Index policies for scheduling problems

Maialen Larrañaga
maialen.larranaga@laas.fr

U. Ayesta (CNRS, LAAS), I.M. Verloop (CNRS, IRIT).

YEQT, November 3.
Motivation

Optimal class selection:

\[ N_1(t) \]  
\[ \text{Server 1} \]

\[ N_2(t) \]  
\[ \text{Server 2} \]

\[ N_3(t) \]  
\[ \text{Server 3} \]

\[ N_K(t) \]  
\[ \ldots \]

\[ \text{Server K} \]

Arrivals

Loss/Abandonments

Service completion
Motivation

Optimal class selection:

\[ N_1(t) \]
\[ \text{Server 1} \]

\[ N_2(t) \]
\[ \text{Server 2} \]

\[ N_3(t) \]
\[ \text{Server 3} \]

\[ \vdots \]

\[ N_K(t) \]
\[ \text{Server K} \]

Arrivals
Loss/Abandonments
Service completion
Motivation

Optimal class selection:

\[ N_1(t) \]  \quad \text{Server 1}

\[ N_2(t) \]  \quad \text{Server 2}

\[ N_3(t) \]  \quad \text{Server 3}

\[ \ldots \]

\[ N_K(t) \]  \quad \text{Server K}

- Green: Arrivals
- Red: Loss/Abandonments
- Blue: Service completion
Motivation

Optimal class selection:

\[ N_1(t) \quad Server 1 \]
\[ N_2(t) \quad Server 2 \]
\[ N_3(t) \quad Server 3 \]
\[ N_K(t) \quad Server K \]

Arrivals
Loss/Abandonments
Service completion
Motivation

Optimal class selection:

\[ N_1(t) \]
\[ N_2(t) \]
\[ N_3(t) \]
\[ \ldots \]
\[ N_K(t) \]

Arrivals
Loss/Abandonments
Service completion
Motivation

Optimal class selection:

\[ N_1(t) \]
\[ N_2(t) \]
\[ N_3(t) \]
\[ \vdots \]
\[ N_K(t) \]

Server 1
Server 2
Server 3
Server K

Arrivals
Loss/Abandonments
Service completion
Motivation

Optimal class selection:

\[ N_1(t) \]
\[ N_2(t) \]
\[ N_3(t) \]
\[ \vdots \]
\[ N_K(t) \]

Arrivals
Loss/Abandonments
Service completion
Motivation

Optimal class selection:

\[ N_1(t) \]

\[ N_2(t) \]

\[ N_3(t) \]

\[ \cdots \]

\[ N_K(t) \]

Arrivals

Loss/Abandonments

Service completion
Motivation

Optimal class selection:

\begin{align*}
N_1(t) & \quad \text{Server 1} \\
N_2(t) & \quad \text{Server 2} \\
N_3(t) & \quad \text{Server 3} \\
\vdots & \quad \vdots \\
N_K(t) & \quad \text{Server } K
\end{align*}

- Arrivals
- Loss/Abandonments
- Service completion
Motivation

Optimal class selection:

\[ N_1(t) \]
\[ N_2(t) \]
\[ N_3(t) \]
\[ \vdots \]
\[ N_K(t) \]

Arrivals
Loss/Abandonments
Service completion
Motivation

Optimal class selection:

\[ N_1(t) \quad \text{Server 1} \]
\[ N_2(t) \quad \text{Server 2} \]
\[ N_3(t) \quad \text{Server 3} \]
\[ \vdots \]
\[ N_K(t) \quad \text{Server K} \]

- Arrivals
- Loss/Abandonments
- Service completion
Motivation

Optimal class selection:

\[ N_1(t) \]
\[ N_2(t) \]
\[ N_3(t) \]
\[ \vdots \]
\[ N_K(t) \]

Server 1
Server 2
Server 3
Server K

Arrivals
Loss/Abandonments
Service completion

Which M classes activate?
Motivation

Optimal class selection:

\[ N_1(t) \]  \( \square \) Server 1

\[ N_2(t) \]  \( \square \) Server 2

\[ N_3(t) \]  \( \square \) Server 3

\[ N_K(t) \]  \( \square \) Server K

Which M classes activate?

- Arrivals
- Loss/Abandonments
- Service completion
Motivation

Optimal class selection:

\[ N_1(t) \]
\[ N_2(t) \]
\[ N_3(t) \]
\[ \vdots \]
\[ N_K(t) \]

Which M classes activate?

Type of problems

- Call centers.
- Wireless downlink problem.
Motivation

Optimal load balancing:

- $N_1(t)$
  - Server 1
- $N_2(t)$
  - Server 2
- $N_3(t)$
  - Server 3
- $N_K(t)$
  - Server K

Legend:
- Arrivals
- Loss/Abandonments
- Service completion
Motivation

Optimal load balancing:

- $N_1(t)$
- $N_2(t)$
- $N_3(t)$
- $N_K(t)$

Arrivals
Loss/Abandonments
Service completion
Motivation

Optimal load balancing:

\[ N_1(t) \]

\[ N_2(t) \]

\[ N_3(t) \]

\[ \vdots \]

\[ N_K(t) \]

- **Arrivals**
- **Loss/Abandonments**
- **Service completion**
Motivation

Optimal load balancing:

Arrivals
Loss/Abandonments
Service completion
Motivation

Optimal load balancing:

Arrivals
Loss/Abandonments
Service completion

$N_1(t)$
$N_2(t)$
$N_3(t)$
$N_K(t)$

Server 1
Server 2
Server 3
Server K
Motivation

Optimal load balancing:

- $N_1(t)$
- $N_2(t)$
- $N_3(t)$
- $N_K(t)$

Arrivals
- Loss/Abandonments
- Service completion
Motivation

Optimal load balancing:

Arrivals
Loss/Abandonments
Service completion
Motivation

Optimal load balancing:

To which M servers to dispatch?

Arrivals
Loss/Abandonments
Service completion
Motivation

Optimal load balancing:

To which M servers to dispatch?

- Arrivals
- Loss/Abandonments
- Service completion
Motivation

Optimal load balancing:

To which M servers to dispatch?

- Arrivals
- Loss/Abandonments
- Service completion
Motivation

Optimal load balancing:

To which $M$ servers to dispatch?

- Arrivals
- Loss/Abandonments
- Service completion
Motivation

Optimal load balancing:

To which M servers to dispatch?

- Arrivals
- Loss/Abandonments
- Service completion
Motivation

Optimal load balancing:

To which M servers to dispatch?

Type of problems

- Power-aware server farm.
- Routing + Admission with impatience.
Objective

The objective is to find the scheduling policies that minimize the average-cost criteria

$$\sum_{k=1}^{K} \mathbb{E} \left[ C_k(N^\phi_k, S^\phi_k(N^\phi)) \right]$$

where $C_k(\cdot, \cdot)$ is the cost function under policy $\phi$.

- $N^\phi_k$ number of class-$k$ customers in system.
- $S^\phi_k(\vec{N}) = 0$, if $\phi$ prescribes to be passive, 1 otherwise.
- $\sum_k S^\phi_k(\vec{N}(t)) \leq M$. 

How to solve?

1.- Heuristic: stochastic indices

Original problem:

$$\min_{\phi} \sum_{k=1}^{K} \mathbb{E} \left[ C_k(N_k^\phi, S_k^\phi(N^\phi)) \right]$$

$$\sum_{k=1}^{K} S_k^\phi(\tilde{N}^\phi(t)) \leq M$$
How to solve?

1. Heuristic: stochastic indices

Relax the constraint:

$$\min_{\phi} \sum_{k=1}^{K} \mathbb{E} \left[ C_k(N^\phi_k, S^\phi_k(N^\phi)) \right]$$

$$\mathbb{E} \left( \sum_{k=1}^{K} S^\phi_k(N^\phi) \right) \leq M$$
How to solve?

1.- Heuristic: stochastic indices

Relaxed problem:

$$\min_{\phi} \sum_{k=1}^{K} \mathbb{E} \left[ C_k(N_k^\phi, S_k^\phi(N^\phi)) \right] - W \left( M - \mathbb{E} \left( \sum_{k=1}^{K} S_k^\phi(N^\phi) \right) \right)$$
How to solve?

1.- Heuristic: stochastic indices

Relaxed problem:

$$\min_{\phi} \sum_{k=1}^{K} \mathbb{E} \left[ C_k (N_k^{\phi}, S_k^{\phi}(N^{\phi})) \right] - W \left( M - \mathbb{E} \left( \sum_{k=1}^{K} S_k^{\phi}(N^{\phi}) \right) \right)$$

K-dimensional problem $\Rightarrow$ unidimensional problem
How to solve?

1.- Heuristic: stochastic indices

Relaxed problem:

\[
\min_{\phi} \sum_{k=1}^{K} \mathbb{E} \left[ C_k(N^\phi_k, S^\phi_k(N^\phi)) \right] - W \left( M - \mathbb{E} \left( \sum_{k=1}^{K} S^\phi_k(N^\phi) \right) \right)
\]

K-dimensional problem \implies\ unidimensional problem

\[
\min_{\phi} \mathbb{E} \left[ C(N^\phi, S^\phi(N^\phi)) \right] - W \mathbb{E} \left( 1_{S^\phi(N^\phi)=0} \right)
\]
Heuristic for original problem

Index type of policies are optimal for relaxed problem.
Heuristic for original problem

Index type of policies are optimal for relaxed problem.

The solution to relaxation **NOT feasible** for original problem.

⇓

Need to define a **heuristic** for original model.
Example

Class 1 $\rightarrow W_1(N_1) = 5N_1$,
Class 2 $\rightarrow W_2(N_2) = 4$,
Class 3 $\rightarrow W_3(N_3) = 2N_3^2$,
Example

Class 1 $\rightarrow W_1(N_1) = 5N_1$,
Class 2 $\rightarrow W_2(N_2) = 4$,
Class 3 $\rightarrow W_3(N_3) = 2N_3^2$,
Set $W = 3$, then solution of relaxed

<table>
<thead>
<tr>
<th>State $N(0) = (1, 1, 1)$</th>
<th>$W_1(N_1)$</th>
<th>$W_2(N_2)$</th>
<th>$W_3(N_3)$</th>
<th>Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>Serve Class 1 and 2</td>
</tr>
</tbody>
</table>
Example

Class 1 → \( W_1(N_1) = 5N_1 \),
Class 2 → \( W_2(N_2) = 4 \),
Class 3 → \( W_3(N_3) = 2N_3^2 \),

Set \( W = 3 \), then solution of relaxed

<table>
<thead>
<tr>
<th>State</th>
<th>( W_1(N_1) )</th>
<th>( W_2(N_2) )</th>
<th>( W_3(N_3) )</th>
<th>Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N(0) = (1, 1, 1) )</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>Serve Class 1 and 2</td>
</tr>
<tr>
<td>( N(t_1) = (0, 1, 1) )</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>Serve Class 2</td>
</tr>
</tbody>
</table>
Example

Class 1 → \( W_1(N_1) = 5N_1 \),
Class 2 → \( W_2(N_2) = 4 \),
Class 3 → \( W_3(N_3) = 2N_3^2 \),

Set \( W = 3 \), then solution of relaxed

<table>
<thead>
<tr>
<th>State</th>
<th>( W_1(N_1) )</th>
<th>( W_2(N_2) )</th>
<th>( W_3(N_3) )</th>
<th>Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N(0) = (1, 1, 1) )</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>Serve Class 1 and 2</td>
</tr>
<tr>
<td>( N(t_1) = (0, 1, 1) )</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>Serve Class 2</td>
</tr>
<tr>
<td>( N(t_2) = (0, 1, 2) )</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td>Serve Class 2 and 3</td>
</tr>
</tbody>
</table>
Example

Class 1 → $W_1(N_1) = 5N_1$,
Class 2 → $W_2(N_2) = 4$,
Class 3 → $W_3(N_3) = 2N_3^2$,

Set $W = 3$, then solution of relaxed

<table>
<thead>
<tr>
<th>State</th>
<th>$W_1(N_1)$</th>
<th>$W_2(N_2)$</th>
<th>$W_3(N_3)$</th>
<th>Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N(0) = (1, 1, 1)$</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>Serve Class 1 and 2</td>
</tr>
<tr>
<td>$N(t_1) = (0, 1, 1)$</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>Serve Class 2</td>
</tr>
<tr>
<td>$N(t_2) = (0, 1, 2)$</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td>Serve Class 2 and 3</td>
</tr>
</tbody>
</table>

Heuristic

<table>
<thead>
<tr>
<th>State</th>
<th>$W_1(N_1)$</th>
<th>$W_2(N_2)$</th>
<th>$W_3(N_3)$</th>
<th>Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N(0) = (1, 1, 1)$</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>Serve Class 1</td>
</tr>
</tbody>
</table>
Example

Class 1  $\rightarrow W_1(N_1) = 5N_1$,
Class 2  $\rightarrow W_2(N_2) = 4$,
Class 3  $\rightarrow W_3(N_3) = 2N_3^2$,
Set  $W = 3$, then solution of relaxed

<table>
<thead>
<tr>
<th>State</th>
<th>$W_1(N_1)$</th>
<th>$W_2(N_2)$</th>
<th>$W_3(N_3)$</th>
<th>Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N(0) = (1, 1, 1)$</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>Serve Class 1 and 2</td>
</tr>
<tr>
<td>$N(t_1) = (0, 1, 1)$</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>Serve Class 2</td>
</tr>
<tr>
<td>$N(t_2) = (0, 1, 2)$</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td>Serve Class 2 and 3</td>
</tr>
</tbody>
</table>

Heuristic

<table>
<thead>
<tr>
<th>State</th>
<th>$W_1(N_1)$</th>
<th>$W_2(N_2)$</th>
<th>$W_3(N_3)$</th>
<th>Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N(0) = (1, 1, 1)$</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>Serve Class 1</td>
</tr>
<tr>
<td>$N(t_1) = (0, 1, 1)$</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>Serve Class 2</td>
</tr>
</tbody>
</table>
Example

Class 1 $\rightarrow W_1(N_1) = 5N_1$,
Class 2 $\rightarrow W_2(N_2) = 4$,
Class 3 $\rightarrow W_3(N_3) = 2N_3^2$,
Set $W = 3$, then solution of relaxed

<table>
<thead>
<tr>
<th>State</th>
<th>$W_1(N_1)$</th>
<th>$W_2(N_2)$</th>
<th>$W_3(N_3)$</th>
<th>Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N(0) = (1, 1, 1)$</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>Serve Class 1 and 2</td>
</tr>
<tr>
<td>$N(t_1) = (0, 1, 1)$</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>Serve Class 2</td>
</tr>
<tr>
<td>$N(t_2) = (0, 1, 2)$</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td>Serve Class 2 and 3</td>
</tr>
</tbody>
</table>

Heuristic

<table>
<thead>
<tr>
<th>State</th>
<th>$W_1(N_1)$</th>
<th>$W_2(N_2)$</th>
<th>$W_3(N_3)$</th>
<th>Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N(0) = (1, 1, 1)$</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>Serve Class 1</td>
</tr>
<tr>
<td>$N(t_1) = (0, 1, 1)$</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>Serve Class 2</td>
</tr>
<tr>
<td>$N(t_2) = (0, 1, 2)$</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td>Serve Class 3</td>
</tr>
</tbody>
</table>
Outline

1. Lagrangian Relaxation

2. Fluid Model of Relaxed Problem

3. Case studies
   - Scheduling a multi-class queue with impatient users
   - Routing/blocking in a power-aware server-farm

4. Conclusion
Whittle’s Index: relaxed model

\[
\min_{\phi} \sum_{k=1}^{K} \left( \mathbb{E} \left[ C_k(N_k, S_k(N)) \right] - W \left( M - \mathbb{E} \left( \sum_{k=1}^{K} S_k(N) \right) \right) \right)
\]
Whittle’s Index: relaxed model

\[
\min_{\phi} \sum_{k=1}^{K} \left( \mathbb{E} \left[ C_k(N_k^\phi, S_k^\phi(N^\phi)) \right] - W \left( M - \mathbb{E} \left( \sum_{k=1}^{K} S_k^\phi(N^\phi) \right) \right) \right)
\]

Where \( W \) serves as subsidy for passivity or penalization for activity.
Whittle’s Index: relaxed model

\[
\min_{\phi} \sum_{k=1}^{K} \left( \mathbb{E} \left[ C_k(N^\phi_k, S^\phi_k(N^\phi)) \right] - W \left( M - \mathbb{E} \left( \sum_{k=1}^{K} S^\phi_k(N^\phi) \right) \right) \right)
\]

Where \( W \) serves as *subsidy for passivity* or penalization for activity. Drop dependency on \( k \), classes are independent of each other.

\[
\min_{\phi} \left( \mathbb{E} \left[ C(N^\phi, S^\phi(N^\phi)) \right] - W \mathbb{E} \left( 1_{S^\phi(N^\phi)=0} \right) \right)
\]

for each class \( k \) compute \( W_k(N_k) \), amount of subsidy so as indifferent of the action in state \( N_k \).
Consider Birth-and-Death processes.
Consider Birth-and-Death processes.

Proof optimality of threshold policies, denote by $\phi = n$. 

\begin{align*}
\mathbb{E} & \left[ C \left( N_n, S_n \right) \right] - W \mathbb{E} \left[ 1 \right] = 0 \\
\sum_{n=m} \pi_n (m) & - \sum_{n-1=m} \pi_{n-1} (m),
\end{align*}

where $\pi_n (m)$ is the steady-state distribution under threshold policy $n$. 


Consider Birth-and-Death processes.

Proof optimality of threshold policies, denote by \( \phi = n \).

\[
\left( \mathbb{E} [C(N^n, S^n(N^n))] - W \mathbb{E} (1_{S^n(N^n)=0}) \right) \\
= \left( \mathbb{E} [C(N^{n-1}, S^{n-1}(N^{n-1}))] - W \mathbb{E} (1_{S^{n-1}(N^{n-1})=0}) \right)
\]
Consider Birth-and-Death processes.

Proof optimality of threshold policies, denote by $\phi = n$.

\[
(\mathbb{E}[C(N^n, S^n(N^n))] - W\mathbb{E}(\mathbf{1}_{S^n(N^n)=0}))
= (\mathbb{E}[C(N^{n-1}, S^{n-1}(N^{n-1}))] - W\mathbb{E}(\mathbf{1}_{S^{n-1}(N^{n-1})=0}))
\]

**Proposition: Whittle’s index**

\[
W(n) = \frac{\mathbb{E}(C(N^n, S^n(N^n))) - \mathbb{E}(C(N^{n-1}, S^{n-1}(N^{n-1})))}{\sum_{m=0}^{n} \pi^n(m) - \sum_{m=0}^{n-1} \pi^{n-1}(m)},
\]

where $\pi^n(m)$ is the steady-state distribution under threshold policy $n$. 
Heuristic

- Solution of the relaxed problem is to serve all classes such that

\[ W_k(N_k) \geq W. \]

- Solution of relaxation infeasible for original model.
Heuristic

- Solution of the relaxed problem is to serve all classes such that
  \[ W_k(N_k) \geq W. \]

- Solution of relaxation **infeasible** for original model.

\[\Downarrow\]

- **Heuristic**: serve the class having the **highest non-negative** Whittle’s index \( W_k(n_k). \)
Heuristic

- Solution of the relaxed problem is to serve all classes such that
  \[ W_k(N_k) \geq W. \]

- Solution of relaxation infeasible for original model.

  \[ \Downarrow \]

- **Heuristic:** serve the class having the highest non-negative Whittle’s index \( W_k(n_k) \).

Disadvantages of Whittle’s index:
Heuristic

- Solution of the relaxed problem is to serve all classes such that

$$W_k(N_k) \geq W.$$ 

- Solution of relaxation infeasible for original model.

↓

- **Heuristic:** serve the class having the highest non-negative Whittle’s index $W_k(n_k)$.

Disadvantages of Whittle’s index:

- **Indexability** and **monotonicity** might be hard to prove.
Heuristic

- Solution of the relaxed problem is to serve all classes such that
  \[ W_k(N_k) \geq W. \]

- Solution of relaxation infeasible for original model.

  \[ \downarrow \]

- **Heuristic**: serve the class having the highest non-negative Whittle’s index \( W_k(n_k) \).

Disadvantages of Whittle’s index:

- **Indexability** and **monotonicity** might be hard to prove.
- Very hard to obtain **qualitative insights**.
Fluid model of Relaxed Problem

Transition of stochastic model:

\[ q_k^a(N_k, N_k + 1) = b_k^a(N_k), \quad q_k^a(N_k, N_k - 1) = d_k^a(N_k), \]

where \( a \in \{0, 1\} \).
Fluid model of Relaxed Problem

Transition of stochastic model:

\[ q_k^a(N_k, N_k + 1) = b_k^a(N_k), \quad q_k^a(N_k, N_k - 1) = d_k^a(N_k), \]

where \( a \in \{0, 1\} \).

Define the drifts

\[ f^0(m) := b^0(m) - d^0(m), \]
\[ f^1(m) := b^1(m) - d^1(m), \]
Fluid model of Relaxed Problem

Transition of stochastic model:

\[ q^a_k(N_k, N_k + 1) = b^a_k(N_k), \quad q^a_k(N_k, N_k - 1) = d^a_k(N_k), \]

where \( a \in \{0, 1\} \).

Define the drifts

\[ f^0(m) := b^0(m) - d^0(m), \]
\[ f^1(m) := b^1(m) - d^1(m), \]

Approximate the relaxed stochastic model by the deterministic:

\[ \frac{dm^u(t)}{dt} = (1 - s^u(t))f^0(m^u(t)) + s^u(t)f^1(m^u(t)), \]

where \( u \) is the fluid control that determines \( s^u(t) \in \{0, 1\} \).
On statedy-state the stochastic process resembles the fluid approximation.
Objective of fluid approximation

Objective: find $u$ so as to minimize bias-optimal criteria:

\[
\min_s \left( \int_0^\infty \left( C(m(t), s(t)) - W(1 - s(t)) \right) - EC^*(W) \right) dt
\]

where $EC^*(W)$ is the cost at equilibrium, that is,

\[
EC^*(W) := (1 - s^* C(m^*, 0)) + s^* C(m^*, 1) - W(1 - s^*).
\]
Objective of fluid approximation

**Objective:** find $u$ so as to minimize **bias-optimal** criteria:

$$
\min_{s(t)} \int_0^\infty (C(m(t), s(t)) - W(1 - s(t))) - EC^*(W) \, dt
$$
Objective of fluid approximation

**Objective:** find $u$ so as to minimize **bias-optimal** criteria:

$$\min_{s(t)} \int_0^\infty (C(m(t), s(t)) - W(1 - s(t))) - EC^*(W) \, dt$$

where $EC^*(W)$ is the cost at equilibrium, that is,

$$EC^*(W) := (1 - s^*) C(m^*, 0) + s^* C(m^*, 1) - W(1 - s^*).$$
Objective of fluid approximation

**Objective**: find $u$ so as to minimize **bias-optimal** criteria:

$$\min_{s(t)} \int_0^\infty \left( C(m(t), s(t)) - W(1 - s(t)) \right) - EC^*(W) \, dt$$

where $EC^*(W)$ is the cost at equilibrium, that is,

$$EC^*(W) := (1 - s^*) C(m^*, 0) + s^* C(m^*, 1) - W(1 - s^*).$$

and the system evolves as the following ODE

$$\frac{dm^u(t)}{dt} = (1 - s^u(t)) f^0(m^u(t)) + s^u(t) f^1(m^u(t)),$$
**Proposition: an optimal solution**

Let all possible equilibria be in \([m^1, m^0]\),
Propostition: an optimal solution

Let all possible equilibria be in \([m^1, m^0]\),

then, \(s(t) = 1\), if \(w(m) \geq W\) and \(s(t) = 0\) otherwise. Where

\[
w(m) = C(m, 0) - C(m, 1) + \begin{cases} 
  w^{(1)}(m) & \text{if } m < m^1, \\
  w^{(2)}(m) & \text{if } m \in [m^1, m^0], \\
  w^{(3)}(m) & \text{if } m > m^0,
\end{cases}
\]
### Proposition: an optimal solution

Let all possible equilibria be in \([m^1, m^0]\),

\[
\text{possible equilibria}
\]

\[
0 \quad m_k^1 \quad m_k^0 \quad m
\]

\[
w^{(1)}(m) = (f^1(m) - f^0(m)) \frac{C(m, 1) - C(m^1, 1)}{f^1(m)},
\]

\[
w^{(2)}(m) = (f^1(m) - f^0(m)) \left( f^0(m) \frac{dC(m, 1)}{dm} - f^1(m) \frac{dC(m, 0)}{dm} \right) \frac{f^0(m)}{f^0(m)} - f^1(m) \frac{df^0(m)}{dm},
\]

\[
w^{(3)}(m) = (f^1(m) - f^0(m)) \frac{C(m, 0) - C(m^0, 0)}{f^0(m)}.
\]
Fluid index policy

- Monotonicity is a consequence of $w(\cdot)$ being non-decreasing.
- Indexability is immediate.

**Definition: Fluid index policy**

Assume the system’s state is $\tilde{N}(t) = \tilde{n}$, the fluid index policy prescribes to serve the $M$ bandits having currently the highest non-negative fluid index $w_k(n_k)$. 
Fluid index policy

- Monotonicity is a consequence of $w(\cdot)$ being non-decreasing.
- Indexability is immediate.

**Definition: Fluid index policy**

Assume the system’s state is $\vec{N}(t) = \vec{n}$, the fluid index policy prescribes to serve the $M$ bandits having currently the highest non-negative fluid index $w_k(n_k)$.

**Advantages:**

- non-decreasingness of $w(\cdot)$ easy to check.
- Fluid index in close-form expression.
scheduling a multi-class queue with impatient users

- Only one server can be activated.
- Objective: find the policy to minimize cost of holding + cost for abandoning.
- We compute $W_k(N_k)$ numerically,
- We obtain $w_k(n_k)$ in closed form expression.
Example 1
Routing/blocking in a power-aware server-farm

- We can only dispatch to one queue.
- Speed scaling rule applied: $d_k^a(N_k) = \mu_k N_k^{\alpha}$.
- Objective: find the policy to minimize cost of holding + cost for rejection + power consumption.
- We compute $W_k(N_k)$ numerically.
- We obtain $w_k(n_k)$ in closed form expression.
Case studies
Routing/blocking in a power-aware server-farm

Example 2

Figure: (*) priority to class 1 (class 2). White area users blocked.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>0.1</th>
<th>0.9</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluid index</td>
<td>$0.08704 \times 10^{-7}$</td>
<td>$0.0821 \times 10^{-7}$</td>
<td>$0.06099 \times 10^{-7}$</td>
</tr>
<tr>
<td>Whittle’s index</td>
<td>$0.08704 \times 10^{-7}$</td>
<td>$0.0821 \times 10^{-7}$</td>
<td>$0.06099 \times 10^{-7}$</td>
</tr>
</tbody>
</table>
Conclusion

- Derive an expression for the Whittle index in BD systems.
- Develop a new method: fluid index
  - Monotonicity easy to verify.
  - Closed form expression of $W$. 

Perspectives:
- Get sufficient conditions for the fluid index to be non-decreasing.
- Is the fluid index asymptotically optimal?
- When do the Whittle and the fluid index coincide?
Conclusion

- Derive an expression for the Whittle index in BD systems.
- Develop a new method: fluid index
  - Monotonicity easy to verify.
  - Closed form expression of $W$.

Perspectives:

- Get sufficient conditions for the fluid index to be non-decreasing.
- Is the fluid index asymptotically optimal?
- When do the Whittle and the fluid index coincide?
Thank you!