Curvature effects for infinite particle systems via optimal transport

Matthias Erbar
joint work with M. Huesmann

Scuola Normale Superiore, Pisa

Eindhoven, 13 March, 2014

An infinite particle system

Consider a system of interacting Brownian motions [Osada '11–'13] in $\mathbb{R}^d$:

$$\mathrm{d}X_t^i = \mathrm{d}B_t^i + \frac{\beta}{2} \lim_{R\to\infty} \sum_{|X_t^i - X_t^j| < R} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} \mathrm{d}t \ , \quad i = 1, 2, \dots$$

An infinite particle system

Consider a system of interacting Brownian motions [Osada '11–'13] in $\mathbb{R}^d$:

$$\mathrm{d}X_t^i = \mathrm{d}B_t^i + \frac{\beta}{2} \lim_{R\to\infty} \sum_{|X_t^i - X_t^j| < R} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} \mathrm{d}t \ , \quad i = 1, 2, \ldots$$

For finitely many particles, OT gives powerful tools :
Otto calculus, gradient flow structure, asymptotic behaviour, stability...

An infinite particle system

Consider a system of interacting Brownian motions [Osada '11–'13] in $\mathbb{R}^d$:

$$\mathrm{d}X_t^i = \mathrm{d}B_t^i + \frac{\beta}{2}\lim_{R\to\infty}\sum_{|X_t^i - X_t^j|<R}\frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}\mathrm{d}t\ , \quad i = 1, 2, \dots$$

For finitely many particles, OT gives powerful tools :
Otto calculus, gradient flow structure, asymptotic behaviour, stability...

Aim: Develop analoguous tools for infinite systems

An infinite particle system

Consider a system of interacting Brownian motions [Osada '11–'13] in $\mathbb{R}^d$:

$$\mathrm{d}X_t^i = \mathrm{d}B_t^i + \frac{\beta}{2} \lim_{R\to\infty} \sum_{|X_t^i - X_t^j| < R} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} \mathrm{d}t \; , \quad i = 1, 2, \ldots$$

For finitely many particles, OT gives powerful tools :
Otto calculus, gradient flow structure, asymptotic behaviour, stability...

Aim: Develop analoguous tools for infinite systems

As first step, understand non-interacting system of independent particles and its geometry!

Question: What is the natural state space for the particle system?

The configuration space

Setting: the base space

- $(M, \langle \cdot, \cdot \rangle)$ Riemannian manifold
- $d : M \times M \to \mathbb{R}_+$ Riemannian distance
- m volume measure

The configuration space

Setting: the base space

- $(M, \langle \cdot, \cdot \rangle)$ Riemannian manifold
- $d : M \times M \to \mathbb{R}_+$ Riemannian distance
- m volume measure

- The *configuration space* Υ over M is the set of locally finite counting measures on M, i.e.

$$\Upsilon = \{\gamma \in \mathcal{M}(M) : \gamma(K) \in \mathbb{N}_0 \text{ for all } K \subset M \text{ compact}\} .$$

The configuration space

Setting: the base space

- $(M, \langle\cdot,\cdot\rangle)$ Riemannian manifold
- $d : M \times M \to \mathbb{R}_+$ Riemannian distance
- m volume measure

- The *configuration space* Υ over M is the set of locally finite counting measures on M, i.e.

$$\Upsilon = \{\gamma \in \mathcal{M}(M) : \gamma(K) \in \mathbb{N}_0 \text{ for all } K \subset M \text{ compact}\} .$$

- Any configuration $\gamma \in \Upsilon$ can be represented by a *labelling*:

$$\gamma = \sum_{i=1}^{n} \delta_{x_i}$$

for $x_1, \ldots, x_n \in M$ and $n \in \mathbb{N} \cup \{\infty\}$.

The configuration space

Setting: the base space

- $(M, \langle \cdot, \cdot \rangle)$ Riemannian manifold
- $d : M \times M \to \mathbb{R}_+$ Riemannian distance
- m volume measure

- The *configuration space* Υ over M is the set of locally finite counting measures on M, i.e.

$$\Upsilon = \{\gamma \in \mathcal{M}(M) : \gamma(K) \in \mathbb{N}_0 \text{ for all } K \subset M \text{ compact}\} .$$

- Any configuration $\gamma \in \Upsilon$ can be represented by a *labelling*:

$$\gamma = \sum_{i=1}^{n} \delta_{x_i}$$

  for $x_1, \ldots, x_n \in M$ and $n \in \mathbb{N} \cup \{\infty\}$.
- We equip Υ with vague topology (duality with $C_c(M)$)

Υ as an infinite-dimensional manifold

Albeverio–Kondratiev–Röckner '98:

natural Riemannian structure on Υ via *lifting* from M

Υ as an infinite-dimensional manifold

Albeverio–Kondratiev–Röckner '98:

natural Riemannian structure on Υ via *lifting* from M

- Tangent space: $T_\gamma\Upsilon = \{V : M \to TM : \int |V|_x^2 \mathrm{d}\gamma(x) < \infty\}$
  equipped with inner product: $\langle V_1, V_2\rangle_\gamma = \int \langle V_1, V_2\rangle_x \mathrm{d}\gamma(x)$

Υ as an infinite-dimensional manifold

Albeverio–Kondratiev–Röckner '98:

natural Riemannian structure on Υ via *lifting* from M

- Tangent space: $T_\gamma \Upsilon = \{V : M \to TM : \int |V|_x^2 \mathrm{d}\gamma(x) < \infty\}$ equipped with inner product: $\langle V_1, V_2 \rangle_\gamma = \int \langle V_1, V_2 \rangle_x \mathrm{d}\gamma(x)$
- Gradient: for a cylinder function $F(\gamma) = g\big(\gamma(\varphi_1), \cdots, \gamma(\varphi_n)\big)$ with $g \in C^\infty(\mathbb{R}^n)$ and $\varphi_i \in C_c^\infty(M)$ define

$$\nabla^\Upsilon F(\gamma) = \sum_{i=1}^n \partial_i g\big(\gamma(\varphi_1), \cdots, \gamma(\varphi_n)\big) \nabla \varphi_i$$

Υ as an infinite-dimensional manifold

Albeverio–Kondratiev–Röckner '98:

natural Riemannian structure on Υ via *lifting* from M

- Tangent space: $T_\gamma \Upsilon = \{V : M \to TM : \int |V|_x^2 \mathrm{d}\gamma(x) < \infty\}$ equipped with inner product: $\langle V_1, V_2 \rangle_\gamma = \int \langle V_1, V_2 \rangle_x \mathrm{d}\gamma(x)$
- Gradient: for a cylinder function $F(\gamma) = g\big(\gamma(\varphi_1), \cdots, \gamma(\varphi_n)\big)$ with $g \in C^\infty(\mathbb{R}^n)$ and $\varphi_i \in C_c^\infty(M)$ define

$$\nabla^\Upsilon F(\gamma) = \sum_{i=1}^n \partial_i g\big(\gamma(\varphi_1), \cdots, \gamma(\varphi_n)\big) \nabla \varphi_i$$

- Divergence: for cylindrical vector field $W(\gamma) = \sum_{i=1}^n F_i(\gamma) V_i$ with $F_i \in \mathrm{Cyl}^\infty(\Upsilon)$ and $V_i : M \to TM$ vector fields define

$$\mathrm{div}^\Upsilon W(\gamma) = \sum_{i=1}^n \langle \nabla^\Upsilon F_i, V_i \rangle_\gamma + F_i(\gamma) \langle \mathrm{div}\, V_i, \gamma \rangle$$

Υ as an infinite-dimensional manifold

Albeverio–Kondratiev–Röckner '98:

natural Riemannian structure on Υ via *lifting* from M

- Tangent space: $T_\gamma \Upsilon = \{V : M \to TM : \int |V|_x^2 \mathrm{d}\gamma(x) < \infty\}$ equipped with inner product: $\langle V_1, V_2 \rangle_\gamma = \int \langle V_1, V_2 \rangle_x \mathrm{d}\gamma(x)$
- Gradient: for a cylinder function $F(\gamma) = g\big(\gamma(\varphi_1), \cdots, \gamma(\varphi_n)\big)$ with $g \in C^\infty(\mathbb{R}^n)$ and $\varphi_i \in C_c^\infty(M)$ define

$$\nabla^\Upsilon F(\gamma) = \sum_{i=1}^n \partial_i g\big(\gamma(\varphi_1), \cdots, \gamma(\varphi_n)\big) \nabla \varphi_i$$

- Divergence: for cylindrical vector field $W(\gamma) = \sum_{i=1}^n F_i(\gamma) V_i$ with $F_i \in \mathrm{Cyl}^\infty(\Upsilon)$ and $V_i : M \to TM$ vector fields define

$$\mathrm{div}^\Upsilon W(\gamma) = \sum_{i=1}^n \langle \nabla^\Upsilon F_i, V_i \rangle_\gamma + F_i(\gamma) \langle \mathrm{div}\, V_i, \gamma \rangle$$

- Laplace opterator: $\Delta^\Upsilon F(\gamma) = \mathrm{div}^\Upsilon \nabla^\Upsilon F(\gamma)$

Υ as ∞-dim. mfd: distance

Define a distance on Υ via

$$d_{\Upsilon}^2(\gamma_1, \gamma_2) = \inf \left\{ \sum_{i=1}^{n} d^2(x_i, y_i) : \gamma_1 = \sum_{i=1}^{n} \delta_{x_i}, \gamma_2 = \sum_{i=1}^{n} \delta_{y_i} \right\}$$

Υ as ∞-dim. mfd: distance

Define a distance on Υ via

$$d^2_\Upsilon(\gamma_1, \gamma_2) = \inf\left\{\sum_{i=1}^n d^2(x_i, y_i) : \gamma_1 = \sum_{i=1}^n \delta_{x_i}, \gamma_2 = \sum_{i=1}^n \delta_{y_i}\right\}$$

- L^2-transportation distance for non-normalized measures.

Υ as ∞-dim. mfd: distance

Define a distance on Υ via

$$d_\Upsilon^2(\gamma_1, \gamma_2) = \inf \left\{ \sum_{i=1}^n d^2(x_i, y_i) : \gamma_1 = \sum_{i=1}^n \delta_{x_i}, \gamma_2 = \sum_{i=1}^n \delta_{y_i} \right\}$$

- L^2-transportation distance for non-normalized measures.
- It is a pseudo distance, typically $d_\Upsilon(\gamma_1, \gamma_2) = +\infty$

Υ as ∞-dim. mfd: distance

Define a distance on Υ via

$$d_\Upsilon^2(\gamma_1, \gamma_2) = \inf \left\{ \sum_{i=1}^n d^2(x_i, y_i) : \gamma_1 = \sum_{i=1}^n \delta_{x_i}, \gamma_2 = \sum_{i=1}^n \delta_{y_i} \right\}$$

- L^2-transportation distance for non-normalized measures.
- It is a pseudo distance, typically $d_\Upsilon(\gamma_1, \gamma_2) = +\infty$
- It is the induced Riemannian distance on Υ:

$$d_\Upsilon^2(\gamma_0, \gamma_1) = \inf \left\{ \int_0^1 |V_t|_{\gamma_t}^2 \mathrm{d}t : \gamma_t = \sum_i \delta_{x_i(t)}, \dot{x}_i(t) = V_t(x_i) \right\}$$

Υ as ∞-dim. mfd: measure

The natural reference measure on Υ (in the independent case $\beta = 0$) is the *Poisson measure* π with intensity measure *m*:

Υ as ∞-dim. mfd: measure

The natural reference measure on Υ (in the independent case $\beta = 0$) is the *Poisson measure* π with intensity measure m:

Under π γ becomes a Υ-valued random variable, s.t.

- $\gamma(A) \sim Poi(m(A))$ for all $A \subset M$, i.e. $\pi_\mu\big[\gamma(A) = n\big] = \mathrm{e}^{-m(A)}\, m(A)^n/n!$
- $\gamma(A_1), \dots, \gamma(A_n)$ independent for all $A_1, \dots, A_n \subset M$ disjoint

Υ as ∞-dim. mfd: measure

The natural reference measure on Υ (in the independent case $\beta = 0$) is the *Poisson measure* π with intensity measure m:

Under π γ becomes a Υ-valued random variable, s.t.

- $\gamma(A) \sim Poi(m(A))$ for all $A \subset M$, i.e.
  $\pi_\mu\big[\gamma(A) = n\big] = \mathrm{e}^{-m(A)}\, m(A)^n/n!$
- $\gamma(A_1), \ldots, \gamma(A_n)$ independent for all $A_1, \ldots, A_n \subset M$ disjoint

π is the unique probability measure on Υ (up to mixtures) such that ∇^Υ and div^Υ are adjoint in $L^2(\pi)$ (AKR '98), i.e.

$$\int F(\gamma)\, \mathrm{div}^\Upsilon W(\gamma) \mathrm{d}\pi(\gamma) = -\int \langle \nabla^\Upsilon F, W\rangle_\gamma \mathrm{d}\pi(\gamma)$$

Dirichlet form and particle process on Υ

Define a Dirichlet form on Υ with domain $\mathrm{Cyl}^\infty(\Upsilon)$ via

$$\mathcal{E}(F) = \int \langle \nabla^\Upsilon F, \nabla^\Upsilon F \rangle_\gamma \mathrm{d}\pi(\gamma) = -\int F(\gamma) \Delta^\Upsilon F(\gamma) \mathrm{d}\pi(\gamma)$$

associated semigroup: $P_t^\Upsilon = \mathrm{e}^{t\Delta^\Upsilon}$

Dirichlet form and particle process on Υ

Define a Dirichlet form on Υ with domain $\mathrm{Cyl}^\infty(\Upsilon)$ via

$$\mathcal{E}(F) = \int \langle \nabla^\Upsilon F, \nabla^\Upsilon F \rangle_\gamma \mathrm{d}\pi(\gamma) = -\int F(\gamma)\Delta^\Upsilon F(\gamma)\mathrm{d}\pi(\gamma)$$

associated semigroup: $P_t^\Upsilon = \mathrm{e}^{t\Delta^\Upsilon}$

$\mathcal{E}$ gives rise to a canonical diffusion process $(\mathbf{B}_t^\gamma)_{t\geq 0}$ starting in a.e. $\gamma \in \Upsilon$.

Dirichlet form and particle process on Υ

Define a Dirichlet form on Υ with domain $\mathrm{Cyl}^{\infty}(\Upsilon)$ via

$$\mathcal{E}(F) = \int \langle \nabla^{\Upsilon} F, \nabla^{\Upsilon} F \rangle_{\gamma} \mathrm{d}\pi(\gamma) = -\int F(\gamma) \Delta^{\Upsilon} F(\gamma) \mathrm{d}\pi(\gamma)$$

associated semigroup: $P_t^{\Upsilon} = \mathrm{e}^{t\Delta^{\Upsilon}}$

$\mathcal{E}$ gives rise to a canonical diffusion process $(\mathbf{B}_t^{\gamma})_{t \geq 0}$ starting in a.e. $\gamma \in \Upsilon$.

Lemma (Identification with the independent particle system):

If $Ric_M \geq K$, there ex. $\Theta \subset \Upsilon$ with $\pi_{\mu}(\Theta) = 1$ s.t. $(\mathbf{B}_t)$ is realized on Θ as $\mathbf{B}_t^{\gamma} = \sum_i \delta_{B_t^{x_i}}$, where $\gamma = \sum_i \delta_{x_i}$ and $B_t^{x_i}$ are independent BMs starting in x_i.

Explicit representation of the semigroup:

$$\widetilde{P}_t^{\Upsilon} F(\gamma) = E\big[F(\sum_i \delta_{B_t^{x_i}})\big] = \int F(\sum_i \delta_{y_i}) \Pi_i p_t(x_i, \mathrm{d}y_i), \ \gamma = \sum_i \delta_{x_i}$$

Goal

Question:

Can we say more about the geometry of the configuration space Υ? In particular, what are its curvature properties?

Sectional curvature bounds

A lower bound on the sectional curvature, $\sec(M) \geq K$, is equivalent to:

Sectional curvature bounds

A lower bound on the sectional curvature, $\sec(M) \geq K$, is equivalent to:

- Alexandrov triangle comparison: triangles in M are "fatter" than triangles in the model space of constant sectional curvature K, $S^2(\sqrt{K}), \mathbb{R}^2, H^2(\sqrt{-K})$

Sectional curvature bounds

A lower bound on the sectional curvature, $\sec(M) \geq K$, is equivalent to:

- Alexandrov triangle comparison: triangles in M are "fatter" than triangles in the model space of constant sectional curvature K, $S^2(\sqrt{K}), \mathbb{R}^2, H^2(\sqrt{-K})$
- Quadruple comparison: For all $x_0, x_1, x_2, x_3 \in M$ we have:

$$\sum_{i=1}^{3} d^2(x_0, x_i) \geq \frac{1}{6} \sum_{i,j=1}^{3} d^2(x_i, x_j) \, , \qquad (K = 0) \, ,$$

$$\sum_{i=1}^{3} \cos\left(\sqrt{K} d(x_0, x_i)\right) \leq \sum_{i,j=1}^{3} \cos\left(\sqrt{K} d(x_i, x_j)\right) \, , \qquad (K > 0) \, ,$$

$$\sum_{i=1}^{3} \cosh\left(\sqrt{-K} d(x_0, x_i)\right) \geq \sum_{i,j=1}^{3} \cosh\left(\sqrt{-K} d(x_i, x_j)\right) \, , \quad (K < 0) \, .$$

Ricci curvature bounds

A lower bound on the Ricci curvature, $\mathrm{Ric}_M \geq K$, is equivalent to:

Ricci curvature bounds

A lower bound on the Ricci curvature, $\mathrm{Ric}_M \geq K$, is equivalent to:

- Bochner's inequality: for all $u : M \to \mathbb{R}$ smooth:

$$\frac{1}{2}\Delta|\nabla u|^2 - \langle \nabla u, \Delta \nabla u \rangle = \mathrm{Ric}[\nabla u] + \|\operatorname{Hess} u\|_{HS}^2 \geq K|\nabla u|^2$$

Ricci curvature bounds

A lower bound on the Ricci curvature, $\mathrm{Ric}_M \geq K$, is equivalent to:

- Bochner's inequality: for all $u : M \to \mathbb{R}$ smooth:

$$\frac{1}{2}\Delta|\nabla u|^2 - \langle \nabla u, \Delta \nabla u \rangle = \mathrm{Ric}[\nabla u] + \|\,\mathrm{Hess}\, u\|_{HS}^2 \geq K|\nabla u|^2$$

- Gradient estimates for the heat semigroup $P_t = \mathrm{e}^{t\Delta}$:

$$|\nabla P_t u|^2 \leq \mathrm{e}^{-2Kt}\, P_t|\nabla u|^2$$

Ricci curvature bounds

A lower bound on the Ricci curvature, $\mathrm{Ric}_M \geq K$, is equivalent to:

- Bochner's inequality: for all $u : M \to \mathbb{R}$ smooth:

$$\frac{1}{2}\Delta|\nabla u|^2 - \langle \nabla u, \Delta \nabla u \rangle = \mathrm{Ric}[\nabla u] + \|\operatorname{Hess} u\|_{HS}^2 \geq K|\nabla u|^2$$

- Gradient estimates for the heat semigroup $P_t = \mathrm{e}^{t\Delta}$:

$$|\nabla P_t u|^2 \leq \mathrm{e}^{-2Kt} P_t |\nabla u|^2$$

(proof: look at $\varphi(s) = \mathrm{e}^{-2Ks} P_s |\nabla P_{t-s} u|^2$ and derivate in s)

Ricci curvature bounds and OT

Cordero–McCann–Schmuckenschläger '01, vRenesse–Sturm '05

$\mathrm{Ric}_M \geq K \;\Leftrightarrow\;$ Entropy K-convex along L^2-transport geodesics

Ricci curvature bounds and OT

Cordero–McCann–Schmuckenschläger '01, vRenesse–Sturm '05

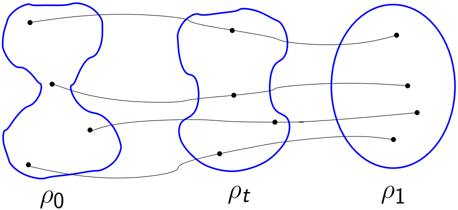
$\mathrm{Ric}_M \geq K \;\Leftrightarrow\;$ Entropy K-convex along L^2-transport geodesics

- Relative entropy:

$$\mathrm{Ent}(\rho|m) = \begin{cases} \int u \log u \,\mathrm{d}m \ , & \rho = um << \mathrm{vol} \ , \\ +\infty \ , & \text{else} \ . \end{cases}$$

- L^2-transport distance between probability measures ρ_0, ρ_1:

$$W_2^2(\rho_0, \rho_1) = \inf \left\{ \int_{M \times M} d(x,y)^2 \mathrm{d}\gamma(x,y) \ : \ \gamma \text{ coupling } \rho_0, \rho_1 \right\}$$

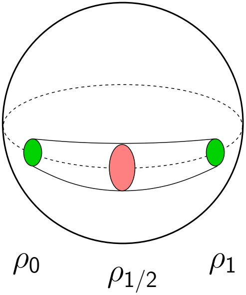


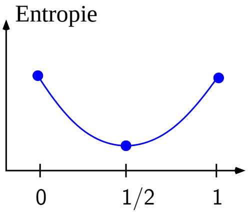
Ricci curvature bounds and OT

Cordero–McCann–Schmuckenschläger '01, vRenesse–Sturm '05

$\mathrm{Ric}_M \geq K \;\Leftrightarrow\;$ Entropy K-convex along L^2-transport geodesics

Heuristics: Transport on the sphere





Ricci curvature bounds and OT II

A lower bound on the Ricci curvature, $\mathrm{Ric}(M) \geq K$ is also equivalent to:

Ricci curvature bounds and OT II

A lower bound on the Ricci curvature, $\mathrm{Ric}(M) \geq K$ is also equivalent to:

- Wasserstein expansion bounds: for all $\rho_1, \rho_2 \in \mathcal{P}_2(M)$ and $t > 0$

$$W_2(P_t\rho_1, P_t\rho_2) \leq \mathrm{e}^{-Kt}\, W_2(\rho_1, \rho_2)$$

Ricci curvature bounds and OT II

A lower bound on the Ricci curvature, $\mathrm{Ric}(M) \geq K$ is also equivalent to:

- Wasserstein expansion bounds: for all $\rho_1, \rho_2 \in \mathcal{P}_2(M)$ and $t > 0$

$$W_2(P_t\rho_1, P_t\rho_2) \leq \mathrm{e}^{-Kt}\, W_2(\rho_1, \rho_2)$$

- Pathwise expansion bounds: for all $x, y \in M$ there ex. a coupling (B_t^x, B_t^y) of Brownian motions starting in x, y s.t. almost surely

$$d(B_t^x, B_t^y) \leq \mathrm{e}^{-Kt}\, d(x, y)\ \forall t > 0$$

Ricci curvature bounds and OT II

A lower bound on the Ricci curvature, $\mathrm{Ric}(M) \geq K$ is also equivalent to:

- Wasserstein expansion bounds: for all $\rho_1, \rho_2 \in \mathcal{P}_2(M)$ and $t > 0$

$$W_2(P_t\rho_1, P_t\rho_2) \leq \mathrm{e}^{-Kt}\, W_2(\rho_1, \rho_2)$$

- Pathwise expansion bounds: for all $x, y \in M$ there ex. a coupling (B_t^x, B_t^y) of Brownian motions starting in x, y s.t. almost surely

$$d(B_t^x, B_t^y) \leq \mathrm{e}^{-Kt}\, d(x, y)\ \forall t > 0$$

- Evolution Variational Inequality: for all $\rho, \eta \in \mathcal{P}_2(M)$ and $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(P_t\rho, \eta) - \frac{K}{2}W_2^2(P_t\rho, \eta) \leq \mathrm{Ent}(\eta|m) - \mathrm{Ent}(P_t\rho|m)$$

EVI encodes simultanuously convexity of the entropy, contraction and that P_t is the gradient flow of Ent w.r.t. W_2

Ricci bounds for metric measure spaces

Let (X, d, m) be a geodesic mms. Synthetic notion of (Riemannian) Ricci curvature bound:

Ricci bounds for metric measure spaces

Let (X, d, m) be a geodesic mms. Synthetic notion of (Riemannian) Ricci curvature bound:

- $\mathrm{CD}(K, \infty)$ (Sturm '06, Lott–Villani '09):
  $\forall \rho_0, \rho_1 \in \mathcal{P}_2(X)$ ex. W_2-geodesic $(\rho_t)_{t \in [0,1]}$ s.t.:

  $$\mathrm{Ent}(\rho_t|m) \leq (1-t)\,\mathrm{Ent}(\rho_0|m) + t\,\mathrm{Ent}(\rho_1|m) - \frac{K}{2} t(1-t) W_2^2(\rho_0, \rho_1)\ .$$

Ricci bounds for metric measure spaces

Let (X, d, m) be a geodesic mms. Synthetic notion of (Riemannian) Ricci curvature bound:

- $\mathrm{CD}(K, \infty)$ (Sturm '06, Lott–Villani '09):
  $\forall \rho_0, \rho_1 \in \mathcal{P}_2(X)$ ex. W_2-geodesic $(\rho_t)_{t\in[0,1]}$ s.t.:

$$\mathrm{Ent}(\rho_t|m) \leq (1-t)\,\mathrm{Ent}(\rho_0|m) + t\,\mathrm{Ent}(\rho_1|m) - \frac{K}{2}t(1-t)W_2^2(\rho_0, \rho_1)\ .$$

- $\mathrm{RCD}(K, \infty)$ (Ambrosio–Gigli–Savaré '11):
  $\forall \rho_0 \in \mathcal{P}_2(X)$ ex. solution $(\rho_t)_{t\geq 0}$ to the EVI, i.e. $\forall \eta \in \mathcal{P}_2(X), t > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\rho_t, \eta) - \frac{K}{2}W_2^2(\rho_t, \eta) \leq \mathrm{Ent}(\eta|m) - \mathrm{Ent}(\rho_t|m)\ .$$

Ricci bounds for metric measure spaces

Let (X, d, m) be a geodesic mms. Synthetic notion of (Riemannian) Ricci curvature bound:

- $\mathrm{CD}(K, \infty)$ (Sturm '06, Lott–Villani '09):
  $\forall \rho_0, \rho_1 \in \mathcal{P}_2(X)$ ex. W_2-geodesic $(\rho_t)_{t\in[0,1]}$ s.t.:

  $$\mathrm{Ent}(\rho_t|m) \leq (1-t)\,\mathrm{Ent}(\rho_0|m) + t\,\mathrm{Ent}(\rho_1|m) - \frac{K}{2}t(1-t)W_2^2(\rho_0, \rho_1)\ .$$

- $\mathrm{RCD}(K, \infty)$ (Ambrosio–Gigli–Savaré '11):
  $\forall \rho_0 \in \mathcal{P}_2(X)$ ex. solution $(\rho_t)_{t\geq 0}$ to the EVI, i.e. $\forall \eta \in \mathcal{P}_2(X), t > 0$:

  $$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\rho_t, \eta) - \frac{K}{2}W_2^2(\rho_t, \eta) \leq \mathrm{Ent}(\eta|m) - \mathrm{Ent}(\rho_t|m)\ .$$

$\mathrm{RCD}(K, \infty)$ is equivalent to a suitable form of Bochner inequality and to gradient estimates

Ricci bounds for metric measure spaces

Let (X, d, m) be a geodesic mms. Synthetic notion of (Riemannian) Ricci curvature bound:

- $\mathrm{CD}(K, \infty)$ (Sturm '06, Lott–Villani '09):
  $\forall \rho_0, \rho_1 \in \mathcal{P}_2(X)$ ex. W_2-geodesic $(\rho_t)_{t\in[0,1]}$ s.t.:

$$\mathrm{Ent}(\rho_t|m) \leq (1-t)\,\mathrm{Ent}(\rho_0|m) + t\,\mathrm{Ent}(\rho_1|m) - \frac{K}{2} t(1-t) W_2^2(\rho_0, \rho_1)\,.$$

- $\mathrm{RCD}(K, \infty)$ (Ambrosio–Gigli–Savaré '11):
  $\forall \rho_0 \in \mathcal{P}_2(X)$ ex. solution $(\rho_t)_{t\geq 0}$ to the EVI, i.e. $\forall \eta \in \mathcal{P}_2(X), t > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2} W_2^2(\rho_t, \eta) - \frac{K}{2} W_2^2(\rho_t, \eta) \leq \mathrm{Ent}(\eta|m) - \mathrm{Ent}(\rho_t|m)\,.$$

$\mathrm{RCD}(K, \infty)$ is equivalent to a suitable form of Bochner inequality and to gradient estimates

Problem: The configuration space $(\Upsilon, d_\Upsilon, \pi)$ is only an extended mms

Sectional curvature bounds for Υ

Recall the distance on Υ:

$$d^2_{\Upsilon}(\gamma, \sigma) = \inf \left\{ \sum_i d^2(x_i, y_i) : \gamma = \sum_i \delta_{x_i}, \sigma = \sum_i \delta_{y_i} \right\}$$

Sectional curvature bounds for Υ

Recall the distance on Υ:

$$d_\Upsilon^2(\gamma,\sigma) = \inf\left\{\sum_i d^2(x_i,y_i) : \gamma = \sum_i \delta_{x_i}, \sigma = \sum_i \delta_{y_i}\right\}$$

Theorem (Alexandrov bounds):

If $\sec(M) \geq K$, then we have $\sec(\Upsilon) \geq K \wedge 0$ in the Alexandrov sense, i.e. for all $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ with $d_\Upsilon(\gamma_0,\gamma_i) < \infty$:

$$\sum_{i=1}^3 d_\Upsilon^2(\gamma_0,\gamma_i) \geq \frac{1}{6}\sum_{i,j=1}^3 d_\Upsilon^2(\gamma_i,\gamma_j)\,.$$

(if $K = 0$, analogous quadruple comparison for $K \neq 0$)

Sectional curvature bounds for Υ

Recall the distance on Υ:

$$d_\Upsilon^2(\gamma,\sigma) = \inf\left\{\sum_i d^2(x_i,y_i) : \gamma = \sum_i \delta_{x_i}, \sigma = \sum_i \delta_{y_i}\right\}$$

Theorem (Alexandrov bounds):

If $\sec(M) \geq K$, then we have $\sec(\Upsilon) \geq K \wedge 0$ in the Alexandrov sense, i.e. for all $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ with $d_\Upsilon(\gamma_0,\gamma_i) < \infty$:

$$\sum_{i=1}^{3} d_\Upsilon^2(\gamma_0,\gamma_i) \geq \frac{1}{6}\sum_{i,j=1}^{3} d_\Upsilon^2(\gamma_i,\gamma_j)\,.$$

(if $K=0$, analogous quadruple comparison for $K \neq 0$)

Note:

If $K<0$ the Wasserstein space $(\mathcal{P}_2(M), W_2)$ does NOT have a Alexandrov curvature bound!

Bochner and Gradient estimates on Υ

Recall gradient of cylinder function $F(\gamma) = g\big(\gamma(\varphi_1), \cdots, \gamma(\varphi_n)\big)$:
$\nabla^{\Upsilon} F(\gamma) = \sum_{i=1}^{n} \partial_i g\big(\gamma(\varphi_1), \cdots, \gamma(\varphi_n)\big) \nabla \varphi_i$

Bochner and Gradient estimates on Υ

Recall gradient of cylinder function $F(\gamma) = g\big(\gamma(\varphi_1), \cdots, \gamma(\varphi_n)\big)$:
$\nabla^\Upsilon F(\gamma) = \sum_{i=1}^n \partial_i g\big(\gamma(\varphi_1), \cdots, \gamma(\varphi_n)\big) \nabla \varphi_i$

Proposition (Bochner inequ.):

If $\mathrm{Ric}(M) \geq K$, then Bochner's inequality holds on Υ, i.e. for all $F \in \mathrm{Cyl}^\infty(\Upsilon)$ and $\gamma \in \Upsilon$:

$$\frac{1}{2}\Delta^\Upsilon |\nabla^\Upsilon F|^2(\gamma) - \langle \nabla^\Upsilon F, \Delta^\Upsilon \nabla^\Upsilon F\rangle_\gamma \geq K|\nabla^\Upsilon F|_\gamma^2 \,.$$

Bochner and Gradient estimates on Υ

Recall gradient of cylinder function $F(\gamma) = g\big(\gamma(\varphi_1), \cdots, \gamma(\varphi_n)\big)$:
$\nabla^\Upsilon F(\gamma) = \sum_{i=1}^n \partial_i g\big(\gamma(\varphi_1), \cdots, \gamma(\varphi_n)\big) \nabla\varphi_i$

Proposition (Bochner inequ.):

If $\mathrm{Ric}(M) \geq K$, then Bochner's inequality holds on Υ, i.e. for all $F \in \mathrm{Cyl}^\infty(\Upsilon)$ and $\gamma \in \Upsilon$:

$$\frac{1}{2}\Delta^\Upsilon |\nabla^\Upsilon F|^2(\gamma) - \langle \nabla^\Upsilon F, \Delta^\Upsilon \nabla^\Upsilon F\rangle_\gamma \geq K|\nabla^\Upsilon F|^2_\gamma \,.$$

Theorem (gradient estimate):

If $\mathrm{Ric}(M) \geq K$, then for all $G \in \mathcal{D}(\mathcal{E})$ and π-a.e. γ:

$$|\nabla^\Upsilon P_t^\Upsilon G|^2_\gamma \leq \mathrm{e}^{-2Kt}\, P_t^\Upsilon |\nabla^\Upsilon G|^2(\gamma) \,.$$

Bochner and Gradient estimates on Υ

Recall gradient of cylinder function $F(\gamma) = g\big(\gamma(\varphi_1), \cdots, \gamma(\varphi_n)\big)$:
$\nabla^\Upsilon F(\gamma) = \sum_{i=1}^n \partial_i g\big(\gamma(\varphi_1), \cdots, \gamma(\varphi_n)\big)\nabla\varphi_i$

Proposition (Bochner inequ.):

If $\mathrm{Ric}(M) \geq K$, then Bochner's inequality holds on Υ, i.e. for all $F \in \mathrm{Cyl}^\infty(\Upsilon)$ and $\gamma \in \Upsilon$:

$$\frac{1}{2}\Delta^\Upsilon|\nabla^\Upsilon F|^2(\gamma) - \langle\nabla^\Upsilon F, \Delta^\Upsilon\nabla^\Upsilon F\rangle_\gamma \geq K|\nabla^\Upsilon F|_\gamma^2 \,.$$

Theorem (gradient estimate):

If $\mathrm{Ric}(M) \geq K$, then for all $G \in \mathcal{D}(\mathcal{E})$ and π-a.e. γ:

$$|\nabla^\Upsilon P_t^\Upsilon G|_\gamma^2 \leq \mathrm{e}^{-2Kt}\, P_t^\Upsilon|\nabla^\Upsilon G|^2(\gamma) \,.$$

(proof uses representation of P_t^Υ as infinite product of 1 part. semigroups)

Expansion bounds

Wasserstein (pseudo-)distance: for $\rho_1, \rho_2 \in \mathcal{P}(\Upsilon)$ define

$$W_2^2(\rho_1, \rho_2) = \inf \left\{ \int d_\Upsilon^2(\gamma_1, \gamma_2) \mathrm{d}q(\gamma_1, \gamma_2) : q \text{ coupling of } \rho_1, \rho_2 \right\} .$$

We denote the fiber of π by $\mathcal{P}_\pi(\Upsilon) = \{\rho \in \mathcal{P}(\Upsilon) : W_2(\rho, \pi) < \infty\}$.
For $\rho = F\pi_\mu$ define $P_t^\Upsilon \rho = (P_t^\Upsilon F)\pi$.

Expansion bounds

Wasserstein (pseudo-)distance: for $\rho_1, \rho_2 \in \mathcal{P}(\Upsilon)$ define

$$W_2^2(\rho_1, \rho_2) = \inf \left\{ \int d_\Upsilon^2(\gamma_1, \gamma_2) \mathrm{d}q(\gamma_1, \gamma_2) : q \text{ coupling of } \rho_1, \rho_2 \right\} .$$

We denote the fiber of π by $\mathcal{P}_\pi(\Upsilon) = \{\rho \in \mathcal{P}(\Upsilon) : W_2(\rho, \pi) < \infty\}$.
For $\rho = F\pi_\mu$ define $P_t^\Upsilon \rho = (P_t^\Upsilon F)\pi$.

Theorem (expansion):

If $\mathrm{Ric}(M) \geq K$, then for all $\rho_1, \rho_2 \in \mathcal{P}(\Upsilon)$ with $\rho_i << \pi_\mu$:

$$W_2(P_t^\Upsilon \rho_1, P_t^\Upsilon \rho_2) \leq \mathrm{e}^{-Kt} W_2(\rho_1, \rho_2) .$$

Moreover, for all $\gamma_1, \gamma_2 \in \Theta \subset \Upsilon$ ex. a coupling of the infinite independent particle processes s.t. a.s.:

$$d_\Upsilon(\mathbf{B}_t^{\gamma_1}, \mathbf{B}_t^{\gamma_2}) \leq \mathrm{e}^{-Kt} d_\Upsilon(\gamma_1, \gamma_2) \ \forall t > 0 .$$

Expansion bounds

Wasserstein (pseudo-)distance: for $\rho_1, \rho_2 \in \mathcal{P}(\Upsilon)$ define

$$W_2^2(\rho_1, \rho_2) = \inf \left\{ \int d_\Upsilon^2(\gamma_1, \gamma_2) \mathrm{d}q(\gamma_1, \gamma_2) : q \text{ coupling of } \rho_1, \rho_2 \right\} .$$

We denote the fiber of π by $\mathcal{P}_\pi(\Upsilon) = \{\rho \in \mathcal{P}(\Upsilon) : W_2(\rho, \pi) < \infty\}$.
For $\rho = F\pi_\mu$ define $P_t^\Upsilon \rho = (P_t^\Upsilon F)\pi$.

Theorem (expansion):

If $\mathrm{Ric}(M) \geq K$, then for all $\rho_1, \rho_2 \in \mathcal{P}(\Upsilon)$ with $\rho_i << \pi_\mu$:

$$W_2(P_t^\Upsilon \rho_1, P_t^\Upsilon \rho_2) \leq \mathrm{e}^{-Kt} W_2(\rho_1, \rho_2) .$$

Moreover, for all $\gamma_1, \gamma_2 \in \Theta \subset \Upsilon$ ex. a coupling of the infinite independent particle processes s.t. a.s.:

$$d_\Upsilon(\mathbf{B}_t^{\gamma_1}, \mathbf{B}_t^{\gamma_2}) \leq \mathrm{e}^{-Kt} d_\Upsilon(\gamma_1, \gamma_2) \ \forall t > 0 .$$

Under $\mathrm{Ric}(M) \geq K$, $\mathcal{P}_\pi(\Upsilon)$ belongs to closure of $\{\rho << \pi\}$ w.r.t W_2

Synthetic Ricci bounds for Υ

Theorem (EVI on Υ):

If $\mathrm{Ric}(M) \geq K$, then for all $\rho, \eta \in \mathcal{P}_\pi(\Upsilon)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(P_t^\Upsilon \rho, \eta) - \frac{K}{2}W_2^2(P_t^\Upsilon \rho, \eta) \leq \mathrm{Ent}(\eta|\pi) - \mathrm{Ent}(P_t\rho|\pi)\ .$$

Synthetic Ricci bounds for Υ

Theorem (EVI on Υ):

If $\mathrm{Ric}(M) \geq K$, then for all $\rho, \eta \in \mathcal{P}_\pi(\Upsilon)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(P_t^\Upsilon \rho, \eta) - \frac{K}{2}W_2^2(P_t^\Upsilon \rho, \eta) \leq \mathrm{Ent}(\eta|\pi) - \mathrm{Ent}(P_t\rho|\pi) \ .$$

Corollary (geodesic convexity of entropy):

If $\mathrm{Ric}(M) \geq K$, then for any W_2-geodesic $(\rho_t)_{t\in[0,1]}$ in $\mathcal{P}_\pi(\Upsilon)$:

$$\mathrm{Ent}(\rho_t|\pi) \leq (1-t)\,\mathrm{Ent}(\rho_0|\pi) + t\,\mathrm{Ent}(\rho_1|\pi) - \frac{K}{2}t(1-t)W_2^2(\rho_0, \rho_1) \ .$$

Synthetic Ricci bounds for Υ

Theorem (EVI on Υ):

If $\mathrm{Ric}(M) \geq K$, then for all $\rho, \eta \in \mathcal{P}_\pi(\Upsilon)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(P_t^\Upsilon \rho, \eta) - \frac{K}{2}W_2^2(P_t^\Upsilon \rho, \eta) \leq \mathrm{Ent}(\eta|\pi) - \mathrm{Ent}(P_t\rho|\pi) \ .$$

Corollary (geodesic convexity of entropy):

If $\mathrm{Ric}(M) \geq K$, then for any W_2-geodesic $(\rho_t)_{t\in[0,1]}$ in $\mathcal{P}_\pi(\Upsilon)$:

$$\mathrm{Ent}(\rho_t|\pi) \leq (1-t)\,\mathrm{Ent}(\rho_0|\pi) + t\,\mathrm{Ent}(\rho_1|\pi) - \frac{K}{2}t(1-t)W_2^2(\rho_0, \rho_1) \ .$$

- In this sense $(\Upsilon, d_\Upsilon, \pi)$ is an extended (R)CD(K, ∞) mms
- Dirichlet form $\mathcal{E}$ coincides with Cheeger energy Ch_{d_Υ}
- proof starts from gradient estimate

Gradient flow structure of the particle system

Observation

Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ smooth and convex. For $u : \mathbb{R}_+ \to \mathbb{R}^d$ TFAE:

1. u solves the gradient flow equation $u'(t) = -\nabla\varphi(u(t))$,
2. u solves the evolution variational inequality

$$\frac{1}{2}\frac{d}{dt}|u(t) - y|^2 \leq \varphi(y) - \varphi(u(t)) \qquad \forall y \in \mathbb{R}^d \; .$$

Gradient flow structure of the particle system

Observation

Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ smooth and convex. For $u : \mathbb{R}_+ \to \mathbb{R}^d$ TFAE:

1. u solves the gradient flow equation $u'(t) = -\nabla\varphi(u(t))$,
2. u solves the evolution variational inequality

$$\frac{1}{2}\frac{d}{dt}|u(t) - y|^2 \leq \varphi(y) - \varphi(u(t)) \qquad \forall y \in \mathbb{R}^d \; .$$

Corollary

The semigroup of the infinite independent particle sytem P_t^Υ is the W_2-gradient flow of the entropy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(P_t^\Upsilon\rho, \eta) - \frac{K}{2}W_2^2(P_t^\Upsilon\rho, \eta) \leq \mathrm{Ent}(\eta|\pi) - \mathrm{Ent}(P_t\rho|\pi)$$

Future goals

Study the interacting infinite particle systems in $\mathbb{R}^n$ with a pair interaction potential $\Phi : \mathbb{R}^n \to \mathbb{R}$

$$\mathrm{d}X_t^i = \mathrm{d}B_t^i + \lim_{R\to\infty} \sum_{|X_t^i - X_t^j| < R} \nabla\Phi(X_t^i - X_t^j)\mathrm{d}t\,, \quad i = 1, 2, \ldots$$

Future goals

Study the interacting infinite particle systems in $\mathbb{R}^n$ with a pair interaction potential $\Phi : \mathbb{R}^n \to \mathbb{R}$

$$\mathrm{d}X_t^i = \mathrm{d}B_t^i + \lim_{R\to\infty} \sum_{|X_t^i - X_t^j| < R} \nabla\Phi(X_t^i - X_t^j)\mathrm{d}t\ , \quad i = 1, 2, \dots$$

Associated to Dirichlet form

$$\mathcal{E}(F) = \int |\nabla^\Upsilon F|_\gamma^2 \mathrm{d}\pi_\Phi(\gamma)$$

for a Gibbs measure π_Φ

Future goals

Study the interacting infinite particle systems in $\mathbb{R}^n$ with a pair interaction potential $\Phi : \mathbb{R}^n \to \mathbb{R}$

$$\mathrm{d}X_t^i = \mathrm{d}B_t^i + \lim_{R\to\infty} \sum_{|X_t^i - X_t^j| < R} \nabla\Phi(X_t^i - X_t^j)\mathrm{d}t\ , \quad i = 1, 2, \ldots$$

Associated to Dirichlet form

$$\mathcal{E}(F) = \int |\nabla^\Upsilon F|_\gamma^2 \mathrm{d}\pi_\Phi(\gamma)$$

for a Gibbs measure π_Φ

Question

Can we extract curvature bounds on $(\Upsilon, d_\Upsilon, \pi_\Phi)$ from properties of Φ?

Thank you for your attention!