Resource sharing with logarithmic weights

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joint with P. Robert.

Q°: J clients sharing a single server, how can we allocate reasonable fractions of capacity of the server in such a way that none of them monopolizes the resource when it is too big?

More precisely, suppose that each client j is represented by a queue $L_j(t)$ and

$$L_j \to L_j + 1 \quad \text{at rate } \lambda_j$$

$$L_j \to L_j - 1 \quad \text{at rate } \mu_j \frac{f(L_j)}{\sum_{i=1}^{J} f(L_j)}$$

for some function f (that should be increasing, tending to $+\infty$ as the argument $\to \infty$).

How do we choose f for the allocation process to be fair in some sense?

Classical discipline: $f(\ell) = \ell$ or $f(\ell) = \ell^\alpha$.

But if some queues become much bigger than others, the latter will receive a very tiny fraction of the capacity.

$f(\ell) = \lg(1+\ell)$? That's the case we'll consider from now on.

Suppose $J = 2$, and $L_1(0) = 0$, $L_2(0) = N$ (where N will tend to ∞). To set the notation:

$$\rho_i = \frac{\lambda_i}{\mu_i} \quad \text{load of node/queue } i$$

<u>Q°</u>: Behavior of the system (L_1, L_2) as $N \to \infty$?

<u>Main result (but many variants exist)</u>: Suppose $\rho_1 < \frac{1}{2}$ and $\rho_1 + \rho_2 < 1$, then we have the convergence in distribution

$$\lim_{N\to\infty} \left(\left(\frac{L_1(Nt)}{N^{\alpha^*}}, \frac{L_2(Nt)}{N}\right)\right)_{0<t<t_0} = \left(\left(\gamma(t)^{\alpha^*}, \gamma(t)\right)\right)_{0<t<t_0}$$

where

$$\alpha^* = \frac{\rho_1}{1-\rho_1} < 1 \text{ since } \rho_1 < \frac{1}{2}$$

$$\gamma(t) = 1 - \mu_2(1-\rho_1-\rho_2)t = 1 + \mu_2(\rho_2 - (1-\rho_1))t$$
$$= 1 + (\lambda_2 - \mu_2(1-\rho_1))t$$

$$t_0 = \frac{1}{\mu_2(1-\rho_1-\rho_2)}$$

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① We have to understand what happens at the very beginning (observe that $t=0$ is forbidden here)

② Then on the fluid scale we have $L_1 = L_2^{\alpha^*}$ until the time $t_0 N$ at which L_2 reaches "0".

<u>Very beginning</u>: Interval of time $[0, N^\beta]$ with $\beta < 1$.
Then since $L_2(0) = N$ and at most $O(N^t)$ arrivals and departures in queue 2, we can consider that $L_2(s) \approx N$ for $s \in [0, N^\beta]$.

Means that growth rate of L_1 is $\Delta L_1 = \lambda_1 - \mu_1 \dfrac{\lg L_1}{\lg L_1 + \lg N}$

$$> 0 \text{ as long as } \frac{\lg L_1}{\lg N} < \frac{\rho_1}{1-\rho_1} = \alpha^*.$$

So $L_1(N^s) \propto N^s$ for $s < \alpha^*$ and using this information in the evolution equation of L_1, you can show that

$$\left(\frac{L_1(N^s)}{N^s}\right)_{s\in(0,\alpha^*)} \xrightarrow[N\to\infty]{} \left(\lambda_1 - \mu_1 \frac{s}{1+s}\right)_{s\in(0,\alpha^*)}$$ (unif. over compact intervals)

The r.h.s tends to 0 as $s \to \alpha^*$, but it is possible to show that L_1 reaches δN^{α^*} in $\Theta(N^{\alpha^*} \lg N)$ for any $\delta < 1$.

Coming back to the drift: $\Delta L_1 < 0$ if $L_1 > N^{\alpha^*}$
$\Delta L_1 > 0$ if $L_1 < N^{\alpha^*}$

but potentially large fluctuations due to the Poissonian arrival and departures.

who wins ?

Equilibrium: To understand what happens we need to consider the timescale $(N^{\alpha^*} \lg N\, t,\ t \geq 0)$. Then the reasoning "propagates" (with some efforts!)

Write $L_1(N^{\alpha^*} \lg N\, t) = h(t)\, N^{\alpha^*}$ with $h(0) = \delta \in (0,1]$.

Then $\Delta L_1(N^{\alpha^*} \lg N\, t) = \mu_1(N^{\alpha^*} \lg N)\left(\rho_1 - \frac{\lg h(t) + \alpha^* \lg N}{\lg h(t) + (\alpha^*+1)\lg N}\right)$.

Assuming that $h(t)$ is $\Theta(1)$, a Taylor expansion gives

$$= -\frac{\mu_1 N^{\alpha^*} \cancel{\lg N}}{(1+\alpha^*)^2} \frac{\lg h(t)}{\cancel{\lg N}}$$

so that $\underset{\substack{\downarrow \\ h'(t)}}{\Delta h(t)} = -\underbrace{\frac{\mu_1}{(1+\alpha^*)^2}}_{C>0} \lg h(t)$

$\Rightarrow$ if $\delta = 1$, $h(t) = 1$ and if $\delta < 1$, $h'(t) = -C \lg h(t)$.

In the end, we indeed have

$$\left(\frac{L_1(N^{\alpha^*}\lg N\, t)}{N^{\alpha^*}}\right)_{t\geq 0} \xrightarrow[N\to\infty]{(d)} (h(t))_{t\geq 0} \quad \text{described above}$$

and we can even show that

$$\left(\frac{L_1(N^{\alpha^*}\lg N\, t) - h_N(t)\, N^{\alpha^*}}{\sqrt{N^{\alpha^*}\lg N}}\right)_{t\geq 0} \longrightarrow \text{OU - like process}$$

where everything in this statement will remain vague and $h_N(t) = h(t) + \mathcal{O}\left(\frac{1}{\lg N}\right)$.

Conclusion: the force that brings L_1 back to $L_2^{\alpha^*}$ wins over the Poissonian fluctuations and the relation $L_1 \approx L_2^{\alpha^*}$ remains valid as long as L_2 is large.

Fairness: L_1 is stable and receives a fraction ρ_1 of the capacity even though $L_1 \ll L_2$.