A quantitative approach for hydrodynamic limits.

Max Fathi

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Hydrodynamic limits

2 possible descriptions of a gas :

- Atomistic description : N particles in interaction (Newton's laws, etc...)
- Continuum description : (system of) PDE(s) (Navier-Stokes, Euler,...)

Hydrodynamic limit : deriving a continuous model as a scaling limit of discrete models.

In this talk, we shall consider the more simple situation of scaling limits for stochastic models.

We are interested in diffusions of the form

$$dx_t = -\psi'(x_t)dt + \sqrt{2}dB_t.$$

We consider a large number N of such diffusions, and add a nearest-neighbor interaction, such that

$$dx_i(t) = N^2(\psi'(x_{i+1}) + \psi'(x_{i-1}) - 2\psi'(x_i))dt + \sqrt{2}N(dB_t^{i+1} - dB_t^i)$$

where the B^i are independant Brownian motions. The additional factor N corresponds to a scaling in time. The quantity

$$\frac{1}{N}\sum x_i$$

is conserved, and the fluctuations around this mean tend to diminish in time.

Some assumptions

We can write the dynamic as

$$dX_t = -A\nabla H(X_t)dt + \sqrt{2A}dB_t.$$

The phase state is

$$X_{N,m} := \left\{ x \in \mathbb{R}^N \, ; \, \sum x_i = Nm \right\},$$

which we equip with the usual ℓ^2 norm. The Hamiltonian H is given by

$$H(x) := \sum_{i=1}^{N} \psi(x_i),$$

with

$$\psi(x) = |x|^p + \delta(x), \;\; p \geq 2, \;\; ||\delta||_{C^2} < \infty.$$

Finally, A is the discrete Laplacian

$$A_{i,j} := N^2(2\delta_{i,j} - \delta_{i,j-1} - \delta_{i,j+1}),$$

where N^2 is a scaling in time of the evolution.

This dynamic is a reversible stochastic dynamic on an affine Euclidean subspace X of $\mathbb{R}^N$, with an (unique) invariant probability measure

$$\mu(dx) := \exp(-H(x))dx$$

The law of this evolution is given by the PDE

$$\frac{\partial}{\partial t}(f\mu) = \nabla \cdot (A\nabla f\mu)$$

where f is the density with respect to μ.
We are interested in the behavior of the probability measure $f\mu$ when the dimension N is large.

Notion of convergence

It will be convenient to see our system as a "‘random function"’. We associate to each vector $x \in X_N$ a step function $\bar{x}$ on the torus, defined by

$$\bar{x}(\theta) := x_i \; ; \; \theta \in \left[\frac{i-1}{N}, \frac{i}{N}\right).$$

We will say a sequence ν_N converges to a macroscopic profile $\rho \in L^2(\mathbb{T})$ if the quantities

$$\frac{1}{N}\sum J(i/N)x_i \approx \int_{\mathbb{T}} J(\theta)\bar{x}(\theta)d\theta$$

converge in probability to

$$\int_{\mathbb{T}} J(\theta)\rho(\theta)d\theta$$

for every smooth test functions J.

It has been proven by Guo, Papanicolaou and Varadhan that, if the initial data converges to a macroscopic profile, then the dynamics behave deterministically at later times, and the behavior is given by the solution of the PDE

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2}{\partial \theta^2} \varphi'(\rho)$$

where φ is given by

$$\varphi(m) := \sup_{\sigma} \left(\sigma m - \log \int \exp(\sigma x - \psi(x)) dx \right).$$

Useful property : φ is uniformly convex

A gradient flow interpretation

This dynamic is a gradient flow of the entropy with respect to the entropy Ent_μ, for the Wasserstein distance associated to the Euclidean distance $\langle A^{-1}\cdot, \cdot\rangle$;
The PDE is the gradient flow of $\rho \longrightarrow \int \varphi(\rho) d\theta$ on the space of functions on the torus, for the H^{-1} norm.

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We can use gradient flow stability arguments to go from one to the other.

The functional point of view allows to embed all the spaces X_N into a single space, which is the space H^{-1}, dual of the Sobolev space H^1. Then we have

$$\frac{1}{N}\langle A^{-1}x, x\rangle \approx ||\bar{x}||^2_{H^{-1}}.$$

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Moreover,

$$\frac{1}{N}\,\mathrm{Ent}_\mu(\cdot) \xrightarrow{\Gamma} \int \varphi(\cdot)d\theta - \varphi\left(\int \cdot d\theta\right).$$

These statements allow us to show that, for well-prepared initial data that weakly converge to a deterministic profile ρ, and such that

$$\frac{1}{N}\,\text{Ent}_\mu(f_{0,N}\mu_N) \longrightarrow \int \varphi(\rho_0)d\theta - \varphi\left(\int \rho_0 d\theta\right),$$

then, at positive times, the solutions $f_N(t)\mu_N$ weakly converge to the solution of the PDE.

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We wish for two improvements :

- Weaken the assumption on the initial data ;
- Obtain explicit quantitative rates of convergence.

Theorem (Grunewald, Otto, Villani, Westdickenberg, 2009, F.-Menz 2013)

If $\int_{X_N} f_0 \log f_0 d\mu_N \leq CN$ *and*

$$\int_X ||\bar{x} - \rho_0||^2_{H^{-1}} f_{0,N}(x)\mu_N(dx) \longrightarrow 0,$$

then there is a nice step-function approximation $\eta(t)$ *of the solution of the PDE, with mesh size* $N^{-1/2}$*, such that for any* $T > 0$*,*

$$\sup_{t\in[0,T]} \frac{1}{N} W_{2,A^{-1}}(f(t)\mu, \delta_{\eta(t)})^2 \longrightarrow 0$$

with quantitative bounds of order $N^{-1/2}$*.*

Method of proof

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For N large enough, $\bar{H}$ is uniformly convex
$\longrightarrow$ functional inequalities in positive curvature (Log-Sobolev, Talagrand,...)

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When computing $\frac{d}{dt}W_{2,A^{-1}}(f(t)\mu, \delta_{\eta(t)})^2$, we can separate the expression we obtain into a macroscopic component and a fluctuations component. The macroscopic component can be handled using functional inequalities, and the fluctuation component doesn't contribute too much to $||\bar{x} - \eta||^2_{H^{-1}}$.

Local Gibbs behavior

Theorem (F., 2013)

Fr any time $t > 0$, we have

$$\frac{1}{N}\,\mathrm{Ent}_{\mu_N}(f_N(t)\mu_N) \longrightarrow \int_{\mathbb{T}} \varphi(\rho(t))d\theta - \varphi\left(\int \rho(t)d\theta\right)$$

with quantitative bounds of order $\frac{\log N}{N^{1/4}}$.

This result means that, even though we didn't assume the initial data was well-prepared, we have convergence of the free energy at any positive time.

Sketch of proof

There exists a sequence of measures of the form $\tilde{\mu}(dx) = \frac{1}{Z}\exp(\lambda(t)\cdot Px)\mu(dx)$ which converge to $\rho(t)$, and are well-prepared.

Since the associated macroscopic measures are log-concave, we can use the HWI inequality at macroscopic level :

$$\mathrm{Ent}_{\tilde{\mu}}(\bar{f}\bar{\mu}) \leq W_2(\bar{f}\bar{\mu}, \tilde{\mu})\sqrt{I_{\tilde{\mu}}(\bar{f}\bar{\mu})}.$$

We have $\frac{1}{N}W_2(\bar{f}\bar{\mu}, \tilde{\mu})^2 \leq \frac{C}{\sqrt{N}}$, and bounds of order N on the (time-integrated) relative Fisher information.

Convergence of $\frac{1}{N}\mathrm{Ent}_{\mu_N}(f_N(t)\mu_N)$ can then be deduced.

Non-reversible dynamics

The method can be extended to cover non-reversible dynamics of the form

$$dX_t = -A\nabla H(X_t)dt + J\nabla H(X_t)dt + \sqrt{2A}dB_t$$

where J is an antisymmetric matrix such that $-J^2 \leq cA$.
In the case where J is a discrete derivation, we obtain the scaling limit

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2}{\partial \theta^2}\varphi'(\rho) + \frac{\partial}{\partial \theta}\varphi'(\rho).$$

(Joint work with M.H. Duong)

Two open problems

Two main questions remain :

- How can we extend the method to cover genuinely Riemannian situations ? We are interested in dynamics of the form

$$dX_t = -A(X_t)\nabla H(X_t)dt - \mathrm{div}(A)(X_t)dt + \sqrt{2A(X_t)}dB_t,$$

  which are gradient flows of the entropy for the Riemannian metric with tensor $A^{-1}(x)$. These can lead to hydrodynamic limits of the form

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial \theta}(a(\rho)\frac{\partial}{\partial \theta}\varphi'(\rho))$$

  with a coefficient $a(m)$ that depends on both $A(x)$ and ψ ;
- Is there a similar method that works for discrete dynamics, such as interacting particle systems ?