A transport problem for two measures in the convex order

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Outline

Definition: martingale transport plan

A probability measure π on $\mathbb{R}\times\mathbb{R}$ is termed a martingale transport plan if $\pi = \mathrm{Law}(X, Y)$ where (X, Y) is a two-times martingale process: $\mathbb{E}(Y|X) = X$.

Equivalently if $(\pi_x)_{x\in\mathbb{R}}$ is a disintegration of π (or "conditional laws", or "Markov kernel"), it has to satisfy

$$\mathrm{Barycenter}(\pi_x) = \int y\,\mathrm{d}\pi_x(y) = x$$

for $(\mathrm{proj}^x_{\#}\pi)$-almost every x.

Some examples

- $\pi = \text{Law}(x, Y)$ where $x = \mathbb{E}(Y)$.
- $\pi = \text{Law}(\mathbb{E}(Y|\mathcal{F}), Y)$ for some $\mathcal{F} \subseteq \sigma(Y)$.
- $\pi = \text{Law}(B, B + B')$ where B' is independent from B and $\mathbb{E}(B') = 0$.
- $\pi_{1/2} = \frac{\pi_0 + \pi_1}{2}$ where π_0, π_1 are martingale transport plans.
- $\pi_t = \sum_{i=1}^{2} \sum_{j=1}^{3} a_{i,j}^t \delta_{(x_i, y_j)}$ where $x \in \{-1, 1\}$ and $y \in \{-2, 0, 2\}$ and

$$(a_{i,j}^t) = \begin{pmatrix} 1/4 & 1/4 & 0 \\ 1/12 & 1/12 & 1/3 \end{pmatrix} + t \begin{pmatrix} 1/12 & -1/6 & 1/12 \\ -1/12 & 1/6 & -1/12 \end{pmatrix}$$

for some $t \in [0, 1]$.

The convex order

Theorem of Strassen

For two probability measures μ and ν,

$$\Pi_M(\mu,\nu) \neq \emptyset \Leftrightarrow \mu \preceq_C \nu.$$

The convex order $\mu \preceq_C \nu$ is defined by

$$\forall \varphi \text{ convex}, \int \varphi d\mu \leq \int \varphi d\nu$$

The model theorem

Model theorem in the classical setting

For μ and ν in $\mathcal{P}_2$ in the convex order and π a transport plan from μ to ν. The following statements are equivalent:

1. The plan π is optimal for the transport problem with $c(x,y) = (y-x)^2$,
2. The plan π is concentrated on a monotone set Γ,
3. The plan π is the quantile coupling.

We (Mathias and I) have proved a theorem similar to this one in the martingale setting.

The extended order and the shadows

Proposition - Definition

- We write $\mu \preceq_E \nu$ and say that μ is smaller than ν in the extended order if

$$F_\mu^\nu := \{\theta : \mu \preceq_C \theta \text{ and } \theta \leq \nu\}$$

  is not empty.
- The partially ordered set $(F_\mu^\nu, \preceq_C)$ has a minimum. We call it the shadow of μ in ν and denote it by $S^\nu(\mu)$.

The extended order and the shadows

Proposition

If $\gamma_1 + \gamma_2 \preceq_E \nu$, then

$$S^\nu(\gamma_1 + \gamma_2) = S^\nu(\gamma_1) + S^{\nu - S^\nu(\gamma_1)}(\gamma_2).$$

As a important consequence, if $\gamma \leq \gamma'$ and $\gamma' \preceq_E \nu$,

$$S^\nu(\gamma) \leq S^\nu(\gamma').$$

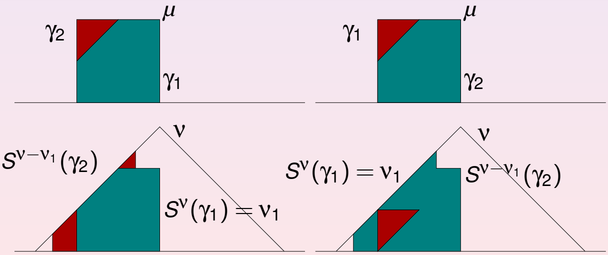

Figure : Shadow of μ in ν and associativity of the shadow projection.

The variational lemma

This lemma is a kind of c-cyclical monotonicity lemma for the martingale setting.

Variational Lemma

Let P be optimal, there exists $\Gamma \subseteq \mathbb{R} \times \mathbb{R}$ such that for any finitely supported measure α with $\alpha(\Gamma) = 1$, the minimum of $\alpha' \in \mathrm{Competitor}(\alpha) \mapsto \int c(x,y)\,\mathrm{d}\alpha'(x,y)$ over

$$\mathrm{Competitor}(\alpha) = \left\{ \alpha' : \begin{array}{l} \alpha' \text{ has the same marginals as } \alpha \\ \text{and } \forall x \in \mathbb{R}, \displaystyle\int y\,\mathrm{d}\alpha_x = \int y\,\mathrm{d}\alpha'_x \end{array} \right\}$$

is obtained in α.

The model theorem

Model theorem in the classical setting

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The general problem

The problem

Minimize

$$f : \pi \in \Pi_M(\mu, \nu) \mapsto \int c(x,y) \mathrm{d}\pi(x,y) \in \mathbb{R}$$

among the martingale transport plans from μ to ν.

For different cost functions *c* we would like to know:

- How do the minimizers look like?
- What are their properties?
- Is there a unique minimizer?

The martingale theorem

Theorem (Beiglboeck–J.)

For μ and ν in $\mathcal{P}_3$ in the convex order and π a martingale transport plan from μ to ν. The following statements are equivalent:

1. The plan π is optimal for the martingale transport problem with cost $c(x,y) = (y-x)^3$,
2. The plan π is concentrated on a martingale-monotone set Γ (*see the figure*),
3. The plan π is the left-curtain coupling (*i.e., transports $\mu_{]-\infty,x]}$ to its shadow*)

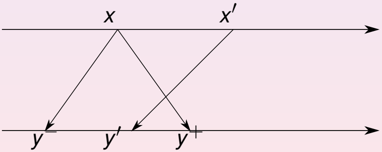

Figure : This configuration of three points (x,y), (x',y^-) and (x',y^+) is forbidden on martingale-monotone sets Γ.

One example

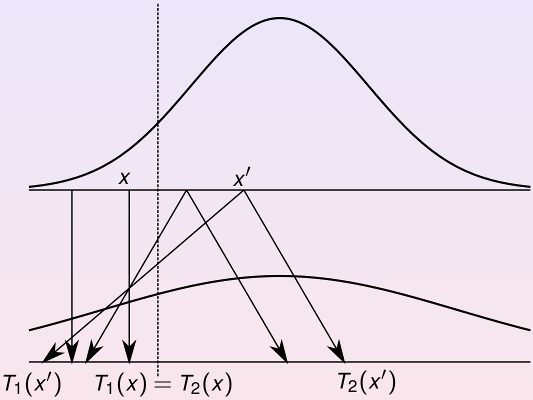

Figure : Optimal transport plan between Gaussian measures.

Corollary

Corollary for μ continuous

If μ is continuous (=no atom), there are $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ such that the optimal π is concentrated on $\mathrm{graph}(T_1) \cup \mathrm{graph}(T_2)$.

The variational lemma is of general use, especially when μ is continuous.

Examples

- $c(x,y) = -|y-x|$
- $c(x,y) = |y-x|$
- $c(x,y) = (y-x)^n$
- $c(x,y) = h(y-x)$ where $h'(x) = ax + b$ has less than n solutions.