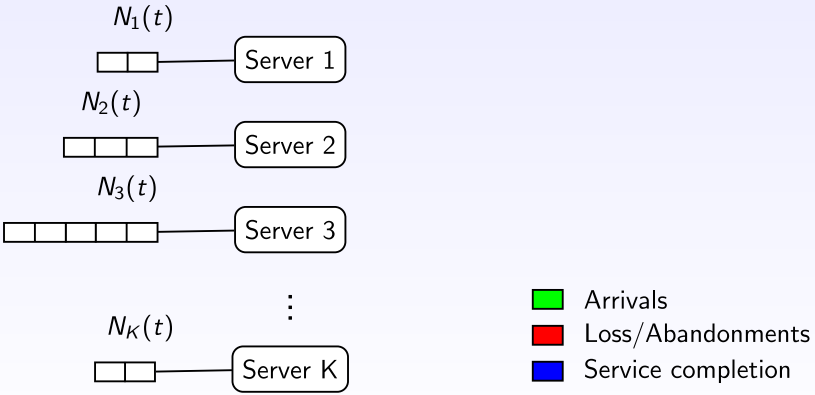
Index policies for scheduling problems

Maialen Larrañaga
maialen.larranaga@laas.fr
U. Ayesta (CNRS, LAAS), I.M. Verloop (CNRS, IRIT).

YEQT, November 3.

Motivation

Optimal class selection:



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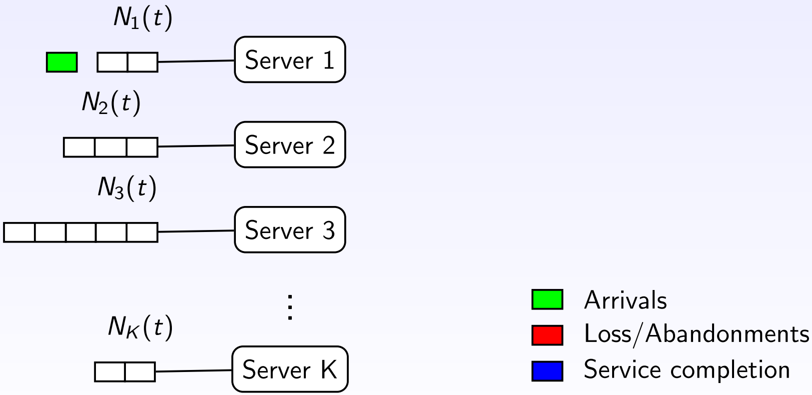

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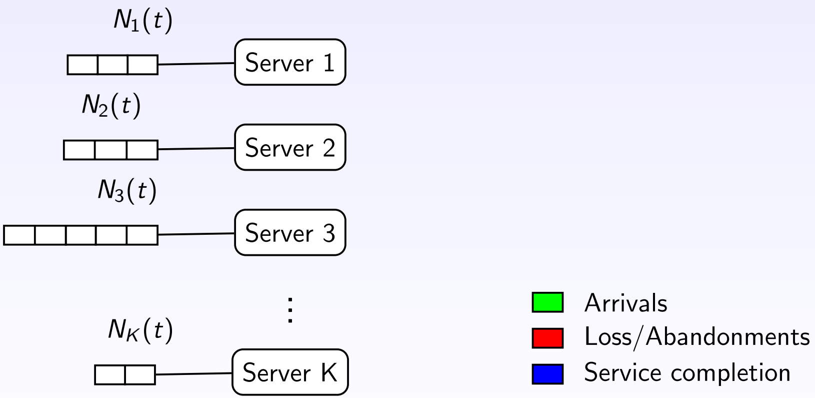

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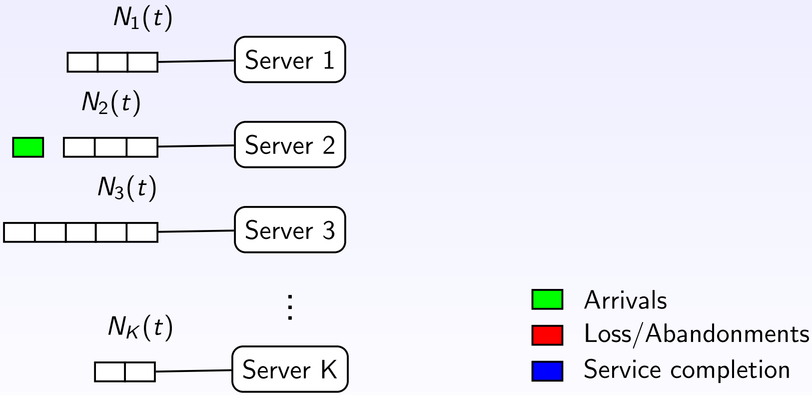

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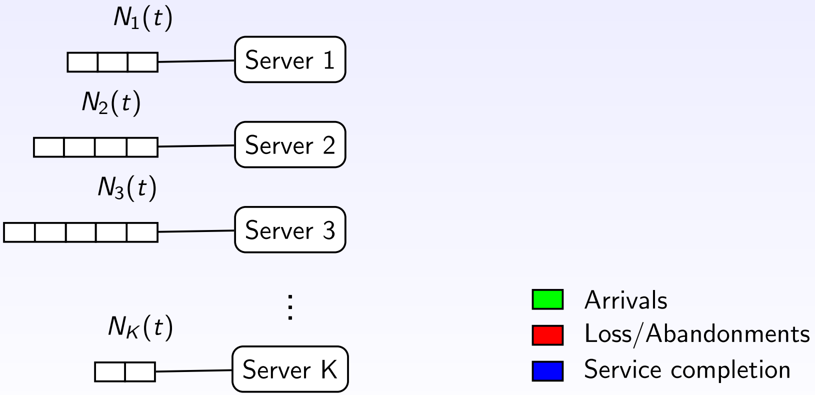

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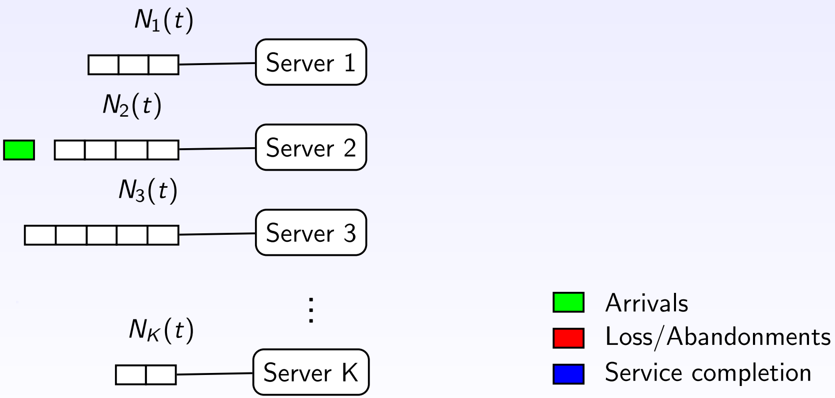

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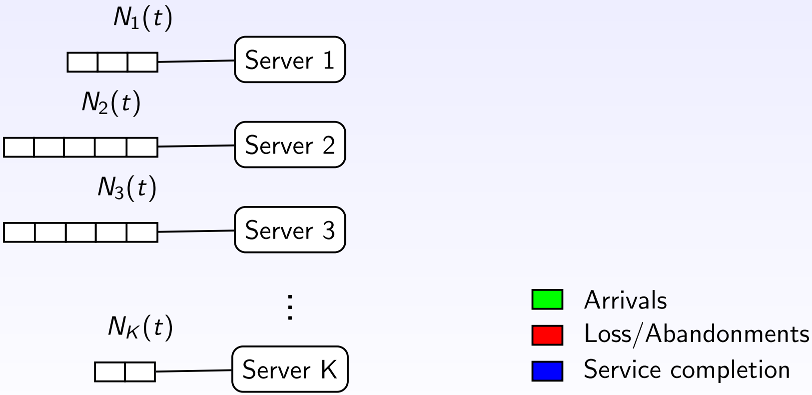

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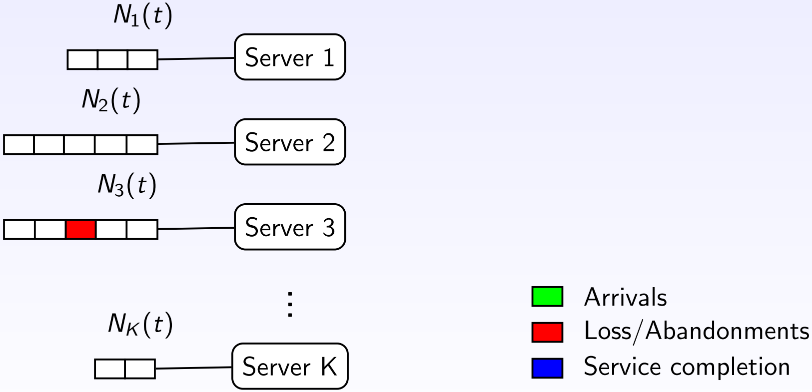

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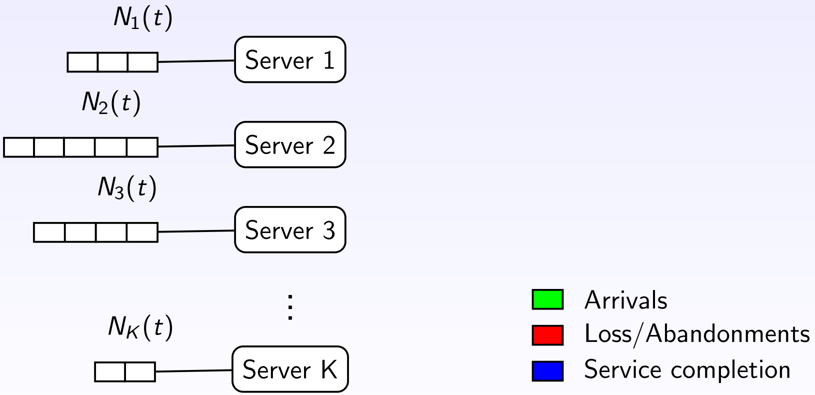

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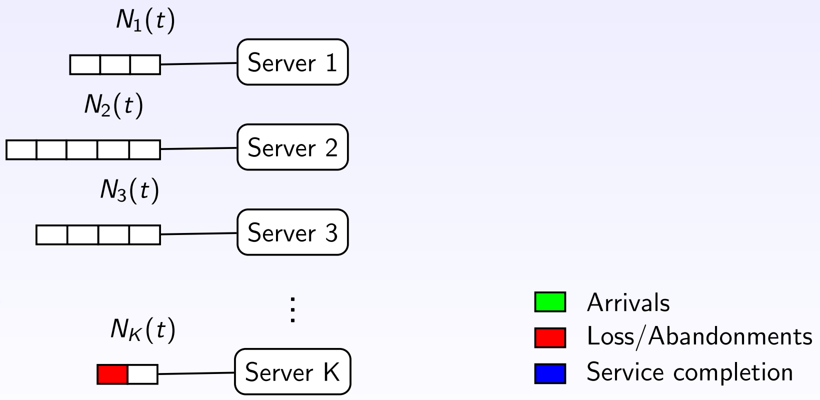

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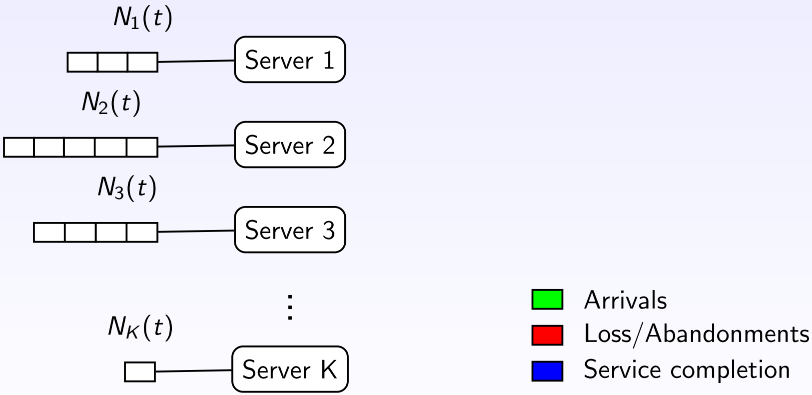

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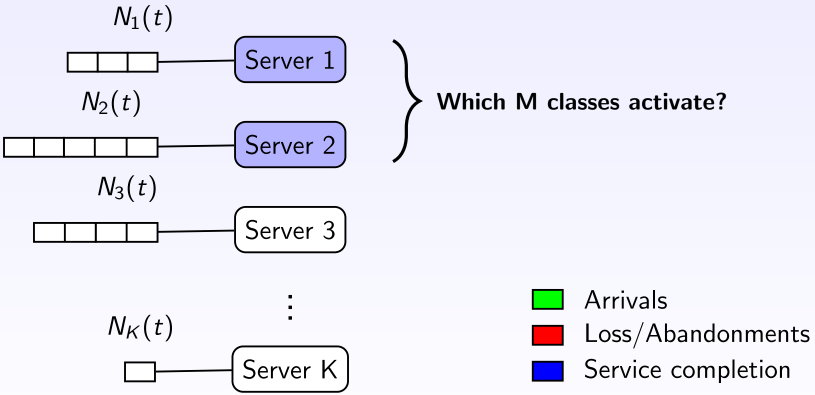

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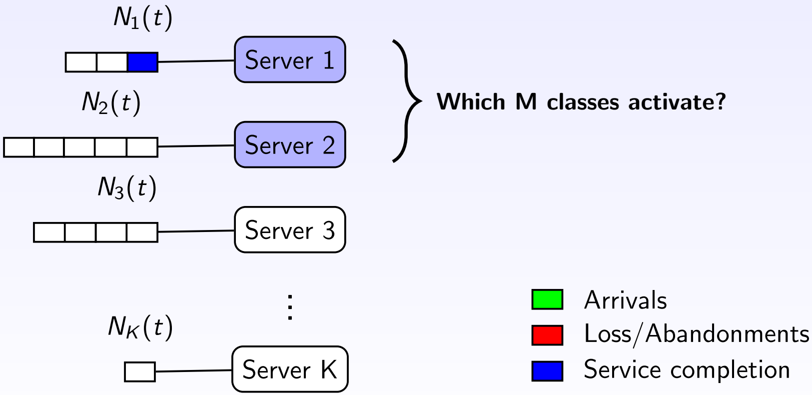

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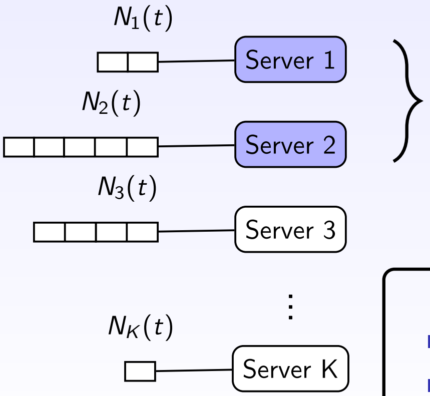


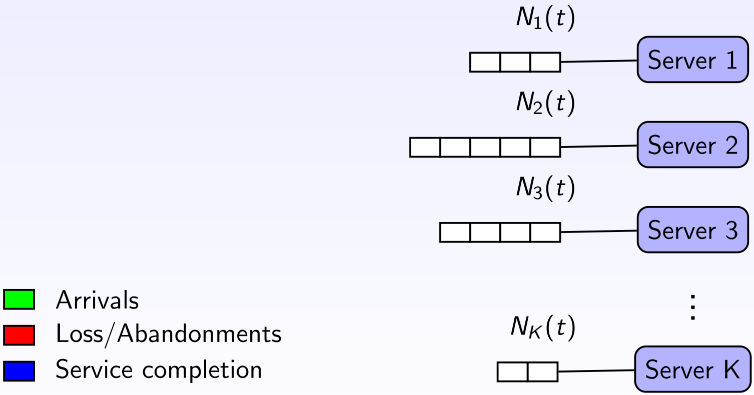
Which M classes activate?

Type of problems

- Call centers.
- Wireless downlink problem.

Motivation

Optimal load balancing:



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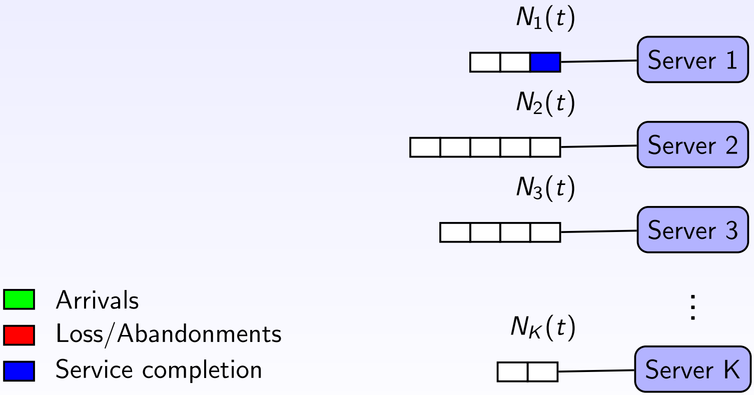

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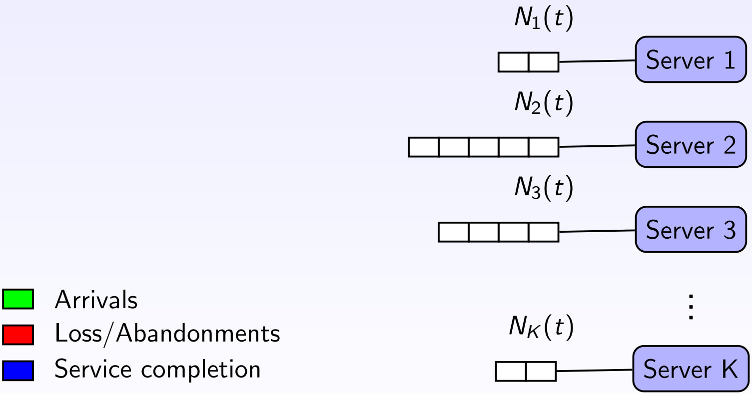

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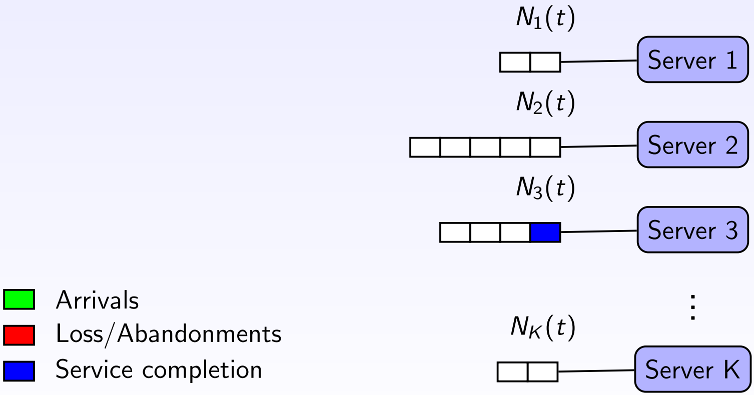

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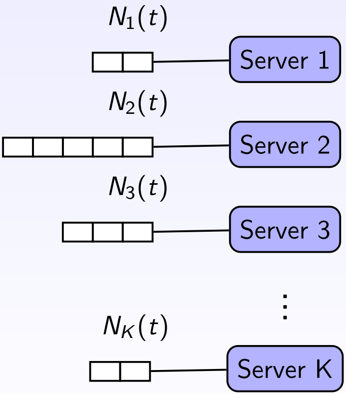

- Arrivals
- Loss/Abandonments
- Service completion

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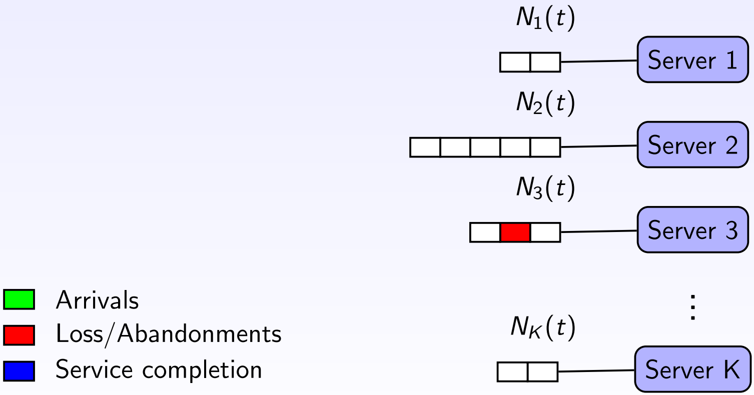

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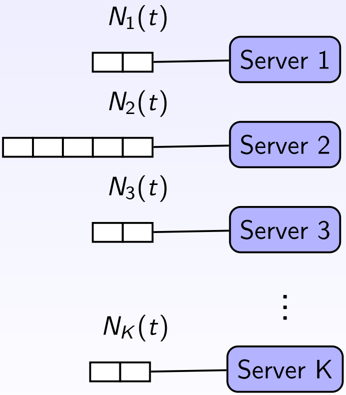

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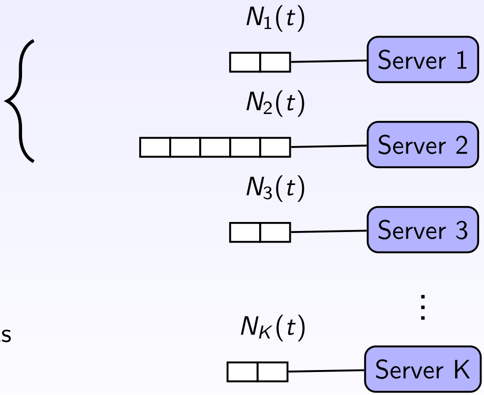
To which M servers to dispatch?



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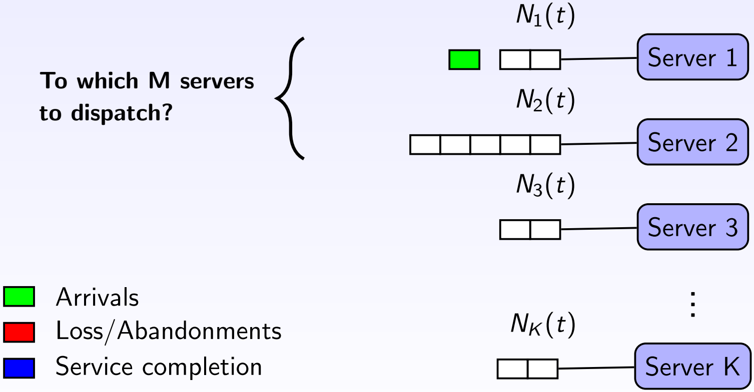

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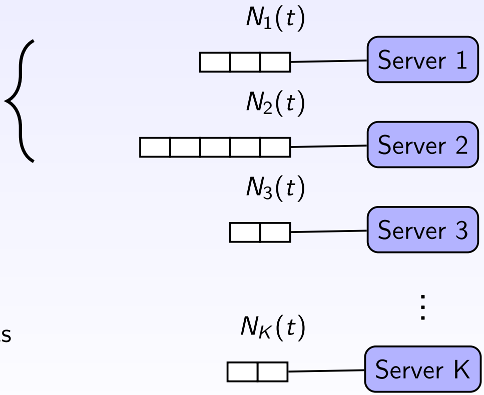

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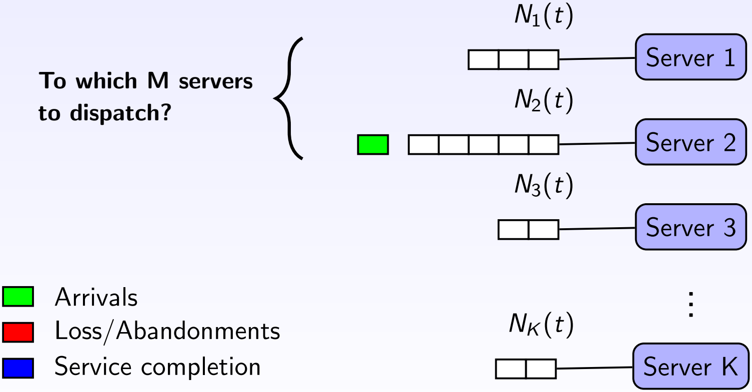

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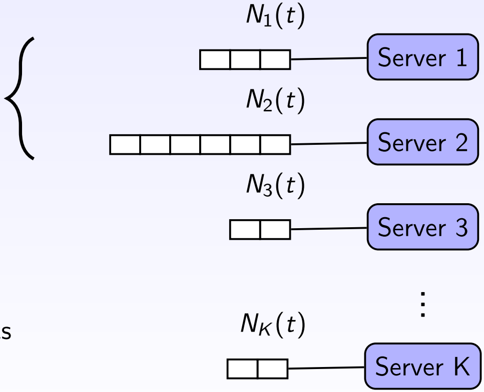

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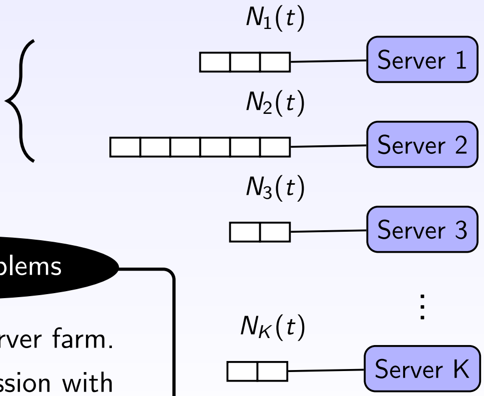

Type of problems

- Power-aware server farm.
- Routing+Admission with impatiency.

Objective

The objective is to find the **scheduling policies** that **minimize** the average-cost criteria

$$\sum_{k=1}^{K} \mathbb{E}\left[C_k(N_k^{\phi}, S_k^{\phi}(N^{\phi}))\right]$$

where $C_k(\cdot,\cdot)$ is the cost function under policy ϕ.

- N_k^{ϕ} number of class-k customers in system.
- $S_k^{\phi}(\vec{N}) = 0$, if ϕ prescribes to be passive, 1 otherwise.
- $\sum_k S_k^{\phi}(\vec{N}(t)) \leq M$.

How to solve?

1.-Heuristic: stochastic indices

Lagrangian relaxation

Original problem:

$$\min_{\phi} \sum_{k=1}^{K} \mathbb{E}\left[C_k(N_k^{\phi}, S_k^{\phi}(N^{\phi}))\right]$$

$$\sum_{k=1}^{K} S_k^{\phi}(\vec{N}^{\phi}(t)) \leq M$$

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Relax the constraint:

$$\min_{\phi} \sum_{k=1}^{K} \mathbb{E}\left[C_k(N_k^{\phi}, S_k^{\phi}(N^{\phi}))\right]$$

$$\mathbb{E}\left(\sum_{k=1}^{K} S_k^{\phi}(N^{\phi})\right) \leq M$$

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Relaxed problem:

$$\min_{\phi} \sum_{k=1}^{K} \mathbb{E}\left[C_k(N_k^{\phi}, S_k^{\phi}(N^{\phi}))\right] - \mathbf{W}\left(\mathbf{M} - \mathbb{E}\left(\sum_{\mathbf{k=1}}^{\mathbf{K}} \mathbf{S}_{\mathbf{k}}^{\phi}(\mathbf{N}^{\phi})\right)\right)$$

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K-dimensional problem $\Longrightarrow$ **unidimensional problem**

$$\min_{\phi} \mathbb{E}\left[C(N^{\phi}, S^{\phi}(N^{\phi}))\right] - \mathbf{W}\mathbb{E}\left(\mathbb{1}_{\mathbf{S}^{\phi}(\mathbf{N}^{\phi})=\mathbf{0}}\right)$$

Heuristic for original problem

Index type of policies are optimal for relaxed problem.

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The solution to relaxation **NOT feasible** for original problem.

$$\Downarrow$$

Need to define a **heuristic** for original model.

Example

$$\begin{aligned}
&\text{Class } 1 \rightarrow W_1(N_1) = 5N_1, \\
&\text{Class } 2 \rightarrow W_2(N_2) = 4, \\
&\text{Class } 3 \rightarrow W_3(N_3) = 2N_3^2,
\end{aligned}$$

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Set $W = 3$, then solution of relaxed

State	$W_1(N_1)$	$W_2(N_2)$	$W_3(N_3)$	Policy
$N(0) = (1, 1, 1)$	5	4	2	Serve Class 1 and 2

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Heuristic

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Outline

Whittle's Index: relaxed model

$$\min_{\phi} \sum_{k=1}^{K} \left(\mathbb{E}\left[C_k(N_k^{\phi}, S_k^{\phi}(N^{\phi})) \right] - W \left(M - \mathbb{E}\left(\sum_{k=1}^{K} S_k^{\phi}(N^{\phi}) \right) \right) \right)$$

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Where W serves as *subsidy for passivity* or penalization for activity. Drop dependency on k, classes are independent of each other.

$$\min_{\phi} \left(\mathbb{E}\left[C(N^{\phi}, S^{\phi}(N^{\phi})) \right] - W\mathbb{E}\left(\mathbb{1}_{S^{\phi}(N^{\phi})=0} \right) \right)$$

for each class k compute $W_k(N_k)$, amount of subsidy so as indifferent of the action in state N_k.

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Proposition: Whittle's index

$$W(n) = \frac{\mathbb{E}(C(N^n, S^n(N^n))) - \mathbb{E}(C(N^{n-1}, S^{n-1}(N^{n-1})))}{\sum_{m=0}^{n} \pi^n(m) - \sum_{m=0}^{n-1} \pi^{n-1}(m)},$$

where $\pi^n(m)$ is the steady-state distribution under threshold policy n.

Heuristic

- Solution of the relaxed problem is to serve all classes such that

$$W_k(N_k) \geq W.$$

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Disadvantages of Whittle's index:

- **Indexability** and **monotonicity** might be hard to prove.
- Very hard to obtain **qualitative insights**.

Fluid model of Relaxed Problem

Transition of stochastic model:

$$q_k^a(N_k, N_k + 1) = b_k^a(N_k), \quad q_k^a(N_k, N_k - 1) = d_k^a(N_k),$$

where $a \in \{0, 1\}$.

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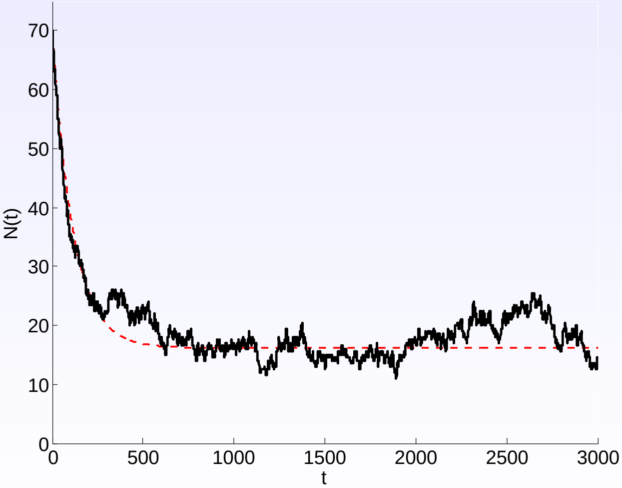
Approximate the relaxed stochastic model by the deterministic:

$$\frac{\mathrm{d}m^u(t)}{\mathrm{d}t} = (1-s^u(t))f^0(m^u(t)) + s^u(t)f^1(m^u(t)),$$

where u is the fluid control that determines $s^u(t) \in \{0,1\}$.

On statedy-state the stochastic process resembles the fluid approximation



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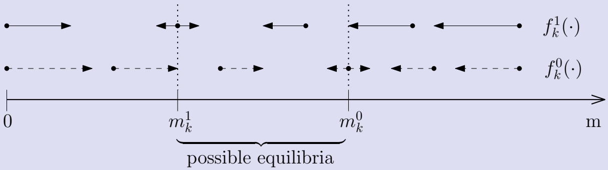
and the system evolves as the following ODE

$$\frac{\mathrm{d}m^u(t)}{\mathrm{d}t} = (1 - s^u(t))f^0(m^u(t)) + s^u(t)f^1(m^u(t)),$$

Propostition: an optimal solution

Let all possible equilibria be in $[m^1, m^0]$,



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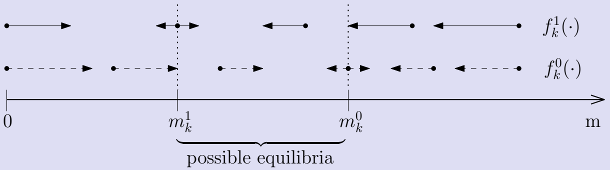

then, $s(t) = 1$, if $w(m) \geq W$ and $s(t) = 0$ otherwise. Where

$$w(m) = C(m,0) - C(m,1) + \begin{cases} w^{(1)}(m) & \text{if } m < m^1, \\ w^{(2)}(m) & \text{if } m \in [m^1, m^0], \\ w^{(3)}(m) & \text{if } m > m^0, \end{cases}$$

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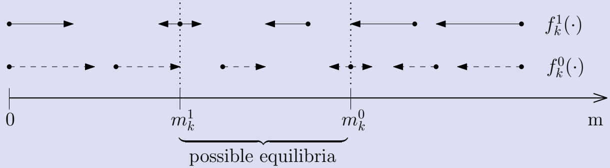

$$w^{(1)}(m) = (f^1(m) - f^0(m))\frac{C(m,1) - C(m^1,1)}{f^1(m)},$$

$$w^{(2)}(m) = \frac{(f^1(m) - f^0(m))(f^0(m)\frac{\mathrm{d}C(m,1)}{\mathrm{d}m} - f^1(m)\frac{\mathrm{d}C(m,0)}{\mathrm{d}m})}{f^0(m)\frac{\mathrm{d}f^1(m)}{\mathrm{d}m} - f^1(m)\frac{\mathrm{d}f^0(m)}{\mathrm{d}m}},$$

$$w^{(3)}(m) = (f^1(m) - f^0(m))\frac{C(m,0) - C(m^0,0)}{f^0(m)}.$$

Fluid index policy

- Monotonicity is a consequence of $w(\cdot)$ being non-decreasing.
- Indexability is immediate.

Definition: Fluid index policy

Assume the system's state is $\vec{N}(t) = \vec{n}$, the fluid index policy prescribes to serve the M bandits having currently the highest non-negative fluid index $w_k(n_k)$.

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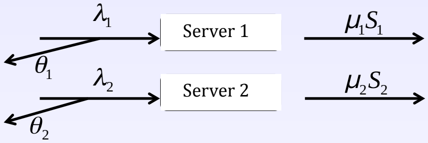
Advantages:

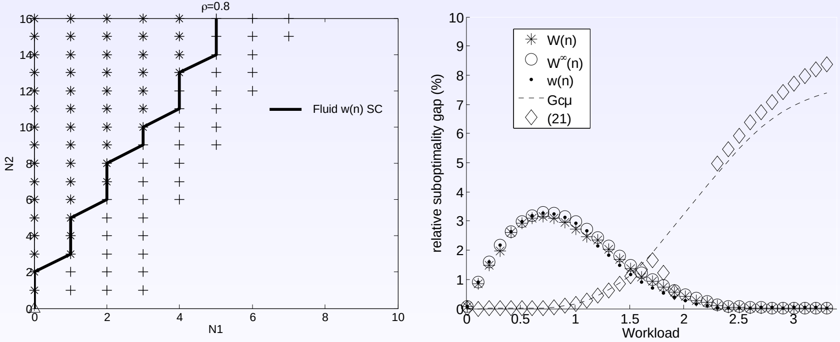
- non-decreasingness of $w(\cdot)$ easy to check.
- Fluid index in close-form expression.

scheduling a multi-class queue with impatient users

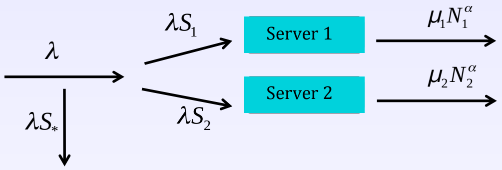


- Only one server can be activated.
- Objective: find the policy to minimize cost of holding + cost for abandoning.
- We compute $W_k(N_k)$ numerically,
- We obtain $w_k(n_k)$ in closed form expression.

Example 1


ρ=0.8
Fluid w(n) SC
N1
N2
W(n)
W∞(n)
w(n)
Gcμ
(21)
relative suboptimality gap (%)
Workload

Routing/blocking in a power-aware server-farm



- We can only dispatch to one queue.
- Speed scaling rule applied: $d_k^a(N_k) = \mu_k N_k^{\alpha}$.
- Objective: find the policy to minimize cost of holding + cost for rejection+power consumption.
- We compute $W_k(N_k)$ numerically,
- We obtain $w_k(n_k)$ in closed form expression.

Example 2

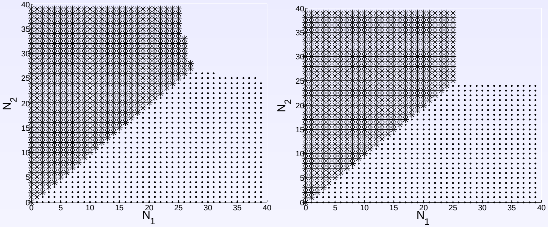


Figure: . (*) priority to class 1 (class 2). White area users blocked.

ρ	0.1	0.9	2
Fluid index	0.08704×10^{-7}	0.0821×10^{-7}	0.06099×10^{-7}
Whittle's index	0.08704×10^{-7}	0.0821×10^{-7}	0.06099×10^{-7}

Conclusion

- Derive an expression for the Whittle index in BD systems.
- Develop a new method: fluid index
  - Monotonicity easy to verify.
  - Closed form expression of W.

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Perspectives:

- Get sufficient conditions for the fluid index to be non-decreasing.
- Is the fluid index asymptotically optimal?
- When do the Whittle and the fluid index coincide?

Thank you!