Convergence rates of Laplace-transform based estimators

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Outline

1. Motivation from practical problem
2. Convergence rates of Laplace-transform based estimators
3. Applications

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Motivation

- Surfnet: network operator for universities and hospitals.

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- Contractual agreements about network usage & performance.
- Validation of these agreements via probing.
- More probes = higher accuracy. But: probes affect speed of regular network traffic.

Question from Surfnet: What is the optimal probing frequency?

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Goal: Estimate $P(W > w)$, for some predefined $w > 0$.

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Disadvantage of empirical estimator:

1. Hard to analyze finite-sample estimation error as function of δ, n (due to correlations).
2. No intervention at rare events $P(W > w)$.

Alternative Estimator

$$Q_i := W_{(i+1)\delta} - (W_{i\delta} - \delta)$$

If $W_{i\delta} \geq \delta$, then $Q_i =$ the amount of arrived work during $((i-1)\delta, i\delta]$.

If $W_{i\delta} < \delta$, then
$Q_i \geq$ the amount of arrived work during $((i-1)\delta, i\delta] \geq W_{i\delta}$.

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Advantage: i.i.d. rvs: easier to analyze.
Disadvantage: neglect censored observations.

Alternative Estimator

Write $\tilde{X}(s) = \mathbb{E}[\exp(-sX)]$, $s \in \mathbb{C}_+$.

Pollaczek-Khinchine formula.

$$\tilde{W}(s) = \frac{s(1-\lambda\mathbb{E}[B])}{s-\lambda+\lambda\tilde{B}(s)}, \quad (s \in \mathbb{C}_+)$$

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Estimator of $P(W \leq w)$:

1. Replace $\mathbb{E}[Q]$ and $\tilde{Q}(s)$ by $n^{-1}\sum_{i=1}^{n} Q_i$ and $n^{-1}\sum_{i=1}^{n} \exp(-sQ_i)$.

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Goal: find useful bounds on estimation error

Related literature

Decompounding: Buchmann and Grübel (2003,2004), Van Es et al. (2007), Bogsted and Pitts (2010), Duval (2013), Comte et al. (2014abc).

Estimation in queueing and insurance models: Glynn and Torres (1996, 1997), Zeevi and Glynn (2004), Hall and Park (2004), Hansen and Pitts (2006), Mnatsakov et al. (2008), Zhang and Xu (2010), Shimizu (2012), Blanchet et al. (2013).

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Outline

Laplace-transform based estimation

General setting:

- $\mathcal{X}$ a collection of nonnegative random variables.
- $\tilde{\mathcal{X}}$ Laplace transforms of $\mathcal{X}$.
- Known map

$$\Psi : \tilde{\mathcal{X}} \to \{g : \mathbb{C}_+ \to \mathbb{C}\},$$

with $\tilde{Y} = \Psi \tilde{X}$ for some unknown X, Y nonnegative.

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  with $\tilde{Y} = \Psi \tilde{X}$ for some unknown X, Y nonnegative.
- Estimate cdf $F^Y(w)$ of Y, for given $w > 0$, based on i.i.d. sample $X_1, \ldots, X_n$ of X.

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1. Is $\tilde{X}_n$ in the domain of Ψ?
2. Is $\tilde{F}_n^Y$ a Laplace transform?

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On $E_n^{\mathfrak{c}}$, define $F_n^Y(w) \in [0, 1]$ arbitrarily.

Main result: conditions

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Assume

(A1) $\mathbb{P}(E_n^{\mathfrak{c}}) = O(n^{-1/2})$,

(A2) $F^Y(y)$ is C^1 on $[0, \infty)$, and twice differentiable at $y = w$,

(A3) There are k_1, k_2, and nonnegative random variables $(Z_n)_{n \in \mathbb{N}}$ with $\sup_{1<p<2} \mathbb{E}[|Z_n|^p] \leq k_2 n^{-1/2}$ for all $n \in \mathbb{N}$, such that, on E_n,

$$|(\Psi \tilde{X}_n)(s) - (\Psi \tilde{X})(s)| \leq k_1 |\tilde{X}_n(s) - \tilde{X}(s)| + Z_n \text{ a.s.}$$

for all $s \in c + \mathrm{i}y$, $n \in \mathbb{N}$ and $y \in [-\sqrt{n}, \sqrt{n}]$.

Main result

Convergence rate

Suppose (A1)-(A3). Then there is a $C > 0$ such that, for all $n \in \mathbb{N}$,

$$\mathbb{E}[|F_n^Y(w) - F^Y(w)|] \leq Cn^{-1/2}\log(n+1).$$

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Note: $n^{-1/2}$ optimal rate (take Ψ identity, apply Cramér Rao bound).

Remarks

Proof based on standard probabilistic inequalities and results on convergence rate of characteristic functions.

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Ill-posedness of the problem (Laplace inversion not continuous operator) does not play a rôle when estimating cdf.

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Application 1: Queueing example

The M/G/1 queueing example, with

- $\mathcal{X} = \{X = \sum_{i=1}^{\text{Poiss}(\lambda\delta)} B_i \mid 0 < \mathbb{E}[X] < \delta, E[B_1^2] < \infty\}$;
- $E_n = \{0 \leq n^{-1} \sum_{i=1}^{n} X_i < \delta\}$;
- $(\Psi\tilde{X})(s) = \frac{s(1-\delta^{-1}\mathbb{E}[X])}{s+\delta^{-1}\mathrm{Log}(\tilde{X}(s))}$;
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satisfies the assumptions (A1)-(A3), and thus yields

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for all n and some $C > 0$.

Application 2: Decompounding

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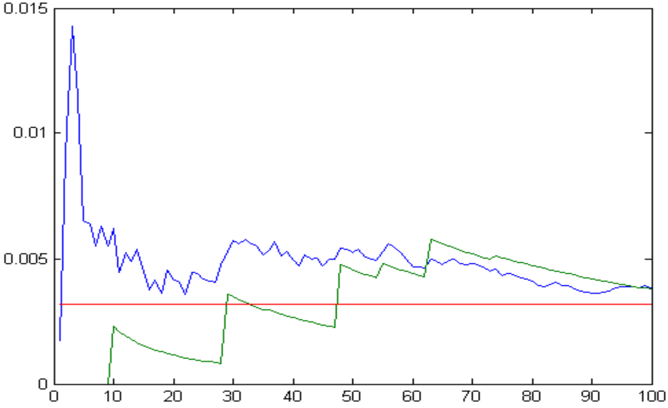
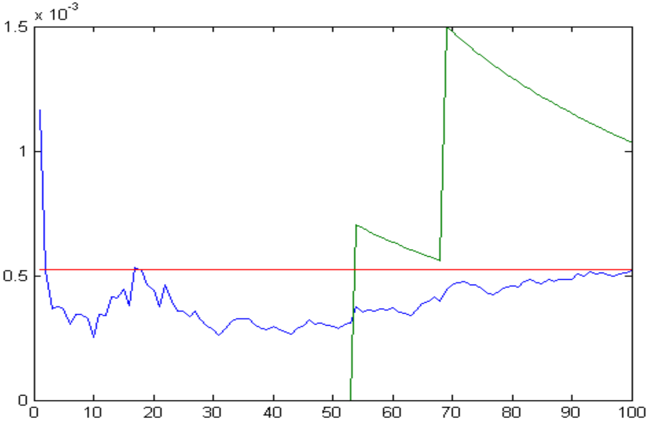
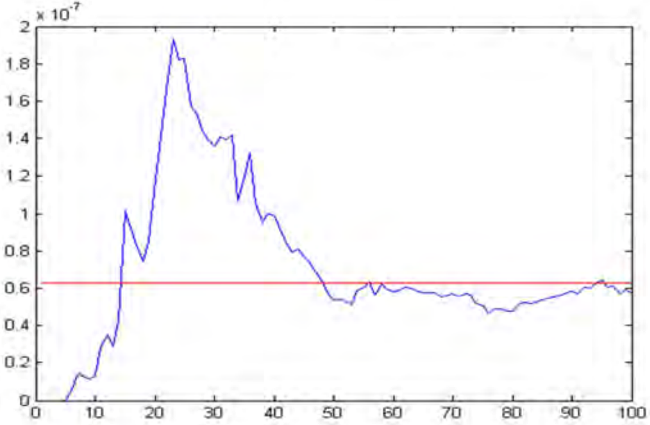
Numerical: *P(W>w)* for *w=90*



Numerical: *P(W>w)* for *w=120*



Numerical: *P(W>w)* for *w=150*



Main message:

If you estimate $P(Y \leq w)$ from a sample $X_1, \ldots, X_n$
and you know a relation between the Laplace transforms:

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Nice future challenge: incorporate interval-observations.