Martingale Transport and Peacocks

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Outline

Outline

1. Peacocks
2. A discrete time martingale transport problem
3. Continuous-time limit
   - Problem formulation
   - Main results
   - Applications

Peacocks

- A peacock is a stochastic process $(X_t, t \geq 0)$, if

  (i) it is integrable, i.e. $\mathbb{E}[|X_t|] < \infty$, $\forall t \geq 0$;

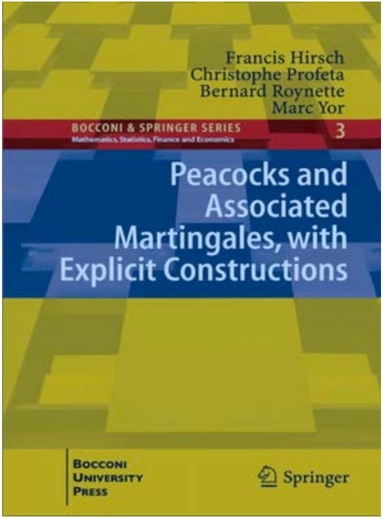
  (ii) it increases in convex order, i.e. for every convex function $\phi : \mathbb{R} \to \mathbb{R}$, the map $t \mapsto \mathbb{E}[\phi(X_t)]$ is increasing.

- PCOC : "Processus Croissant pour l'Ordre Convexe" in French.

- A peacock is determined by the family of marginal distributions.

- Kellerer's theorem : Every peacock has the same one-dimensional marginals as a martingale $(M_t, t \geq 0)$, i.e $X_t \sim M_t$ in law and $\mathbb{E}\big[M_t | M_r, r \in [0, s]\big] = M_s$ for every $s \leq t$.

Peacocks


Francis Hirsch
Christophe Profeta
Bernard Roynette
Marc Yor
BOCCONI & SPRINGER SERIES
Mathematics, Statistics, Finance and Economics
3
Peacocks and
Associated
Martingales, with
Explicit Constructions
Bocconi
University
Press
Springer

Peacocks

A proud peacock spreads

Its tail pretending to be

A martingale.

Extremal martingale peacocks

• Let $(\mu_t, t \geq 0)$ be a peacock, ξ be a reward/cost function on the martingale M, we look for the extremal martingale peacocks :

$$\sup_{M \text{ martingale peacock}} \mathbb{E}\Big[\xi(M_\cdot)\Big].$$

Kellerer's theorem (proof of Hirsch and Roynette)

• Kellerer's theorem : For every peacock $(\mu_t)_{t\geq 0}$, there is a martingale $(M_t, t \geq 0)$ such that $M_t \sim \mu_t$.

- Suppose the marginals μ_t admits a smooth density function $p(t,x)$, denote $C(t,x) := \int_x^\infty (y-x)\mu_t(dy)$. Then $p(t,x)$ solves the Fokker-Planck equation

$$\partial_t p(t,x) \;=\; \frac{1}{2}\partial^2_{xx}\Big(\sigma^2(t,x)p(t,x)\Big),$$

for $\sigma(t,x) = \Big(2\frac{\partial_t C(t,x)}{\partial^2_{xx}C(t,x)}\Big)^{1/2}$.

- The Fokker-Planck equation is related to the diffusion

$$M_t \;=\; M_0 + \int_0^t \sigma(s, Z_s)dB_s.$$

- When C is not smooth, approximate it by smooth functions.

Extremal martingale peacocks

- Let $(\mu_t, t \geq 0)$ be a peacock, ξ be a reward/cost function on the martingale M, we look for the extremal martingale peacocks :

$$\sup_{M \text{ martingale peacock}} \mathbb{E}\Big[\xi(M_\cdot)\Big].$$

- Approximation technique.

Outline

Martingale Transportation Problem

- Monge-Kantorovich's Optimal Transportation Problem :

$$\sup_{\mathbb{P}\in\mathcal{P}(\mu_0,\mu_1)} \mathbb{E}^{\mathbb{P}}\big[c(X_0,X_1)\big]$$

$$= \inf\Big\{\mu_0(\lambda_0)+\mu_1(\lambda_1)\ :\lambda_0(x)+\lambda_1(y)\geq c(x,y)\Big\}.$$

- Martingale Transportation Problem :

$$\sup_{\mathbb{P}\in\mathcal{M}(\mu_0,\mu_1)} \mathbb{E}^{\mathbb{P}}\big[c(X_0,X_1)\big]$$

$$= \inf\Big\{\mu_0(\lambda_0)+\mu_1(\lambda_1)\ :\lambda_0(x)+\lambda_1(y)+h(x)(y-x)\geq c(x,y)\Big\}.$$

Martingale version of Brenier's theorem

• Brenier's theorem (Féchet-Hoeffding coupling) in the one-dimensional case : when $\partial_{xy} c > 0$, the solution is given by the monotone transference plan $T := F_1^{-1} \circ F_0$.

• Martingale version (Beiglbock-Juillet, Henry-Labordère -Touzi) : When $\partial_{xyy} c > 0$, the optimal solution is given by the left-monotone martingale transference plan (which is a binomial model).

• The transition kernel of the binomial model is, with $T_d(x) \le x \le T_u(x)$, $q(x) := \frac{x - T_d(x)}{T_u(x) - T_d(x)}$,

$$T_*(x, dy) := q(x)\delta_{T_u(x)}(dy) + (1 - q(x))\delta_{T_d(x)}(dy).$$

Martingale version of Brenier's theorem

Determinate T_u and T_d : assume that $\delta F := F_1 - F_0$ has only one local maximizer m.

- Coupled ODE, on $[m, \infty)$,

$$d(\delta F \circ T_d) = -(1-q) dF_0, \quad d(F_1 \circ T_u) = q dF_0.$$

- Resolution of ODE : denote $g(x,y) := F_1^{-1}\big(F_0(x) + \delta F(y)\big)$,

$$\int_{-\infty}^{x} \left[F_1^{-1}(F_0(\xi)) - \xi\right] dF_0(\xi) + \int_{-\infty}^{T_d(x)} (g(x,\xi) - \xi) d\delta F(\xi) = 0,$$

$$T_u(x) = g\big(x, T_d(x)\big).$$

Martingale version of Brenier's theorem

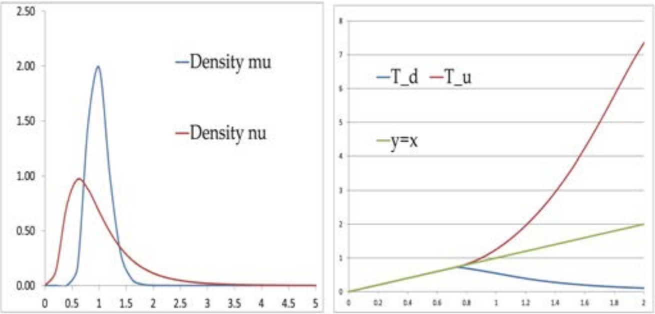


Figure : An example of T_u and T_d.

The optimal dual components

- The dynamic strategy h_* :

$$h'_*(x) = \frac{c_x(x, T_u(x)) - c_x(x, T_d(x))}{T_u(x) - T_d(x)}, \ \forall x \in [m, \infty),$$

$$h_*(x) = h_*\big(T_d^{-1}(x)\big) + c_y(x,x) - c_y\big(T_d^{-1}(x), x\big), \ \forall x \in (-\infty, m).$$

- The static strategy (λ_0, λ_1) :

$$\lambda'_1 = c_y(T^{-1}, \cdot) - h_* \circ T^{-1}, \quad T^{-1} = T_u^{-1} 1_{[m,\infty)} + T_d^{-1} 1_{(-\infty,m)}.$$

$$\lambda_0 = q\big(c(\cdot, T_u) - \lambda_1(T_u)\big) + (1-q)\big(c(\cdot, T_d) - \lambda_1(T_d)\big).$$

The multi-marginals case

- An easy extension to the multi-marginals case

$$\sup_{\mathbb{P}\in\mathcal{M}(\mu_0,\cdots,\mu_n)} \mathbb{E}^{\mathbb{P}}\Big[\sum_{k=1}^{n} c(X_{k-1},X_k)\Big].$$

- The extremal model is a Markov chain (martingale), and the optimal dual strategies are all explicit.

- What happens if $n \to \infty$?
  - Do they "converge" ?
    - the criteria function,
    - the Markov chain,
    - the super hedging strategy.
  - Does the limit keep the optimality ?

Outline

Limit of the criteria function

- Assumption : $c(x,x) = c_y(x,x) = 0$, $c_{xyy}(x,y) > 0$.

- Quadratic variation (Föllmer) of a càdlàg path $\mathbf{x} : [0,1] \to \mathbb{R}$,

$$\sum_{1 \le k \le n} \big(\mathbf{x}_{t_k} - \mathbf{x}_{t_{k-1}}\big)^2 \delta_{t_{k-1}}(dt).$$

- It is proved in Hobson and Klimmek (2012) that

$$\sum_{k=1}^{n} c(\mathbf{x}_{t_{k-1}}, \mathbf{x}_{t_k}) \to C(\mathbf{x}) := \frac{1}{2}\int_0^1 c_{yy}(\mathbf{x}_t, \mathbf{x}_t) d[\mathbf{x}]_t^c + \sum_{0 \le t \le 1} c(\mathbf{x}_{t-}, \mathbf{x}_t).$$

Continuous-time martingale transport

- Let $\mu = (\mu_t)_{0 \le t \le 1}$ be increasing in convex ordering, right-continuous and unif. integrable.

- Let $\Omega := D([0,1], \mathbb{R})$, $\mathcal{M}_\infty$ the set of martingale measures on Ω and $\mathcal{M}_\infty(\mu)$ that subset of measures under which X fits all marginals.
  - $\mathcal{M}_\infty(\mu)$ is non-empty (Kellerer(1972), Hirsch and Roynette (2012)).

- MT problem

$$P_\infty(\mu) := \sup_{\mathbb{P} \in \mathcal{M}_\infty(\mu)} \mathbb{E}^{\mathbb{P}}\big[C(X_\cdot)\big].$$

Peacocks

- Peacock (PCOC "Processus Croissant pour l'Ordre Convex") : Construction of associated martingales (Hirsch, Profeta, Roynette and Yor(2011), etc.)
- Self-similar martingales, when $X_t = \sqrt{t}X$. Madan-Yor (2002), Hamza-Klebaner(2012), etc.
- Fake Brownian motion, when $X \sim N(0,1)$. Oleszkiewicz(2008), Albin (2008), Hobson (2013), etc.

Dual formulation

- Dynamic strategy : $\mathbb{H}_0 : [0,1] \times \Omega \to \mathbb{R}$ denotes the set of all predictable, locally bounded processes,

$$\mathcal{H} := \left\{ H \in \mathbb{H}_0 \ : H \cdot X \text{ is a } \mathbb{P}\text{-supermartingale for every } \mathbb{P} \in \mathcal{M}_\infty \right\}.$$

- $\Lambda := \{\lambda(x, dt) = \lambda^0(t,x)\gamma(dt)$,

$$\Lambda(\mu) := \{\lambda \in \Lambda \ : \mu(|\lambda|) < \infty\}, \quad \mu(\lambda) := \int\int \lambda^0(t,x)\mu_t(dx)\gamma(dt).$$

- Dual problem

$$D_\infty(\mu) := \left\{ (H,\lambda) : \int_0^1 \lambda(X_t, dt) + (H \cdot X)_1 \geq C(X_\cdot), \ \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{M}_\infty \right\}.$$

The limit of Markov chain

- (i) Suppose that $(\mu_t)_{t\in[0,1]}$ admits smooth density functions $f(t,x)$. Denote by $F(t,x)$ the distribution function.
  (ii) $x \mapsto \partial_t F(t,x)$ has only one local maximizer $m(t)$.

- Define $T_d : [0,1) \times [m(t), \infty) \to \mathbb{R}$ by

$$\int_{T_d(t,x)}^{x} (x-\xi)\partial_t f(t,\xi)d\xi = 0$$

$$j_d(t,x) := x - T_d(t,x)$$

$$j_u(t,x) := \frac{\partial_t F(t, T_d(t,x)) - \partial_t F(t,x)}{f(t,x)}.$$

Technical Lemma

Lemma

The functions j_d and j_u are both continuous in (t,x) and locally Lipschitz in x.

Lemma

We have the asymptotic estimates

$$T_u^\varepsilon(t,x) = x + \varepsilon j_u(t,x) + O(\varepsilon^2), \quad T_d^\varepsilon(t,x) = x - j_d(t,x) + O(\varepsilon).$$

The limit of dual component

- Dynamic strategy : $h^* : [0,1) \times \mathbb{R}$ is defined by

$$\partial_x h^*(t,x) := \frac{c_x(x,x) - c_x(x, T_d(t,x))}{j_d(t,x)}, \quad x \geq m(t),$$

$$h^*(t,x) := h^*(t, T_d^{-1}(t,x)) - c_y(T_d^{-1}(t,x),x), \quad x < m(t).$$

- Static strategy : let ψ^* and λ_0^* be defined by
$\partial_x \psi^*(t,x) := -h^*(t,x)$,

$$\lambda_0^* := \partial_t \psi^* + \Big(\partial_x \psi^* j_u + (\psi^*(\cdot) - \psi^*(\cdot - j_d(\cdot) + c(\cdot - j_d(\cdot))\frac{j_u}{j_d}\Big) 1_{x \geq m(t)}.$$

the static strategy is given by

$$\psi^*(1,x) - \psi^*(0,x) + \int_0^1 \lambda_0^*(t,x)dt.$$

Main results

Theorem

Let $(\pi^n)_{n\geq 1}$ be a sequence of partitions of $[0,1]$, and X^n be the associated optimal Markov chain, then the law of X^n converge to $\mathbb{P}^* \in \mathcal{M}_\infty(\mu)$, under which X is local Lévy process

$$X_t = X_0 - \int_0^t 1_{X_{s-} > m(s)} j_d(s, X_{s-}) \big(dN_s - \frac{j_u}{j_d}(s, X_{s-}) ds \big),$$

where N is a pure jump process with predictable compensated process $\frac{j_u}{j_d}$. Under further integrability conditions, we have

$$\mathbb{E}^{\mathbb{P}^*}\big[C(X_\cdot)\big] \;=\; P_\infty(\mu) \;=\; D_\infty(\mu) \;=\; \mu(\lambda^*)$$

$$= \int_0^1 \int_{m(t)}^\infty \frac{j_u}{j_d}(t,x) c(x, x - j_d(t,x)) f(t,x) dx dt.$$

Main results

Theorem

The density function satisfies the PDE

$$\partial_t f(t,x) = -1_{x<m(t)} \frac{j_u f}{j_d(1-\partial_x j_d)}(t, T_d^{-1}(t,x)) - 1_{x \geq m(t)} \Big(\frac{j_u f}{j_d} - \partial_x (j_u f) \Big)(t,x),$$

which is also Kolmogorov–Fokker-Planck forward equation associated to the local Lévy process

$$dX_t = -1_{X_{t-}>m(t)} j_d(t, X_{t-}) \Big(dN_t - \frac{j_u}{j_d}(t, X_{t-}) dt \Big).$$

Robust hedging of variance swap

• The payoff of variance swap : in discrete-time case $\sum_{k=1}^{n} \log^2 \frac{X_{t_k}}{X_{t_{k-1}}}$; in continuous-time case

$$\int_0^1 \frac{1}{X_t^2} d[X]_t^c + \sum_{0<t\leq 1} \log^2 \frac{X_t}{X_{t^-}}.$$

• Application of the main result with $c(x,y) := \log^2(x/y)$, we find an optimal no-arbitrage bounds as well as the super-hedging strategies.

Fake Brownian motion

When $f(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$, we get a more explicit formula for j_d and j_u,

$$e^{-\hat{T}_d(x)^2/2}\Big(1 + \hat{T}_d(x)^2 - x\hat{T}_d(x)\Big) \;=\; e^{-x^2/2},$$

$$\hat{j}_u(x) \;:=\; \frac{1}{2}\Big[x - \frac{\hat{T}_d(x)}{1 + \hat{T}_d(x)^2 - x\hat{T}_d(x)}\Big]$$

and also

$$j_u(t,x) := \sqrt{t}\hat{j}_u(x/\sqrt{t}), \qquad j_d(t,x) := x - \sqrt{t}\,\hat{T}_d(x/\sqrt{t}).$$

Fake Brownian motion

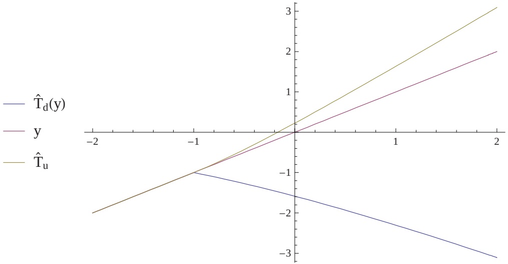


Figure : $\hat{T}_u$ and $\hat{T}_d$ of our fake Brownian motion.

Discussions

- Can we find other extremal martingale Peacocks ?
- SEP (Skorokhod Embedding Problem) approach
  - Monroe's theorem : Every right-continuous martingale can be embedded into a Brownian motion with stopping times.
  - Compactness can be obtained more easily (Beiglbock and Huesmann (2013)).