\left(\frac{x}{2}\right) - f(x)\right),$$

where $r > 0$ *is non decreasing then*

$$\rho = \frac{1}{2}\inf_{x\geq 0}(r(x) - xr'(x)).$$



Problem

Let us consider :

- an irreducible CT Markov chain I, on a finite space $F = \{1, ..., N\}$, with an invariant distribution ν,

Problem

Let us consider :

- an irreducible CT Markov chain I, on a finite space $F = \{1, ..., N\}$, with an invariant distribution ν,
- for each $i \in F$, a Markov process $(X^{(i)})_{t\geq 0}$, with Wasserstein curvature $\rho(i)$, on a Polish space (E, d).

Problem

Let us consider :

- an irreducible CT Markov chain I, on a finite space $F = \{1, ..., N\}$, with an invariant distribution ν,
- for each $i \in F$, a Markov process $(X^{(i)})_{t \geq 0}$, with Wasserstein curvature $\rho(i)$, on a Polish space (E, d).

We consider the Markov process (X, I) such that X moves according to $(X_t^{(I_t)})_{t \geq 0}$.

Problem

Let us consider :

- an irreducible CT Markov chain I, on a finite space $F = \{1, ..., N\}$, with an invariant distribution ν,
- for each $i \in F$, a Markov process $(X^{(i)})_{t \geq 0}$, with Wasserstein curvature $\rho(i)$, on a Polish space (E, d).

We consider the Markov process (X, I) such that X moves according to $(X_t^{(I_t)})_{t \geq 0}$. This couple is generated by

$$\mathbb{L}f(x, i) = \mathcal{L}^{(i)} f(x, i) + \int_F (f(x, j) - f(x, i)) Q(i, dj).$$

Problem

Let us consider :

- an irreducible CT Markov chain I, on a finite space $F = \{1, ..., N\}$, with an invariant distribution ν,
- for each $i \in F$, a Markov process $(X^{(i)})_{t\geq 0}$, with Wasserstein curvature $\rho(i)$, on a Polish space (E, d).

We consider the Markov process (X, I) such that X moves according to $(X_t^{(I_t)})_{t\geq 0}$. This couple is generated by

$$\mathbb{L}f(x,i) = \mathcal{L}^{(i)}f(x,i) + \int_F (f(x,j) - f(x,i))Q(i,dj).$$

Under which conditions X admits an invariant distribution and converges exponentially fast to it ?

Problem

Let us consider :

- an irreducible CT Markov chain I, on a finite space $F = \{1, ..., N\}$, with an invariant distribution ν,
- for each $i \in F$, a Markov process $(X^{(i)})_{t\geq 0}$, with Wasserstein curvature $\rho(i)$, on a Polish space (E, d).

We consider the Markov process (X, I) such that X moves according to $(X_t^{(I_t)})_{t\geq 0}$. This couple is generated by

$$\mathbb{L}f(x,i) = \mathcal{L}^{(i)}f(x,i) + \int_F (f(x,j) - f(x,i))Q(i,dj).$$

Under which conditions X admits an invariant distribution and converges exponentially fast to it ? What happens if Q depends on x ?

Motivations

Applications :

- Chemostat (Collet, Martinez, Méléard, San Martin, 2012)
- Gene Network (Crudu, Debussche, Muller, Radulescu, 2012)
- Storage modelling (Boxma, Kaspi, Kella, Perry, 2005)
- Neuronal activity (Genadot, Pakdaman, Thieullen, Wainrib, 2012)
- Molecular biology (Faggionato, Gabrielli, Crivellari, 2008)
- Finance (Herrmann, Vallois, 2010)

Motivations

Applications :

- Chemostat (Collet, Martinez, Méléard, San Martin, 2012)
- Gene Network (Crudu, Debussche, Muller, Radulescu, 2012)
- Storage modelling (Boxma, Kaspi, Kella, Perry, 2005)
- Neuronal activity (Genadot, Pakdaman, Thieullen, Wainrib, 2012)
- Molecular biology (Faggionato, Gabrielli, Crivellari, 2008)
- Finance (Herrmann, Vallois, 2010)

See a series of papers of Benaïm, Le Borgne, Malrieu, Zitt in the special case $\mathcal{L}^{(i)} = F^{(i)} \cdot \nabla$.

An explosive switched vector fields

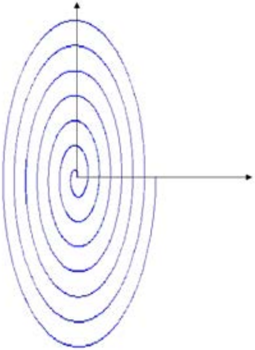


FIGURE: First vector field : $F^{(1)} : x \mapsto A_1 \cdot x$

2. An explosive switched vector fields

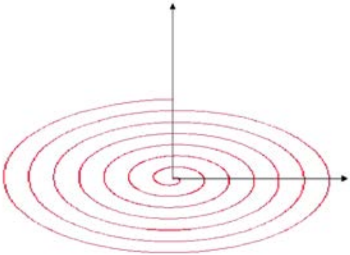


FIGURE: Second vector field : $F^{(2)} : x \mapsto A_2 \cdot x$

An explosive switched vector fields

Let $a > 0$, we consider the following generator :

$$\mathbf{L}f(x,i) = A_i \cdot \nabla_x f(x,i) + a(f(x,1-i) - f(x,i)),$$

where $x \in \mathbb{R}^2$, $i \in \{0,1\}$ and f is smooth.

An explosive switched vector fields

Let $a > 0$, we consider the following generator :

$$\mathbf{L}f(x,i) = A_i \cdot \nabla_x f(x,i) + a(f(x,1-i) - f(x,i)),$$

where $x \in \mathbb{R}^2$, $i \in \{0,1\}$ and f is smooth. If we fix $i \in \{0,1\}$ then the solutions of

$$\forall t \geq 0,\ \partial_t y_t = A_i y_t$$

satisfy

$$\|y_t\| \leq Ce^{-t}\|y_0\|.$$

An explosive switched vector fields

Let $a > 0$, we consider the following generator :

$$\mathbf{L}f(x,i) = A_i \cdot \nabla_x f(x,i) + a(f(x,1-i) - f(x,i)),$$

where $x \in \mathbb{R}^2$, $i \in \{0,1\}$ and f is smooth. If we fix $i \in \{0,1\}$ then the solutions of

$$\forall t \geq 0,\ \partial_t y_t = A_i y_t$$

satisfy

$$\|y_t\| \leq Ce^{-t}\|y_0\|.$$

Nevertheless if a is large enough then

$$\lim_{t\to+\infty} X_t = +\infty.$$

See (Benaïm, Le Borgne, Malrieu, Zitt 12) and (Lawley, Mattingly, Reed 13).

An explosive switched vector fields

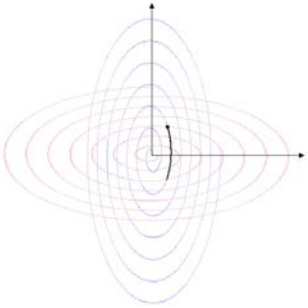


FIGURE: A typical trajectory

An explosive switched vector fields

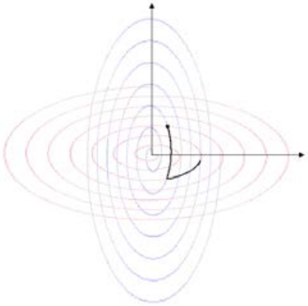


FIGURE: A typical trajectory

An explosive switched vector fields

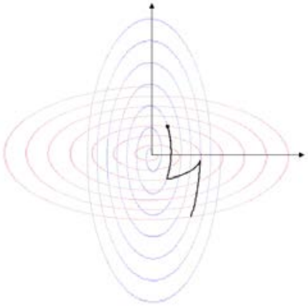


FIGURE: A typical trajectory

An explosive switched vector fields

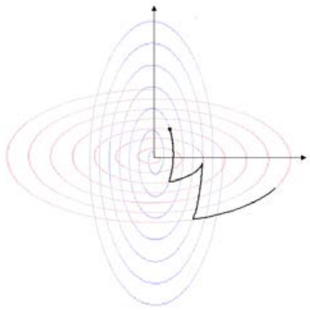


FIGURE: A typical trajectory

The most elementary example

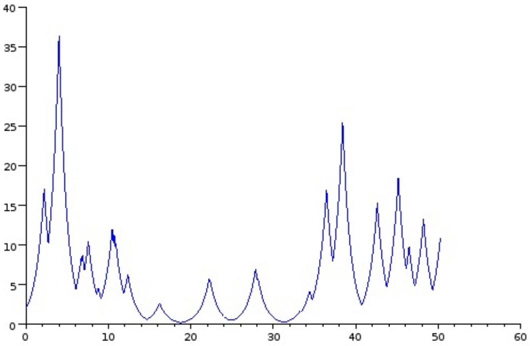


FIGURE: A trajectory of the second example

The most elementary example

Let us consider that I is a Markov Chain on $\{-1, 1\}$, the continuous component belongs to $\mathbb{R}$ and satisfies

$$\partial_t X_t = -I_t X_t .$$

The most elementary example

Let us consider that I is a Markov Chain on $\{-1, 1\}$, the continuous component belongs to $\mathbb{R}$ and satisfies

$$\partial_t X_t = -I_t X_t.$$

We have

$$X_t = e^{-\int_0^t I_s ds}$$

The most elementary example

Let us consider that I is a Markov Chain on $\{-1, 1\}$, the continuous component belongs to $\mathbb{R}$ and satisfies

$$\partial_t X_t = -I_t X_t.$$

We have

$$X_t = e^{-t \times \frac{1}{t} \int_0^t I_s ds}$$

The most elementary example

Let us consider that I is a Markov Chain on $\{-1, 1\}$, the continuous component belongs to $\mathbb{R}$ and satisfies

$$\partial_t X_t = -I_t X_t.$$

We have

$$X_t = e^{-t \times \frac{1}{t} \int_0^t I_s ds}$$

Birkhoff's ergodic theorem gives that

- $X_t \to 0$ if $\sum_i i\nu(i) = \nu(1) - \nu(-1) > 0$,
- $X_t \to +\infty$ if $\sum_i i\nu(i) = \nu(1) - \nu(-1) < 0$.

The most elementary example

Let us consider that I is a Markov Chain on $\{-1,1\}$, the continuous component belongs to $\mathbb{R}$ and satisfies

$$\partial_t X_t = -I_t X_t.$$

We have

$$X_t = e^{-t\times\frac{1}{t}\int_0^t I_s ds}$$

Birkhoff's ergodic theorem gives that

- $X_t \to 0$ if $\sum_i i\nu(i) = \nu(1) - \nu(-1) > 0$,
- $X_t \to +\infty$ if $\sum_i i\nu(i) = \nu(1) - \nu(-1) < 0$.

$\to$ Rates of convergence ?

The most elementary example

Let us consider that I is a Markov Chain on $\{-1, 1\}$, the continuous component belongs to $\mathbb{R}$ and satisfies

$$\partial_t X_t = -I_t X_t.$$

We have

$$X_t = e^{-t \times \frac{1}{t} \int_0^t I_s ds}$$

Birkhoff's ergodic theorem gives that

- $X_t \to 0$ if $\sum_i i\nu(i) = \nu(1) - \nu(-1) > 0$,
- $X_t \to +\infty$ if $\sum_i i\nu(i) = \nu(1) - \nu(-1) < 0$.

$\to$ Rates of convergence ? What is the distance ?

Non convergence with the usual distances

If $X_0 \neq 0$ then $X_t \neq 0, \ \forall t \geq 0$.

Non convergence with the usual distances

If $X_0 \neq 0$ then $X_t \neq 0,\ \forall t \geq 0$. In particular,

$$\forall t \geq 0,\ \|\mathcal{L}(X_t) - \delta_0\|_{TV} = \mathbb{P}(T_0 > t) = 1.$$

Non convergence with the usual distances

If $X_0 \neq 0$ then $X_t \neq 0, \; \forall t \geq 0$. In particular,

$$\forall t \geq 0, \; \|\mathcal{L}(X_t) - \delta_0\|_{TV} = \mathbb{P}(T_0 > t) = 1.$$

In general

$$\lim_{t \to +\infty} E[X_t] = +\infty,$$

Non convergence with the usual distances

If $X_0 \neq 0$ then $X_t \neq 0, \ \forall t \geq 0$. In particular,

$$\forall t \geq 0, \ \|\mathcal{L}(X_t) - \delta_0\|_{TV} = \mathbb{P}(T_0 > t) = 1.$$

In general

$$\lim_{t \to +\infty} E[X_t] = +\infty,$$

and then there is no convergence in L^1-norm and

$$\lim_{t \to +\infty} \mathcal{W}(\mathcal{L}(X_t), \delta_0) = +\infty.$$

Non convergence with the usual distances

If $X_0 \neq 0$ then $X_t \neq 0, \ \forall t \geq 0$. In particular,

$$\forall t \geq 0, \ \|\mathcal{L}(X_t) - \delta_0\|_{TV} = \mathbb{P}(T_0 > t) = 1.$$

In general

$$\lim_{t \to +\infty} E[X_t] = +\infty,$$

and then there is no convergence in L^1-norm and

$$\lim_{t \to +\infty} \mathcal{W}(\mathcal{L}(X_t), \delta_0) = +\infty.$$

$\rightarrow$ We have to modify the distance !

Non convergence with the usual distances

If $X_0 \neq 0$ then $X_t \neq 0, \ \forall t \geq 0$. In particular,

$$\forall t \geq 0, \ \|\mathcal{L}(X_t) - \delta_0\|_{TV} = \mathbb{P}(T_0 > t) = 1.$$

In general

$$\lim_{t \to +\infty} E[X_t] = +\infty,$$

and then there is no convergence in L^1-norm and

$$\lim_{t \to +\infty} \mathcal{W}(\mathcal{L}(X_t), \delta_0) = +\infty.$$

$\rightarrow$ We have to modify the distance !
Convergence of the moment ?

$$\mathbb{E}\left[X_t^p\right] = \mathbb{E}\left[e^{-\int_0^t p I_s ds}\right], \quad p \in (0,1).$$

Moments properties

Feynman-Kac formula :

$$\mathbb{E}\left[e^{-\int_0^t pI_s ds}\right] = \mu_0 e^{t(A-pId)}\mathbf{1}$$

Moments properties

Feynman-Kac formula :

$$\mathbb{E}\left[e^{-\int_0^t pI_s ds}\right] = \mu_0 e^{t(A-pId)}\mathbf{1} \approx e^{-\lambda_p t}.$$

Moments properties

Feynman-Kac formula :

$$\mathbb{E}\left[e^{-\int_0^t pI_s ds}\right] = \mu_0 e^{t(A-pId)}\mathbf{1} \approx e^{-\lambda_p t}.$$

We have $\lambda_0 = 0$

Moments properties

Feynman-Kac formula :

$$\mathbb{E}\left[e^{-\int_0^t pI_s ds}\right] = \mu_0 e^{t(A-pId)}\mathbf{1} \approx e^{-\lambda_p t}.$$

We have $\lambda_0 = 0$ and

$$\partial_p \lambda_p\big|_{p=0} = \sum_{i \in F} i\nu(i).$$

Moments properties

Feynman-Kac formula :

$$\mathbb{E}\left[e^{-\int_0^t pI_s ds}\right] = \mu_0 e^{t(A-pId)}\mathbf{1} \approx e^{-\lambda_p t}.$$

We have $\lambda_0 = 0$ and

$$\partial_p \lambda_p|_{p=0} = \sum_{i\in F} i\nu(i).$$

Hence

$$\sum_{i\in F} i\nu(i) > 0 \Rightarrow \ \exists p > 0, \ \lambda_p > 0.$$

Moments properties

Feynman-Kac formula :

$$\mathbb{E}\left[e^{-\int_0^t pI_s ds}\right] = \mu_0 e^{t(A-pId)}\mathbf{1} \approx e^{-\lambda_p t}.$$

We have $\lambda_0 = 0$ and

$$\partial_p \lambda_p|_{p=0} = \sum_{i\in F} i\nu(i).$$

Hence

$$\sum_{i\in F} i\nu(i) > 0 \Rightarrow \ \exists p > 0,\ \lambda_p > 0.$$

- See (Bardet, Guerin, Malrieu, 2010).

Moments properties

Feynman-Kac formula :

$$\mathbb{E}\left[e^{-\int_0^t pI_s ds}\right] = \mu_0 e^{t(A-pId)}\mathbf{1} \approx e^{-\lambda_p t}.$$

We have $\lambda_0 = 0$ and

$$\partial_p \lambda_p|_{p=0} = \sum_{i \in F} i\nu(i).$$

Hence

$$\sum_{i \in F} i\nu(i) > 0 \Rightarrow \exists p > 0, \ \lambda_p > 0.$$

- See (Bardet, Guerin, Malrieu, 2010).
- Convergence in "L^p-norm" and in a weaker Wasserstein distance.

Wasserstein exponential ergodicity

Theorem (C. and Hairer, 2012)

If

$$\sum_j \nu(j)\alpha(j) > 0,$$

then (X, I) admits an invariant probability measure and it converges exponentially fast to it in a Wasserstein distance.

Wasserstein exponential ergodicity

Theorem (C. and Hairer, 2012)

If

$$\sum_j \nu(j)\alpha(j) > 0,$$

then (X, I) admits an invariant probability measure and it converges exponentially fast to it in a Wasserstein distance.

Here the Wasserstein distance is associated to a concave transformation $(x \mapsto x^p, p \in (0,1))$ of the usual distance.

Wasserstein exponential ergodicity in the non-constant case

If $F = \{-1, 1\}$ and

$$\mathbb{L}f(x,i) = \mathcal{L}^{(i)}f(x,i) + a(x,i)(f(x,-i) - f(x,i)),$$

Wasserstein exponential ergodicity in the non-constant case

If $F = \{-1, 1\}$ and

$$\mathbb{L}f(x,i) = \mathcal{L}^{(i)}f(x,i) + a(x,i)(f(x,-i) - f(x,i)),$$

where we consider $\rho(1) > 0, \rho(-1) < 0$ and

$$\underline{a}(1) = \inf_x a(x,1) \quad \text{and} \quad \bar{a}(-1) = \sup_x a(x,-1)$$

Wasserstein exponential ergodicity in the non-constant case

If $F = \{-1, 1\}$ and

$$\mathbb{L}f(x,i) = \mathcal{L}^{(i)}f(x,i) + a(x,i)(f(x,-i) - f(x,i)),$$

where we consider $\rho(1) > 0, \rho(-1) < 0$ and

$$\underline{a}(1) = \inf_x a(x,1) \quad \text{and} \quad \bar{a}(-1) = \sup_x a(x,-1)$$

Theorem (C. and Hairer, 2013)

If a is Lipschitz and

$$\bar{a}(-1)\rho(1) + \underline{a}(1)\rho(-1) > 0$$

then X admits an invariant probability measure and converges exponentially fast to it in a Wasserstein distance.

A weak form of Harris theorem

Proof based on

Theorem (Hairer, Mattingly, Scheutzow, 09)

Let $(P_t)_{t\geq 0}$ be a Markov semigroup over a Polish space E that admits a Lyapunov function V. Assume furthermore that there exists t_* sufficiently large and a lower semi-continuous metric $d : E \times E \mapsto [0,1]$ such that

- *d^2 is contracting for P_t*
- *level sets of V are $d-$small for P_t.*

Then there exists a unique invariant measure π for P_t and the convergence is exponential.

Others limit theorems

If one of the two assumptions is satisfied :

- $\exists i$ s.t. the process associated to $\mathcal{L}^{(i)}$ "creates density",
- $\forall i,\ \mathcal{L}^{(i)} = F^{(i)} \cdot \nabla$ and the family $(F^{(i)})_i$ verifies an Hörmander-type condition,

then it is enough to find a Lyapunov function to have an exponential decay. And we have

Lemma

If there exists V s.t.

$$\mathcal{L}^{(i)} V \leq -\lambda_i V + K_i,$$

where

$$\sum_{i \in F} \lambda_i \nu(i) > 0,$$

then (X, I) admits a Lyapunov function (and thus converges in total variation distance to an invariant measure).

Thank you for your attention !