Ruhr University
Bochum

Functional Poisson approximation

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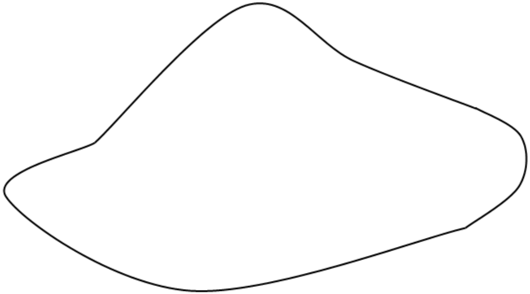
(with L. Decreusefond and M. Schulte)

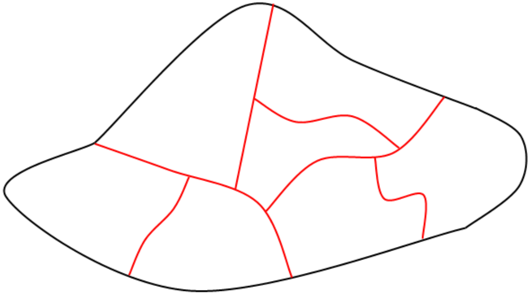
YEP XI – Eindhoven

The Poisson process



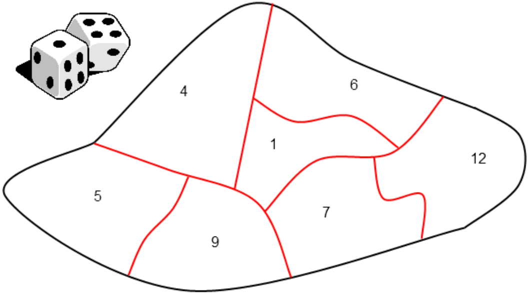
- a measurable space with a σ-finite measure μ

The Poisson process



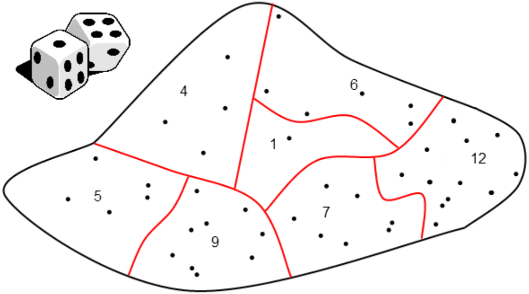
- decomposition into measurable set B_i with finite μ-measure

The Poisson process



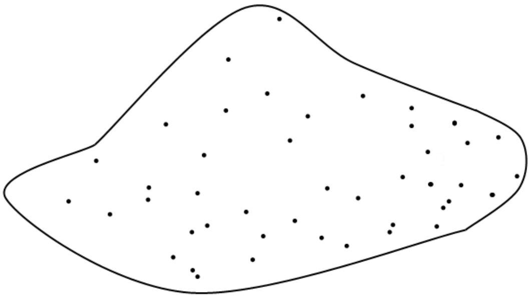
- Poisson random variables N_i with mean $\mu(B_i)$

The Poisson process



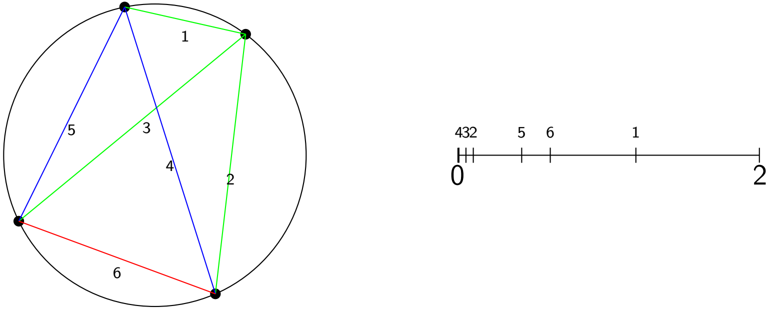
- $X_1, \ldots, X_{N_i}$ random points distributed according to $\mu(\,\cdot\,|B_i)$

The Poisson process



- Poisson process (better: random point field) with control μ

Random polytopes



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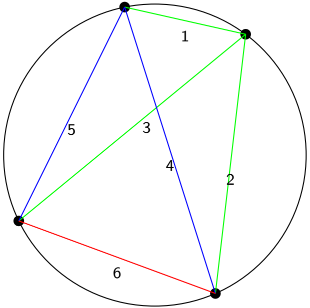

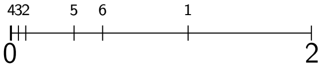

Question

What happens asymptotically with the diameter?

Random polytopes

Hypothesis

Points are distributed to a Poisson process η_t with control $t\mathbf{K}$

Definition

- The number of points is a Poisson rv $(t\,\mathbf{K}(\mathbb{S}^{d-1}))$
- Given the number of points, they are independently drawn with distribution $\mathbf{K}$

Rescaling : $\gamma = 4/(d-1)$

Mecke's formula (special case)

$$\mathbf{E} \sum_{(x_1,\cdots,x_k)\in\eta^k_{t,\neq}} f(x_1,\dots,x_k) = t^k \int f(x_1,\cdots,x_k)\mathbf{K}^k(\mathrm{d}x_1,\cdots,\mathrm{d}x_k)$$

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Mean number of points (after rescaling)

$$\frac{1}{2}\mathbf{E} \sum_{x\neq y\in\eta_t} \mathbf{1}_{2-\|x-y\|\leq\beta t^{-\gamma}} = \frac{t^2}{2} \iint_{\mathbb{S}^{d-1}\otimes\mathbb{S}^{d-1}} \mathbf{1}_{2-\|x-y\|\leq\beta t^{-\gamma}}\mathrm{d}x\mathrm{d}y$$

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Geometry

$$V_{d-1}(\mathbb{S}^{d-1}\cap B^d_{\sqrt{2\beta t^{-\gamma}}}(y)) = \kappa_{d-1}(2\beta t^{-\gamma})^{(d-1)/2} + error$$

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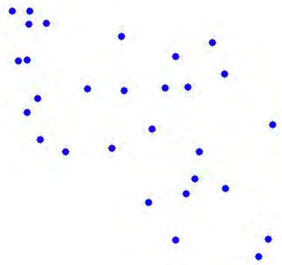
Schulte/T. (2012) based on Peccati (2011)

$$\left|\mathbb{P}(t^{4/(d-1)}\mathrm{diam}_t > x) - e^{-\frac{d}{2}\kappa_d\kappa_{d-1}x^{(d-1)/2}}\right| \leq C\, t^{-\min(4/(d-1),1/2)}$$

Random geometric graphs



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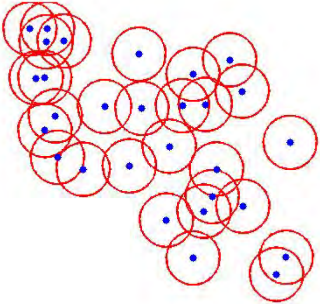

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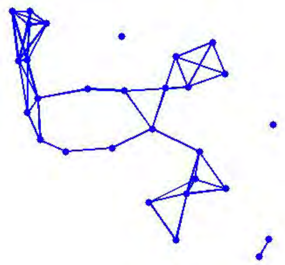

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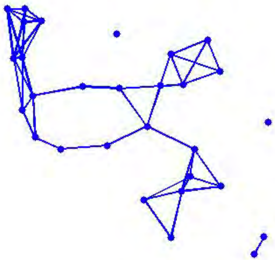

Question

What happens asymptotically with the order-statistic of all edge lengths?

Rescaling : $\gamma = 2/d$

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Geometry

$$V_d(W\cap B^d_{\min(\beta t^{-\gamma},\delta_t)}(y)) = \kappa_d(\min(\beta t^{-\gamma},\delta_t))^d + \mathit{error}$$

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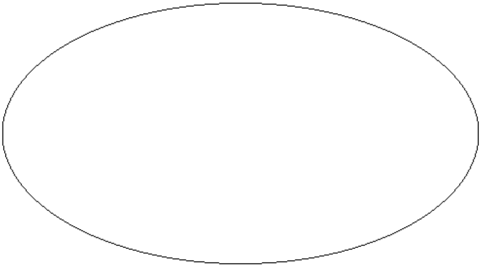
Schulte/T. (2012) based on Peccati (2011)

$$\xi_t := \{\|x-y\| : (x,y) \in \eta^2_{t,\neq}, \|x-y\| \le \delta_t\} \text{ with } t^{2/d}\delta_t \to \infty$$

$$t^{2/d}\xi_t \Longrightarrow \text{suitable PPP}$$

Random simplices



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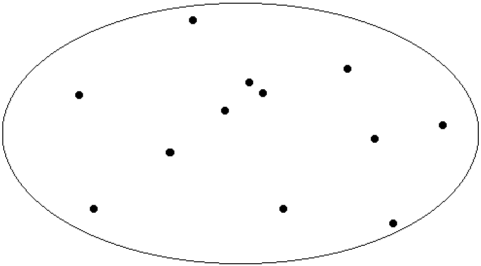

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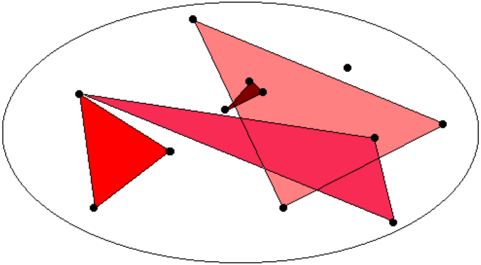

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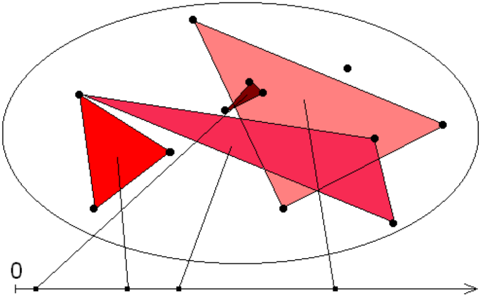

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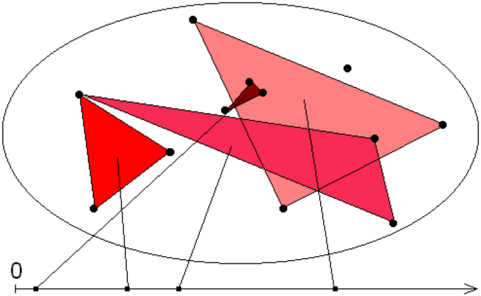

Question (Grimmett/Janson 2003 for minimum and $d = 2$)

What happens asymptotically with the order-statistic of all simplex volumes?

Rescaling : $\gamma = d + 1$

Mecke's formula

$$\mathbf{E} \sum_{(x_1,\cdots,x_k)\in\eta^k_{t,\neq}} f(x_1,\ldots,x_k) = t^k \int f(x_1,\cdots,x_k)\mathbf{K}^k(\mathrm{d}x_1,\cdots,\mathrm{d}x_k)$$

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Mean number of points (after rescaling)

$$\frac{1}{(d+1)!}\mathbf{E} \sum_{(x_1,\ldots,x_{d+1})\in\eta^{d+1}_{t,\neq}} \mathbf{1}_{V_d([x_1,\ldots,x_{d+1}])\leq\beta t^{-\gamma}}$$

$$= \frac{t^{d+1}}{(d+1)!}\int_{W^{\otimes(d+1)}} \mathbf{1}_{V_d([x_1,\ldots,x_{d+1}])\leq\beta t^{-\gamma}}\mathrm{d}(x_1,\ldots,x_{d+1})$$

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Geometry

Fix the first d points and apply Blaschke-Petkanschin $\rightarrow t^{-\gamma}$

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Schulte/T. (2012) based on Peccati (2011)

$$\xi_t := \{V_d([x_1,\dots,x_{d+1}]) : (x_1,\dots,x_{d+1}) \in \eta^{d+1}_{t,\neq}\}$$

$$t^{d+1}\xi_t \Longrightarrow \text{homogeneous PPP with intensity}$$

$$\frac{d\kappa_d}{d+1}\int_{[W]} V_{d-1}(K\cap H)^{d+1}\,\mathrm{d}H \qquad (= 2V_2(W)^2,\ d=2).$$

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What about the speed of convergence measured by a suitable point process distance?

Convergence in distribution

Definition

$(X_n,\ n \geq 1)$ random variables with values in $\mathbb{Y}$ converge in distribution to X whenever

$$\mathbf{E}f(X_n) \xrightarrow{n\to\infty} \mathbf{E}f(X)$$

for all f *continuous and bounded on* $\mathbb{Y}$.

Equivalently

$$\int_{\mathbb{Y}} f(y)\,\mathrm{d}\mathbb{P}_{X_n}(y) \xrightarrow{n\to\infty} \int_{\mathbb{Y}} f(y)\,\mathrm{d}\mathbb{P}_X(y)$$

Configuration space

Definition

A configuration ω is a locally finite set of particles (points) $\{y_1, y_2, \ldots\}$ on a Polish space $\mathbb{Y}$.

$$\omega = \sum_{y \in \omega} \delta_{y_i} \Longrightarrow \int f \mathrm{d}\omega = \sum_{y \in \omega} f(y)$$

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Vague topology

$$\omega_n \xrightarrow{\text{vaguely}} \omega \Longleftrightarrow \int f \mathrm{d}\omega_n \xrightarrow{n \to \infty} \int f \mathrm{d}\omega$$

for all f from $\mathbb{Y}$ to $\mathbb{R}$ *continuous with compact support*

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Be careful!

$$\delta_n \xrightarrow{\text{vaguely}} \emptyset$$

Convergence in distribution

Question

$$\mathbf{E}\Big[F\big(\mathrm{PPP}(\mathbf{M})\big)\Big] - \mathbf{E}\Big[F\big(t^{\gamma}T(\mathrm{PPP}(t\mathbf{K}))\big)\Big] \xrightarrow{t\to\infty} 0$$

for all F bounded and continuous on $\mathbb{N}_{\mathbb{Y}}$?

Convergence in distribution

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for all F bounded and continuous on $\mathbb{N}_{\mathbb{Y}}$?

Theorem

It is sufficient to choose

$$F(\omega) = \psi\Big(\int f \,\mathrm{d}\omega\Big)$$

with $\psi : \mathbb{R} \to \mathbb{R}$ bounded and continuous, $f : \mathbb{Y} \to \mathbb{R}$ continuous with compact support

Lipschitz functionals

Distance between configurations

$\mathsf{dist}_{\mathsf{TV}}(\omega, \eta)$ = number of different points

Lipschitz functionals

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Definition

$F : \mathsf{N}_{\mathbb{Y}} \to \mathbb{R}$ is TV–Lip$_1$ if

$$|F(\omega_1) - F(\omega_2)| \leq \mathsf{dist}_{\mathsf{TV}}(\omega_1, \omega_2)$$

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Examples

- $\omega \mapsto \omega(A)$
- $\omega \mapsto \sum_{x \in \omega} f(x)$ with $\|f\|_\infty \le 1$
- $\omega \mapsto \max_{x \in \omega} f(x)$ with $\|f\|_\infty \le 1/2$

The distance - Link to optimal transport

$\mathbf{P}$, $\mathbf{Q}$ probability measures on $\mathsf{N}_{\mathbb{Y}}$

Definition (Rubinstein distance)

$$\mathrm{d}_{\mathrm{R}}(\mathbf{P},\mathbf{Q}) := \inf_{\mathbf{C}\in\mathrm{Couplings}(\mathbf{P},\mathbf{Q})} \int_{\mathsf{N}_{\mathbb{Y}}\times\mathsf{N}_{\mathbb{Y}}} \mathrm{d}_{\mathrm{TV}}(\omega_1,\omega_2)\,\mathbf{C}(\mathrm{d}(\omega_1,\omega_2))$$

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Definition (after Kantorovitch-Rubinstein duality theorem)

$$\mathrm{d_R}(\mathbf{P},\mathbf{Q}) = \sup_{F\in\mathrm{TV\text{-}Lip}_1} \big(\mathbf{E}_{\mathbf{P}}F - \mathbf{E}_{\mathbf{Q}}F\big),$$

The distance - Link to optimal transport

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Theorem (Decreusefond/Schulte/T.)

$$d_{\mathrm{R}}(\mathbf{P}_n,\mathbf{Q}) \xrightarrow{n\to\infty} 0 \Longrightarrow \mathbf{P}_n \xrightarrow{\mathit{distr.}} \mathbf{Q}$$

Set-up

Our set-up

- $\mathbf{P}_t$: PPP with control $t\mathbf{K}$ on $C \subset \mathbb{X}$
- f : dom f = symm. subset of $C^k \longrightarrow \mathbb{Y}$
- a transformation

$$T\Big(\sum_{x \in \eta_t} \delta_x\Big) = \sum_{(x_1,\cdots,x_k) \in \eta^k_{t,\neq}} \delta_{t^\gamma f(x_1,\cdots,x_k)} := \xi_t(\eta)$$

- $f^\# \mathbf{K}$: image measure of $(t\mathbf{K})^k$ under f
- $\mathbf{M}$: control of the target Poisson PP

Stein's method

What we have to compute

$$\sup_{F\in \mathrm{TV\text{-}Lip}_1} \mathbf{E}\Big[F\big(\mathrm{PPP}(\mathbf{M})\big)\Big] - \mathbf{E}\Big[F\big(T^{\#}(\mathrm{PPP}(t\mathbf{K}))\big)\Big]$$

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The main tool

Construct a Markov process $(X(s), s \geq 0)$

- with values in configuration space
- ergodic with PPP($\mathbf{M}$) as invariant distribution

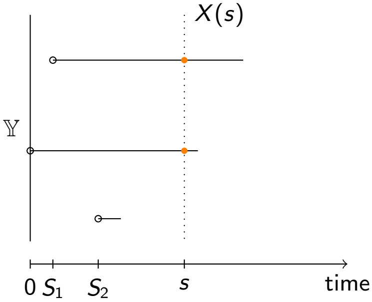
$$X(s) \xrightarrow{distr.} \mathrm{PPP}(\mathbf{M})$$

for all initial condition X(0)

- for which PPP($\mathbf{M}$) is a stationary distribution

$$X(0) \overset{\text{distr.}}{=} \mathrm{PPP}(\mathbf{M}) \Longrightarrow X(s) \overset{\text{distr.}}{=} \mathrm{PPP}(\mathbf{M}),\ \forall s > 0$$

Realization of a Glauber process



- $S_1, S_2, \ldots$: Poisson process with control $\mathbf{M}(\mathbb{Y})\, ds$
- points are placed according to $\mathbf{M}(\,\cdot\,|\mathbb{Y})$
- Lifetimes : Exponential rv with mean 1
- Remark : Number of particles $\sim$ M/M/∞ queue

Properties

Theorem

- $X(s)$=$PPP((1-e^{-s})\mathbf{M})$ + e^{-s}-thinning of the i.c.
- $X(s) \xrightarrow{s\to\infty} PPP(\mathbf{M})$
- If $X(0) \overset{distr.}{=} PPP(\mathbf{M})$ then $X(s) \overset{distr.}{=} PPP(\mathbf{M})$
- Generator

$$LF(\omega) := \int_{\mathbb{Y}} F(\omega+\delta_y) - F(\omega)\,\mathbf{M}(\mathrm{d}y) + \sum_{y\in\omega} F(\omega-\delta_y) - F(\omega)$$

Stein representation formula

Definition

$$P_t F(\omega) = \mathbf{E}[F(X(t)) \mid X(0) = \omega]$$

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Fundamental Lemma

$$\int F(\omega)\mathrm{PPP}_{\mathbf{M}}(\mathrm{d}\omega) - F(\xi) = \int_0^\infty L P_s F(\xi)\mathrm{d}s$$

Stein representation formula

Definition

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Fundamental Lemma

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What we have to compute

Distance representation

$$
\begin{aligned}
&\mathrm{d}_{\mathrm{R}}(\mathrm{PPP}(\mathbf{M}), T^{\#}(\mathrm{PPP}(t\mathbf{K}))) \\
&= \sup_{F \in \mathrm{TV\text{–}Lip}_1} \Big(\mathbf{E} \int_0^\infty \int_{\mathbb{Y}} [P_s F(\xi(\eta_t) + \delta_y) - P_s F(\xi(\eta_t))] \, \mathbf{M}(\mathrm{d}y) \, \mathrm{d}s \\
&\qquad + \mathbf{E} \int_0^\infty \sum_{y \in \xi(\eta_t)} [P_s F(\xi(\eta_t) - \delta_y) - P_s F(\xi(\eta_t))] \, \mathrm{d}s \Big)
\end{aligned}
$$

Mecke formula

Mecke formula (general case)

Let ζ be a $PPP(\mathbf{M})$. Then

$$\mathbf{E}\sum_{y\in\zeta} f(y,\zeta) = \int_{\mathbb{Y}} \mathbf{E}f(y,\zeta+\delta_y)\,\mathbf{M}(\mathrm{d}y)$$

(This is a characterization of $PPP(\mathbf{M})$!)

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Multivariate form

$$\mathbf{E}\sum_{(y_1,\ldots,y_k)\in\zeta_{\neq}}^{k} f(y_1,\ldots,y_k,\zeta)$$

$$= \int_{\mathbb{Y}^k} \mathbf{E}f(y_1,\ldots,y_k,\zeta+\delta_{y_1}+\ldots+\delta_{y_k})\,\mathbf{M}^k(\mathrm{d}(y_1,\ldots,y_k))$$

Generic Theorem

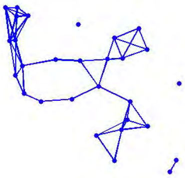
Theorem

$$\mathrm{d}_{\mathrm{R}}(PPP(\mathbf{M}), \xi(\eta)) \quad \leq \quad 2\mathrm{d}_{\mathrm{TV}}(f^{\#}\mathbf{K}^k, \mathbf{M}) \; + \; \frac{2^{k+1}}{k!}\, r(\mathrm{dom} f)$$

where

$$r(\mathrm{dom} f) := \sup_{1\leq\ell\leq k-1} \int_{\mathbb{X}^\ell} \Big(\int_{\mathbb{X}^{k-\ell}} \mathbf{1}((x_1,\ldots,x_\ell,y_1,\ldots$$
$$\ldots,y_{k-\ell}) \in \mathrm{dom} f)\, \mathbf{K}^{k-\ell}(\mathrm{d}(y_1,\ldots,y_{k-\ell}))\Big)^2 \mathbf{K}^k(\mathrm{d}(x_1,\ldots,x_\ell))$$

Random geometric graphs (cont'd)



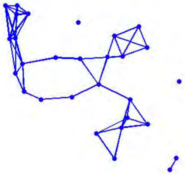
$$\xi_{t,a} := \frac{1}{2} \sum_{(x,y)\in\eta^2_{t,\neq}} \delta_{\|x-y\|} \mathbf{1}(\|x-y\| \leq \min\{\delta_t, t^{-2/d}a\})$$

$$t^{2/d}\delta_t \to \infty$$

Theorem (Edge lengths)

$$\mathrm{d_R}(t^{2/d}\xi_t|_{[0,a]}, PPP(\mathbf{M})|_{[0,a]}) \leq C_a\, t^{-\min\{2/d,1\}}$$

Random geometric graphs (cont'd)



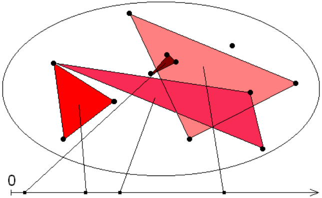
$$\xi_{t,a} := \frac{1}{2} \sum_{(x,y)\in\eta^2_{t,\neq}} \delta_{(x+y)/2}\, \mathbf{1}(\|x-y\| \leq \min\{\delta_t, t^{-2/d}a\})$$

$$t^{2/d}\delta_t \to \infty$$

Theorem (Edge midpoints)

$$\mathrm{d_R}(t^{2/d}\xi_t, PPP(\mathbf{M})) \leq C_a\, t^{-\min\{2/d,1\}}$$

Random simplices (cont'd)

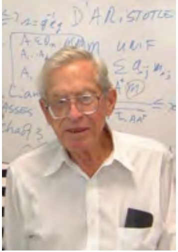


$$\xi_t := \frac{1}{k!} \sum_{(x_1,\ldots,x_{d+1}) \in \eta^{d+1}_{t,\neq}} \delta_{V_d([x_1,\ldots,x_{d+1}])}$$

Theorem

$$\mathrm{d_R}(t^{d+1}\xi_t|_{[0,a]}, PPP(\mathbf{M})|_{[0,a]}) \leq C_a\, t^{-1}$$

Malliavin-Stein method



Final comments

- You can also start with a binomial input.
- You can do this also for certain Gibbs processes.
- The method works perfectly in geometric applications!

Thank you!