Some analytic and geometric properties of infinitesimally Hilbertian metric measure spaces with lower Ricci curvature bounds

Andrea Mondino (ETH-Zurich)

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- Upper/Lower bounds on the sectional curvature are strong assumptions with strong implications E.g. Cartan-Hadamard Theorem (if $K \leq 0$ then the universal cover of M is diffeomorphic to $\mathbb{R}^N$), Topogonov triangle comparison theorem($\rightsquigarrow$ definition of Alexandrov spaces: non smooth spaces with upper/lower bounds on the "sectional curvature"), etc.

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- Upper bounds on the Ricci curvature are very (too) weak assumption for geometric conclusions. E.g. Lokhamp theorem: any compact Riemannian manifold carries a metric with negative Ricci curvature.

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Lower bounds on the Ricci curvature: natural framework for comparison geometry. E.g. Bishop-Gromov volume comparison, Laplacian Comparison, Cheeger-Gromoll splitting, Li-Yau inequalities on heat flow, Anderson-Gallot-Gromov bounds on the topological complexity, etc.

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A fundamental tool in the smooth setting is the Bochner identity: if $f \in C^\infty(M)$ then

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If $dim(M) \leq N$ and $Ric \geq K\, g$ then Dimensional Bochner inequality, also called dimensional Bakry-Emery condition BE(K,N)

$$\frac{1}{2}\Delta|\nabla f|^2 \geq \frac{1}{N}|\Delta f|^2 + K|\nabla f|^2 + g(\nabla \Delta f, \nabla f).$$

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- What would be an intrinsic definition? Like definining the real numbers via Dedekind sections in an axiomatic way:
  What is $\mathbb{R}$? $\mathbb{R}$ is a totally ordered, Dedekind complete field.

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- $(X, \mathrm{d}, \mathfrak{m})$ complete separable metric space with a σ-finite Borel probability measure $\mathfrak{m}$ (more precisely $\mathfrak{m}(B_r(x)) \leq ce^{Ar^2}$); if we fix a point $\bar{x} \in X$, $(X, \mathrm{d}, \mathfrak{m}, \bar{x})$ denotes the corresponding pointed space.

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- $(\mathcal{P}_2(X), W_2)$: metric space of probability measures on X with finite second moment endowed with quadratic transportation distance (Wasserstein)
- Entropy functional $\mathcal{U}_{N,\mathfrak{m}}(\mu)$ if $\mu << \mathfrak{m}$

$$\mathcal{U}_{N,\mathfrak{m}}(\rho\mathfrak{m}) := -N \int \rho^{1-\frac{1}{N}} d\mathfrak{m} \quad \text{if } 1 \leq N < \infty$$

$$\mathcal{U}_{\infty,\mathfrak{m}}(\rho\mathfrak{m}) := \int \rho \log \rho d\mathfrak{m}$$

(if μ is not a.c. then if $N < \infty$ the non a.c. part does not contribute, if $N = +\infty$ then set $\mathcal{U}_{\infty,\mathfrak{m}}(\mu) = \infty$.)

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DEF of $CD(K,N)$ condition [Lott-Sturm-Villani 2006]: fixed $N \in [1,+\infty]$ and $K \in \mathbb{R}$, $(X,\mathsf{d},\mathfrak{m})$ is a $CD(K,N)$-space if the Entropy $\mathcal{U}_{N,\mathfrak{m}}$ is K-convex along geodesics in $(\mathcal{P}_2(X),W_2)$ (for finite N is a "distorted" K-geod. conv.).

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Good properties:

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- GEOMETRIC PROPERTIES: Brunn-Minkoswski inequality, Bishop-Gromov volume growth, Bonnet-Myers diameter bound, Lichnerowictz spectral gap, etc.

Stability of $CD(K,N)$, 1: Lott-Villani Vs Sturm

- Framework of proper spaces (i.e. bounded closed sets are compact), Lott-Villani: $CD(K,N)$ is stable under pointed measured Gromov-Hausdorff convergence (i.e. for every $R > 0$ there is measured Gromov-Hausdorff convergence of balls of radius R around the given points of the space)

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- Framework of probability spaces with finite variance (i.e. $\mathfrak{m}\in\mathcal{P}_2(X)$): Sturm defined a distance

$$\mathbb{D}\left((X_1,\mathsf{d}_1,\mathfrak{m}_1),(X_2,\mathsf{d}_2,\mathfrak{m}_2)\right):=\inf W_2\left((\iota_1)_\sharp\mathfrak{m}_1,(\iota_2)_\sharp\mathfrak{m}_2\right),$$

inf among all metric spaces (Z,d_Z) and all isometric embeddings $\iota_i(\mathrm{supp}(\mathfrak{m}_i),\mathsf{d}_i)\to(Z,\mathsf{d}_Z)$, $i=1,2$. He then proved that $CD(K,N)$ is stable w.r.t. $\mathbb{D}$-convergence.

Stability of $CD(K, N)$, 2: not satisfactory for $N = \infty$

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- In some geometric situations this is not completely satisfactory: when studying blow ups (i.e. tangent cone at a point $\rightsquigarrow$ Cheeger,Colding,Naber) and blow downs (i.e. tangent cones at infinity $\rightsquigarrow$ Cheeger, Colding, Minicozzi, Tian, etc.), assuming $\mathfrak{m}\in\mathcal{P}_2$ is quite unnatural; problems also in dealing with sequences of non compact manifolds with diverging dimensions or more generally with sequences of spaces with diverging doubling constants.

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Q:1) What is a natural notion of convergence in these situations?
2) Is $CD(K,\infty)$ stable w.r.t. this notion?

Pointed measured Gromov (pmG for short) convergence

DEF:(Gigli-M.-Savaré '13) $(X_n, \mathrm{d}_n, \mathfrak{m}_n, \bar{x}_n) \to (X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in pmG-sense if there exist a complete and separable space (Z, d_z) and isometric embeddings $\iota_n : X_n \to Z$, $n \in \bar{N} := \mathbb{N} \cup \{\infty\}$ s.t.

$$\int \varphi \, (\iota_n)_\sharp \mathfrak{m}_n \to \int \varphi \, (\iota_\infty)_\sharp \mathfrak{m}_\infty, \; \forall \varphi \in C_{bs}(Z), \text{ where}$$

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- On normalized spaces of finite variance pmG-convergence is equivalent to $\mathbb{D}$-convergence ($\rightsquigarrow$ consistent with Sturm).
- pmG-convergence no a priori assumption on $(X_n, \mathrm{d}_n, \mathfrak{m}_n)$.

$CD(K,\infty)$ is stable under pmG-convergence

THM(Gigli-M.-Savaré '13): Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \mathbb{N}$, be a sequence of $CD(K,\infty)$ p.m.m. spaces converging to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in the pmG-sense. Then $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ is a $CD(K,\infty)$ space as well.

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Idea of Proof:

1. prove Γ-convegence of the entropies under pmG-convergence

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2. use the compactness of $\mathfrak{m}_n$ to prove compactness of Wasserstein-geodesics in the converging spaces
3. conclude that K-geodesic convexity is preserved.

$\square$

Non completely satisfactiory features of $CD(K, N)$

- Problem 1) the class of $CD(K, N)$ spaces is TOO LARGE: compact Finsler manifolds satisfy $CD(K, N)$ for some $K \in \mathbb{R}$ and $N \geq 1$ [Ohta] (earlier work in this direction by Cordero-Erasquin, Sturm and Villani), but if smooth Finsler manifold M is a mGH-limit of Riemannian manifolds with $Ric \geq K$ then M is Riemannian.

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- Problem 2):LOCAL TO GLOBAL AND TENSORIZATION. It is not clear wether or not the $CD(K,N)$ satisfies the local to global and the tensorization properties

Solving Problem 1: the $RCD(K, \infty)$ condition

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- Definition[Ambrosio-Gigli-Savaré 2011, improved by Ambrosio-Gigli-M.-Rajala 2012]
  $(X, \mathrm{d}, \mathfrak{m})$ is an $RCD(K,\infty)$ space if it an infinitesimally Hilbertian $CD(K,\infty)$ space.

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  $(X, \mathsf{d}, \mathfrak{m})$ is an $RCD(K, \infty)$ space if it an infinitesimally Hilbertian $CD(K, \infty)$ space.
- Question: is $RCD(K, \infty)$ stable under pmG-convergence?

Stability of heat flow under pmG-convergence

- $Ch_{\mathfrak{m}} : L^2(X, \mathfrak{m}) \to \mathbb{R}$ is a convex and l.s.c. functional so (by classical theory of gradient flows, e.g. Brezis) admits a unique gradient flow $(H_t)_{t \geq 0}$ called Heat flow.

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iii) approximate the heat flow by iterated resolvant maps to conclude.

Stability of $RCD(K,\infty)$ under pmG-convergence

Fact: $(X, \mathrm{d}, \mathfrak{m})$ is infinitesimally Hilbertian iff
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ii) since the heat flows of X_n are linear, by the stability of heat flows also the limit heat flow is linear.

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- Same geometric consequence of $CD(K, N)$ (Bishop-Gromov, Bonnet-Myers, Lichnerowicz) but sometimes with slightly worse constants.

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- the case $N=\infty$ was already established by Ambrosio-Gilgli-Savaré '12

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Q: how reinforce $CD^*(K, N)$ to get a stable condition + LTG?

Consequences of Bochner inequality. 1: Local to Global property for $RCD^*(K,N)$ without a priori non-branching assumption

THM[Ambrosio-M.-Savaré '13] Let $(X, \mathsf{d}, \mathfrak{m})$ be a locally compact length space and assume there is a covering $\{U_i\}_{i\in I}$ of X by non empty open subsets s.t. $(\bar{U}_i, \mathsf{d}, \mathfrak{m}\llcorner \bar{U}_i)$ satisfy $RCD(K,\infty)$ (resp. $RCD^*(K,N)$).
Then $(X, \mathsf{d}, \mathfrak{m})$ is an $RCD(K,\infty)$ (resp. $RCD^*(K,N)$) space.

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(iii) Globalize $BI(K,N)$ by using partition of unity and conclude that $RCD^*(K,N)$ holds by applying globally the equivalence theorem □

Consequences of Bochner inequality.
2: Li-Yau and Harnack type inequalities

THM[Garofalo-M. '13] Let $(X, \mathrm{d}, \mathfrak{m})$ be a m.m.s. with $\mathfrak{m}(X) = 1$ and let $f \in L^1(X, \mathfrak{m})$, $f \geq 0$ $\mathfrak{m}$-a.e. . Then

- Li-Yau Inequality: if $(X, \mathrm{d}, \mathfrak{m})$ is an $RCD^*(0, N)$ space then

$$\Delta(\log(H_t f)) \geq -\frac{N}{2t} \quad \mathfrak{m}\text{-a.e.} \quad \forall t > 0$$

- Bakry-Quian Inequality: If $(X, \mathrm{d}, \mathfrak{m})$ is an $RCD^*(K, N)$ space, for some $K > 0$, then

$$\Delta(H_t f) \leq \frac{NK}{4}(H_t f) \quad \mathfrak{m}\text{-a.e.} \quad \forall t > 0$$

- Harnack Inequality: If $(X, \mathrm{d}, \mathfrak{m})$ is an $RCD^*(K, N)$ space, for some $K \geq 0$, then for every $x, y \in X$ and $0 < s < t$ we have

$$(H_t f)(y) \geq (H_s f)(x)\, e^{-\frac{\mathrm{d}^2(x,y)}{4(t-s)e^{\frac{2Ks}{3}}}} \left(\frac{1 - e^{\frac{2K}{3}s}}{1 - e^{\frac{2K}{3}t}} \right)^{\frac{N}{2}}.$$

Euclidean tangents to $RCD^*(K, N)$ spaces

- Cheeger-Colding '97: for limit spaces the local blow ups are a.e. unique and euclidean.

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THM [Gigli-M.-Rajala '13] Let $(X, \mathsf{d}, \mathfrak{m})$ be an $RCD^*(K, N)$ space. Then for $\mathfrak{m}$-a.e. $x \in X$ there exists $n = n(x) \in \mathbb{N}$, $n \leq N$, such that

$$(\mathbb{R}^n, \mathsf{d}_E, \mathcal{L}_n, 0) \in \mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, x),$$

where d_E is the Euclidean distance and $\mathcal{L}_n$ is the n-dimensional Lebesgue measure normalized so that $\int_{B_1(0)} 1 - |x| \, d\mathcal{L}_n(x) = 1$.

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6. repeating the scheme iteratively we conclude.

Bibliography

1. L. Ambrosio, N. Gigli, A. M., T. Rajala, *Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure,* Trans.of the AMS, (in press).
2. N. Gigli, A. M., T. Rajala, *Euclidean spaces as weak tangents of infinitesimally Hilbertian metric spaces with Ricci curvature bounded below*, Crelle's journal (in press).
3. N. Garofalo, A. M., *Li-Yau and Harnack type inequalities in $RCD^*(K,N)$ metric measure spaces* , Nonlinear Analysis: TMA (2014).
4. L. Ambrosio, A. M., G. Savaré *On the Bakry-Émery condition, the gradient estimates and the Local-to-Global property of $RCD^*(K,N)$ met. meas. spaces*, submitted (2013).
5. L. Ambrosio, A. M., G. Savaré *Nonlinear diffusion equations and curvature conditions in metr. meas. spaces*, in prepar.
6. N. Gigli, A. M., G. Savaré, *Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows,* submitted (2013).

!!THANK YOU FOR THE ATTENTION!!