Fluid flow at porous liquid interfaces

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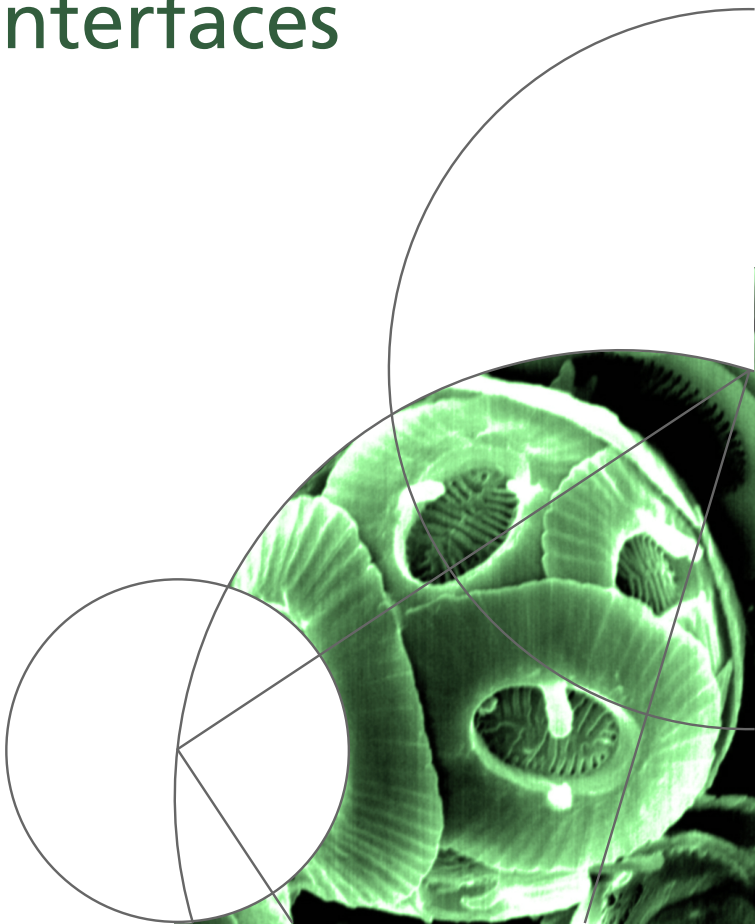
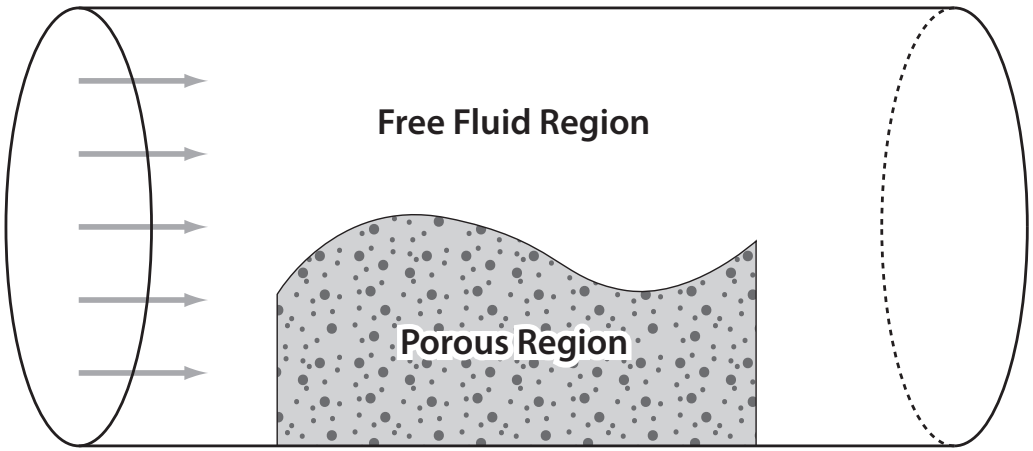


Motivation

- Porous medium inside of a fluid flow
- Interest: Fluid behaviour at the porous-liquid interface of a curved porous medium

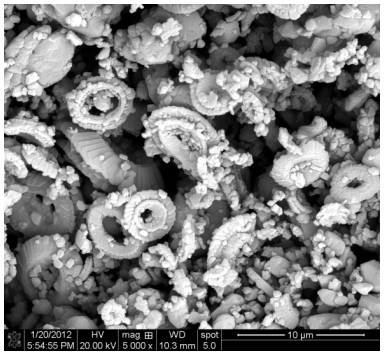


Overview

Idea of periodic homogenization

The concept

Møns Klint



SEM image of chalk

2 different scales:

Macro-Scale

- Size of cm – km
- Modelling is complicated
- Simulations are possible

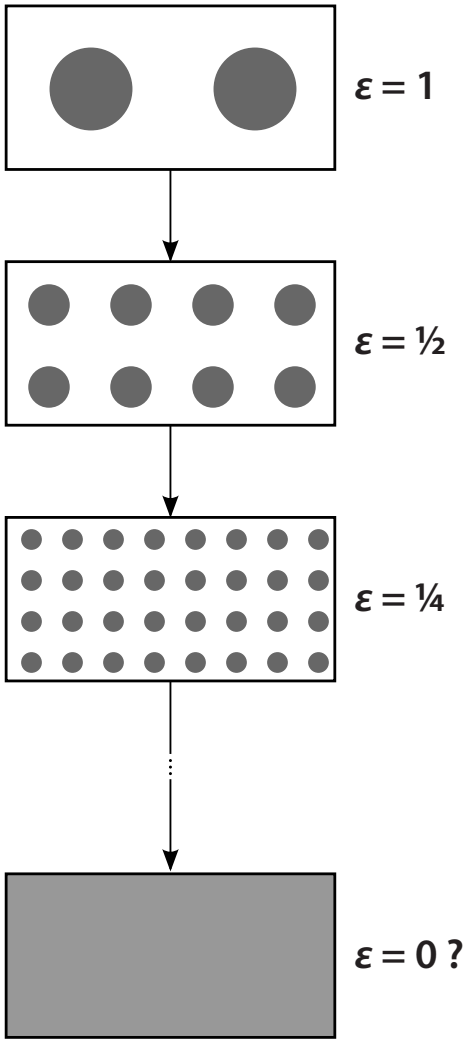
Micro-Scale

- Size of nm – mm
- Modelling is possible
- Simulations are complicated/impossible

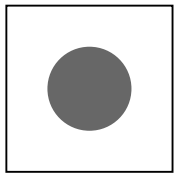
Main interest:
How to get from the micro- to the macro-scale?

Idea of periodic homogenization

The mathematics



- Choose a sequence of scale-parameters $\varepsilon > 0$
- Fill the domain of interest with ε-scaled versions of a reference cell $Y = [0, 1)^n$



- ε fixed: Problem of the form

$$\mathcal{L}^\varepsilon u^\varepsilon = f$$

- We are looking for u^0 and $\mathcal{L}^0$, such that $u^\varepsilon \longrightarrow u^0$ for $\varepsilon \to 0$ and

$$\mathcal{L}^0 u^0 = f$$

$\mathcal{L}^\varepsilon$, $\mathcal{L}^0$: differential operators

Mathematical methods

Methods for deriving the limit equation:

- Asymptotic expansion

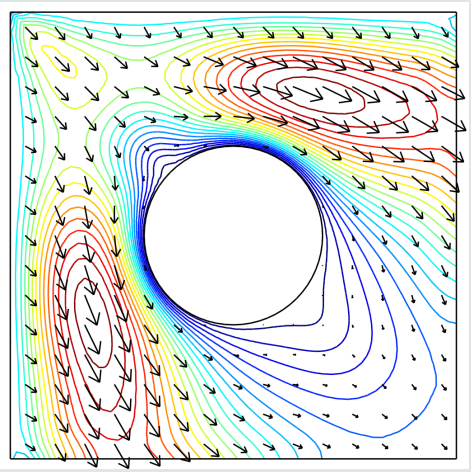
$$u^\varepsilon(x) = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \dots \Big|_{y=\frac{x}{\varepsilon}}$$

  insert in equation; calculate u_0, u_1, ...

- Two-scale convergence
- Periodic Unfolding

Mathematical Description of Fluids

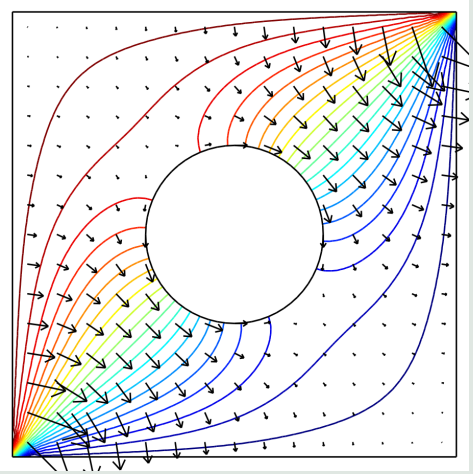
Stokes-Equation



- Incompressible free flow
- Small Reynolds number

$$-\mu \Delta u + \nabla p = f$$
$$\mathrm{div}(u) = 0$$

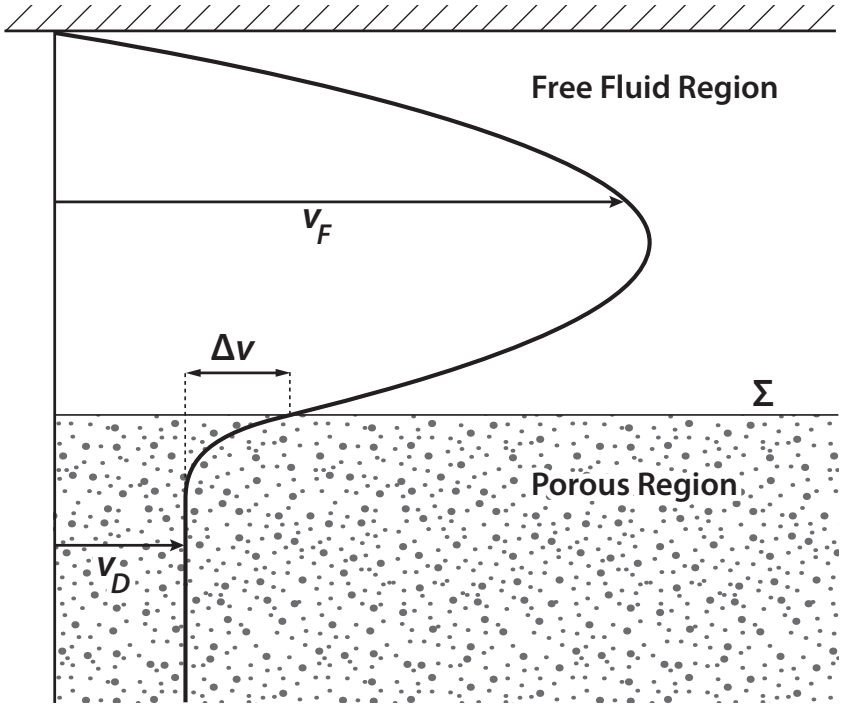
Darcy's law



- Incompressible free fluid in a porous medium
- Effective velocity

$$u = \frac{1}{\mu} K(f - \nabla p)$$
$$\mathrm{div}(u) = 0$$

The Boundary Condition of Beavers and Joseph



- Only for straight interfaces

Beavers/Joseph '67

Velocity normal to Σ is continuous.

Velocity tangential to Σ has a jump:

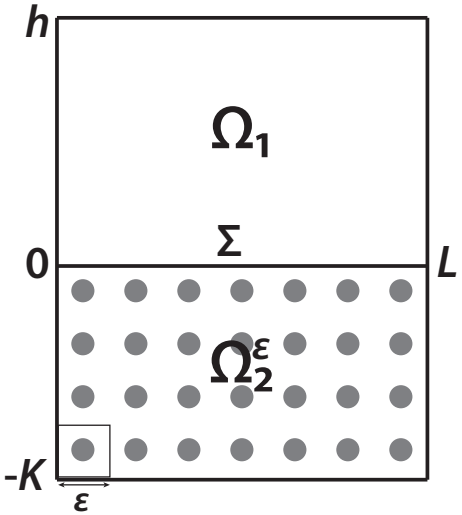
$$\left(v_F - v_D\right) \cdot \tau = \frac{1}{\alpha} K^{\frac{1}{2}} \left(\nabla v_F v\right) \cdot \tau$$

Background

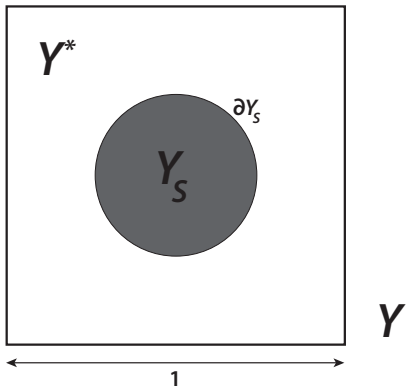
Beavers & Joseph '67	Derivation of the BC via experiments
$\Downarrow$	
Saffman '71	Proof via stochastic arguments
$\Downarrow$	
Jäger & Mikelic '96	Preparatory constructions based on homogenization
$\Downarrow$	
Jäger & Mikelic '00	Interface law for $f = 0$
$\Downarrow$	
Marciniak-Czochra & Mikelic '12	Interface law for general f

The Setting

Domain:



Reference cell:



$$
\begin{aligned}
-\Delta u^\varepsilon + \nabla p^\varepsilon &= f \text{ in } \Omega^\varepsilon \\
\operatorname{div} u^\varepsilon &= 0 \text{ in } \Omega^\varepsilon \\
u^\varepsilon &= 0 \text{ on } \partial\Omega^\varepsilon \backslash (\{x_1 = 0\} \cup \{x_1 = L\})
\end{aligned}
$$

$u^\varepsilon, p^\varepsilon$ are L-periodic in x_1

$f \in \mathcal{C}^\infty(\Omega)^2$, L periodic in x_1.

First approximations

Maciniak-Czochra/Mikelic (2012)

Impermeable interface approximation

$$-\Delta u^0 + \nabla p^0 = f \text{ in } \Omega_1$$

$$\operatorname{div} u^0 = 0 \text{ in } \Omega_1$$

$$u^0 = 0 \text{ on } \partial\Omega_1 \setminus (\{x_1 = 0\} \cup \{x_1 = L\})$$

u^0, p^0 are L-periodic in x_1

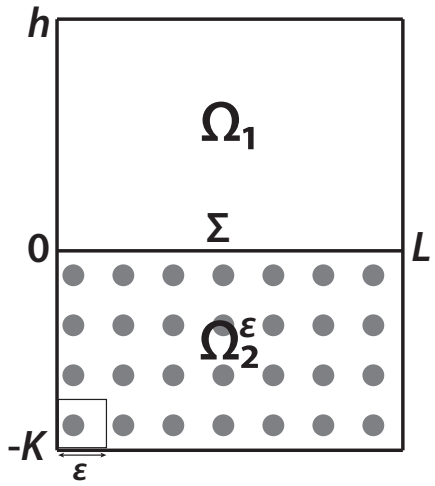
Porous approximation

$$\operatorname{div}(A(f - \nabla p^D)) = 0 \text{ in } \Omega_2$$

$$p^D = p^0 + C_\omega^{bl} \frac{\partial u_1^0}{\partial x_2}(x_1, 0) \text{ on } \Sigma$$

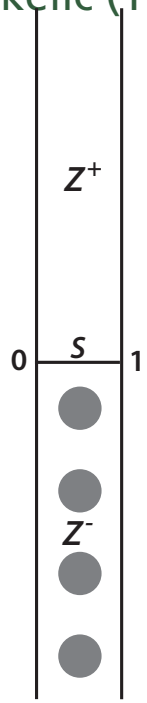
$$A(f - \nabla p^D) \cdot e_2 = 0 \text{ on } \{x_2 = -K\}$$

p^D is L-periodic in x_1



Auxiliary Problem: Boundary Layer

Jäger/Mikelic (1996)



$$-\Delta_y \beta^{bl} + \nabla_y \omega^{bl} = 0 \text{ in } Z^+ \cup Z^-$$

$$\operatorname{div}_y \beta^{bl} = 0 \text{ in } Z^+ \cup Z^-$$

$$[\beta^{bl}]_S(\cdot, 0) = 0 \text{ on } S$$

$$[(\nabla_y \beta^{bl} - \omega^{bl}) e_2]_S(\cdot, 0) = e_1 \text{ on } S$$

$$\beta^{bl} = 0 \text{ on } \bigcup_{k=1}^{\infty} \partial Y_S - \begin{pmatrix} 0 \\ k \end{pmatrix}$$

β^{bl}, ω^{bl} are 1-periodic in y_1

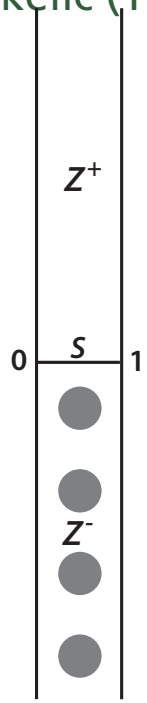
Exponential stabilization

Define $C_1^{bl} = \int_0^1 \beta_1^{bl}(y_1, 0)\, dy_1$, $C_\omega^{bl} = \int_0^1 \omega^{bl}(y_1, 0)\, dy_1$. Then $\exists C, \gamma_0 > 0$ such that

$$\left.\begin{aligned} |\beta^{bl}(y_1, y_2) - (C_1^{bl}, 0)| &\leq C e^{-\gamma_0 |y_2|} \\ |\omega^{bl}(y_1, y_2) - C_\omega^{bl}| &\leq C e^{-\gamma_0 |y_2|} \end{aligned}\right\} \text{ in } Z^+ \qquad \left.\begin{aligned} |\beta^{bl}(y_1, y_2)| &\leq C e^{-\gamma_0 |y_2|} \\ |\omega^{bl}(y_1, y_2)| &\leq C e^{-\gamma_0 |y_2|} \end{aligned}\right\} \text{ in } Z^-$$

Auxiliary Problem: Boundary Layer

Jäger/Mikelic (1996)



$$-\Delta_y \beta^{bl} + \nabla_y \omega^{bl} = 0 \text{ in } Z^+ \cup Z^-$$

$$\operatorname{div}_y \beta^{bl} = 0 \text{ in } Z^+ \cup Z^-$$

$$[\beta^{bl}]_S(\cdot, 0) = 0 \text{ on } S$$

$$[(\nabla_y \beta^{bl} - \omega^{bl}) e_2]_S(\cdot, 0) = e_1 \text{ on } S$$

$$\beta^{bl} = 0 \text{ on } \bigcup_{k=1}^{\infty} \partial Y_S - \begin{pmatrix} 0 \\ k \end{pmatrix}$$

β^{bl}, ω^{bl} are 1-periodic in y_1

Exponential stabilization

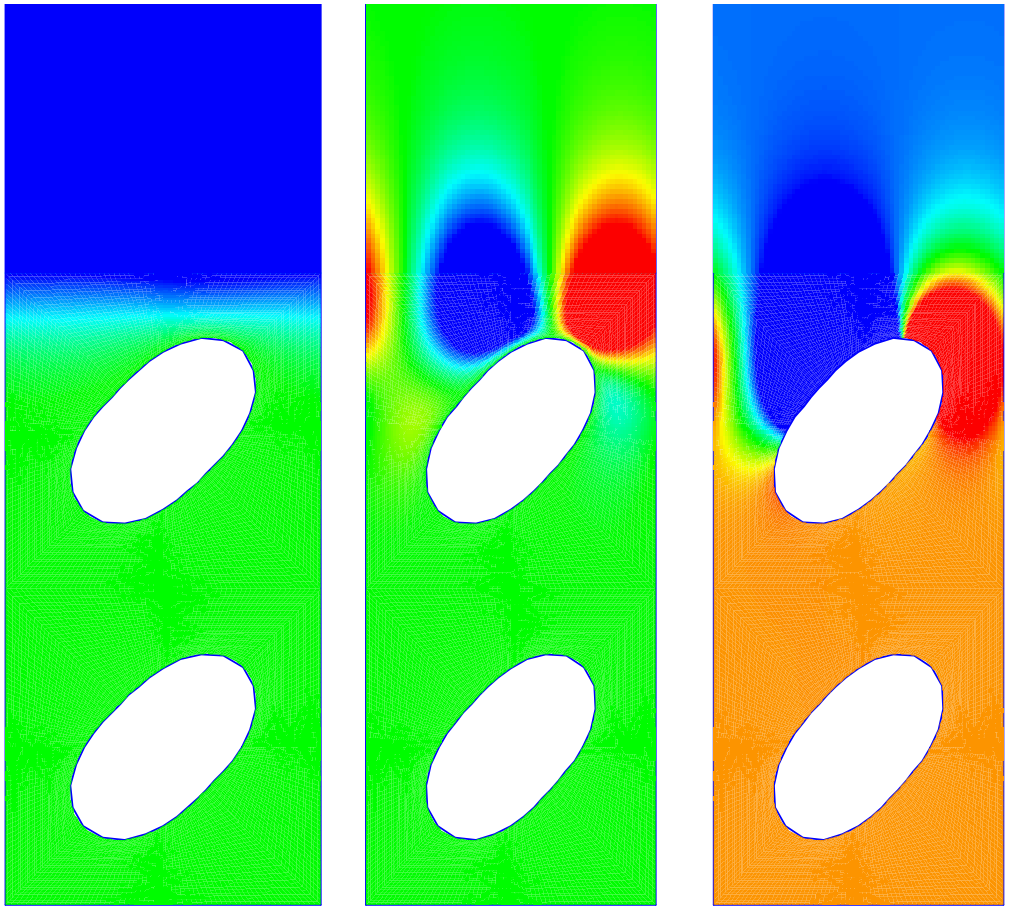
This leads to the estimates

$$\left\| \beta^{bl}\left(\frac{x}{\varepsilon}\right) - (C^{bl}, 0) H(x_2) \right\|_{L^2(\Omega)^2} \leq C\varepsilon^{\frac{1}{2}}$$

$$\left\| \omega^{bl}\left(\frac{x}{\varepsilon}\right) - C^{bl}_{\omega} H(x_2) \right\|_{L^2(\Omega)} + \left\| \nabla \beta^{bl} \right\|_{L^2(\Omega)^4} \leq C\varepsilon^{\frac{1}{2}}$$

Numerical Simulations

(from Jäger, Mikelic, Neuss 2001)

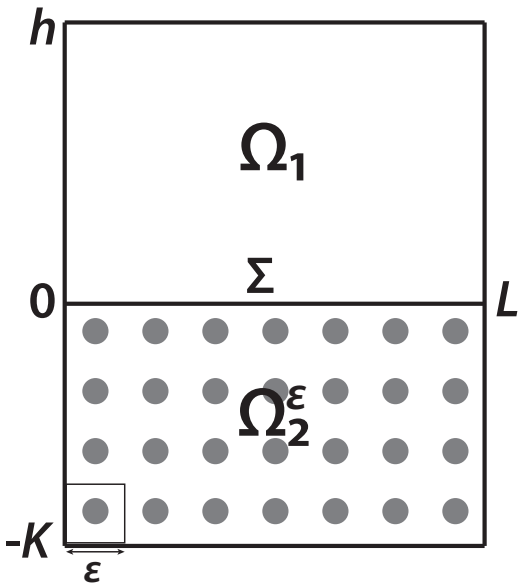


β_1^{bl} β_2^{bl} ω^{bl}

- β_1^{bl}, β_2^{bl} decay to 0 in Z^-
- β_2^{bl} decays to 0 in Z^+
- β_1^{bl} decays to $C_1^{bl} < 0$ in Z^+

Auxiliary Problem: Counterflow

Jäger/Mikelic (1996)



$$-\Delta u^\sigma + \nabla \pi^\sigma = 0 \text{ in } \Omega_1$$
$$\operatorname{div}(u^\sigma) = 0 \text{ in } \Omega_1$$
$$u^\sigma = 0 \text{ on } (0, L) \times \{h\}$$
$$u^\sigma = e_1 \text{ on } \Sigma$$

u^σ, π^σ are L-periodic in x_1

Idea of the Proof

Jäger/Maciniak-Czochra/Mikelic

$$u^\varepsilon(x) \approx u^0(x) - \varepsilon\Big(\beta^{bl}(\frac{x}{\varepsilon}) - (C_1^{bl},0)H(x_2)\Big)\frac{\partial u_1^0}{\partial x_2}(x_1,0)$$
$$- \varepsilon C_1^{bl}\frac{\partial u_1^0}{\partial x_2}(x_1,0)\cdot u^\sigma(x) + \mathcal{O}(\varepsilon^2)$$

$$p^\varepsilon(x) \approx p^0(x)H(x_2) + p^D(x)H(-x_2) - \Big(\omega^{bl}(\frac{x}{\varepsilon}) - H(x_2)C_\omega^{bl}\Big)\frac{\partial u_1^0}{\partial x_2}(x_1,0)$$
$$- \varepsilon C_1^{bl}\frac{\partial u_1^0}{\partial x_2}(x_1,0)\cdot \pi^\sigma H(x_2) + \mathcal{O}(\varepsilon)$$

On Σ:

$$\frac{u_1^\varepsilon}{\varepsilon} = -\beta_1^{bl}(\frac{x_1}{\varepsilon},0)\frac{\partial u_1^0}{\partial x_2}(x_1,0) + \mathcal{O}(\varepsilon)$$

Averaging the right hand side, we expect the following effective behaviour

$$u_1^{\text{eff}} = -\varepsilon C_1^{bl}\frac{\partial u_1^{\text{eff}}}{\partial x_2}(x_1,0)$$

Effective Fluid Behaviour

Maciniak-Czochra/Mikelic (2012)

$$-\Delta u^{\text{eff}} + \nabla p^{\text{eff}} = f \text{ in } \Omega_1$$

$$\operatorname{div} u^{\text{eff}} = 0 \text{ in } \Omega_1$$

$$\int_{\Omega_1} p^{\text{eff}} \, dx = 0$$

$$u^{\text{eff}} = 0 \text{ on } (0, L) \times \{h\}$$

$$u_2^{\text{eff}} = 0 \quad u_1^{\text{eff}} + \varepsilon C_1^{bl} \frac{\partial u_1^{\text{eff}}}{\partial x_2} = 0 \text{ on } \Sigma$$

$u^{\text{eff}}, p^{\text{eff}}$ are L-periodic in x_1

$$\operatorname{div}(A(f - \nabla \tilde{p}^{\text{eff}})) = 0 \text{ in } \Omega_2$$

$$\tilde{p}^{\text{eff}} = p^{\text{eff}} + C_\omega^{bl} \frac{\partial u_1^{\text{eff}}}{\partial x_2}(x_1, 0) \text{ on } \Sigma$$

$$A(f - \nabla \tilde{p}^{\text{eff}})|_{\{x_2 = -K\}} \cdot e_2 = 0$$

$\tilde{p}^{\text{eff}}$ is L-periodic in x_1

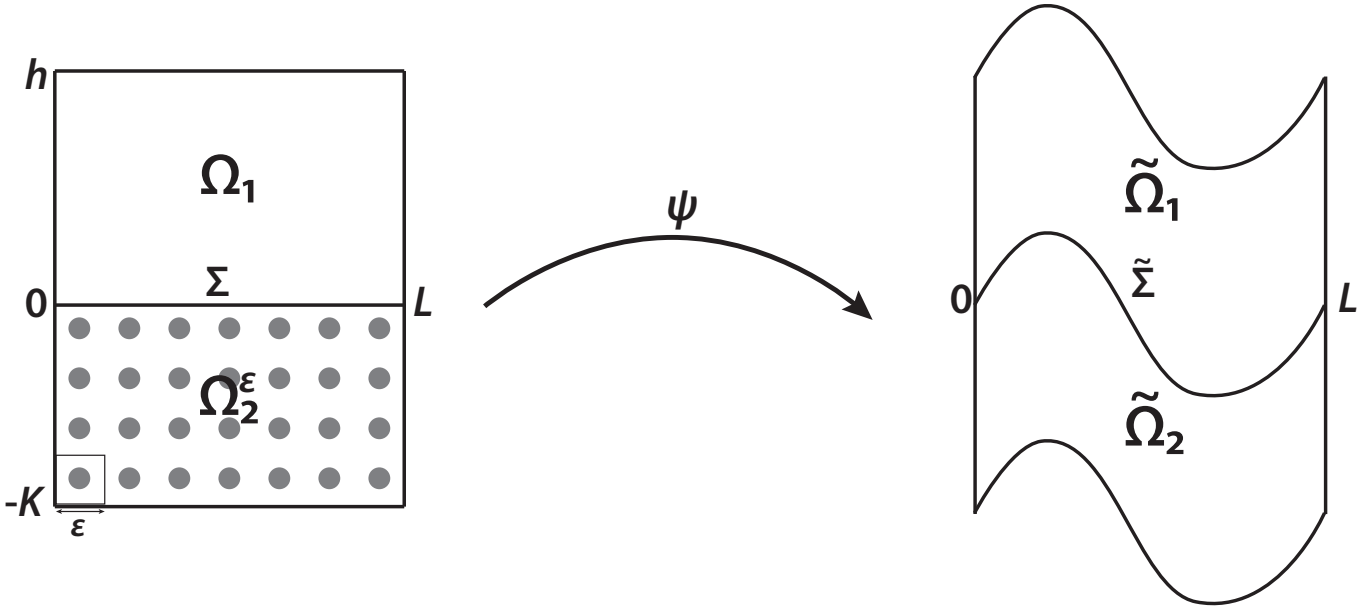
Main estimates

$$\|u^\varepsilon - u^{\text{eff}}\|_{L^2(\Omega_1)^2} \leq C\varepsilon^{\frac{3}{2}}, \qquad p^\varepsilon \longrightarrow \tilde{p}^{\text{eff}} \text{ strongly in } L^2(\Omega_2)$$

$$\|u^\varepsilon - u^{\text{eff}}\|_{H^{\frac{1}{2}}(\Omega_1)^2} + \|p^\varepsilon - p^{\text{eff}}\|_{L^1(\Omega_1)} + \|\nabla(u^\varepsilon - u^{\text{eff}})\|_{L^1(\Omega_1)} \leq C\varepsilon$$

$$\frac{1}{\varepsilon^2} u^\varepsilon - A(f - \nabla \tilde{p}^{\text{eff}}) \rightharpoonup 0 \text{ in } L^2((0, L) \times (-K, -\delta)) \forall \delta > 0$$

Coordinate Transformation



$g \in \mathcal{C}^\infty(\mathbb{R})$ L-periodic: Function describing the interface $\tilde{\Sigma}$

$$\psi : \Omega \longrightarrow \tilde{\Omega}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 + g(z_1) \end{pmatrix}$$

Transformed Differential Operators

F – Jacobian matrix of the transformation ψ

$$F(z) = \begin{bmatrix} 1 & 0 \\ g'(z_1) & 1 \end{bmatrix}$$

Denote coordinates in $\tilde{\Omega}$ by $x = (x_1, x_2)$, in Ω by $z = (z_1, z_2)$

Lemma (Transformation Rules)

Let $\tilde{c} : \tilde{\Omega} \longrightarrow \mathbb{R}$, $\tilde{j} : \tilde{\Omega} \longrightarrow \mathbb{R}^2$ be sufficiently smooth. Define $c(z) := \tilde{c}(\psi(z))$ and $j(z) := \tilde{j}(\psi(z))$, then

- $\nabla_x \tilde{c} = F^{-T} \nabla_z c$
- $\mathrm{div}_x(\tilde{j}) = \mathrm{div}_z(F^{-1} j)$

Transformed Stokes-Equation

$$
\begin{aligned}
-\mathrm{div}(F^{-1}F^{-T}\nabla u^\varepsilon) + F^{-T}\nabla p^\varepsilon &= f \\
\mathrm{div}(F^{-1}u^\varepsilon) &= 0 \\
u^\varepsilon &= 0
\end{aligned}
$$

Transformed Stokes-Equation

$$
\begin{aligned}
-\mathrm{div}(F^{-1}F^{-T}\nabla u^\varepsilon) + F^{-T}\nabla p^\varepsilon &= f && \text{in } \Omega_2^\varepsilon \\
\mathrm{div}(F^{-1}u^\varepsilon) &= 0 && \text{in } \Omega_2^\varepsilon \\
u^\varepsilon &= 0 && \text{on } \partial\Omega_2^\varepsilon
\end{aligned}
$$

Theorem (Transformed Darcy's law)

It holds $\frac{u^\varepsilon}{\varepsilon^2} \longrightarrow u_0$ *in* $L^2(\Omega_2)^2$, $p^\varepsilon \longrightarrow p_0$ *in* $L^2(\Omega_2)$ *with*

$$
\begin{aligned}
u_0 &= A(f - F^{-T}\nabla p_0) && \text{in } \Omega_2 \\
\mathrm{div}(F^{-1}u_0) &= 0 && \text{in } \Omega_2 \\
u_0 \cdot F^{-T}\nu &= 0 && \text{on } \partial\Omega_2
\end{aligned}
\qquad
A_{ij}(x) = \int_Y w_j^i(x,y)\,\mathrm{d}y \in \mathbb{R}^{2\times 2}
$$

Parameter dependent cell problem

$$
\begin{aligned}
-\mathrm{div}_y(F^{-1}(x)F^{-T}(x)\nabla_y w^i(x,y)) + F^{-T}(x)\nabla_y \pi^i(x,y) &= e_i && \text{in } Y^* \\
\mathrm{div}_y(F^{-1}(x)w^i(x,y)) &= 0 && \text{in } Y^* \\
w^i(x,\cdot), \pi^i(x,\cdot) \text{ are } Y\text{-periodic in } y, \qquad w^i(x,y) &= 0 && \text{in } Y_S
\end{aligned}
$$

Transformed Stokes-Equation

$$
\begin{aligned}
-\mathrm{div}(F^{-1}F^{-T}\nabla u^\varepsilon) + F^{-T}\nabla p^\varepsilon &= f && \text{in } \Omega^\varepsilon \\
\mathrm{div}(F^{-1}u^\varepsilon) &= 0 && \text{in } \Omega^\varepsilon \\
u^\varepsilon &= 0 && \text{on } \partial\Omega^\varepsilon \setminus (\{x_1 = 0\} \cup \{x_1 = L\}
\end{aligned}
$$

$u^\varepsilon, p^\varepsilon$ are L-periodic in x_1

First Approximations

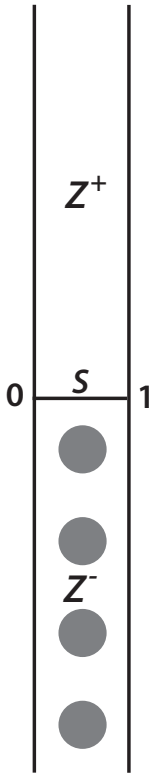
Impermeable interface approximation

$$
\begin{aligned}
-\operatorname{div}(F^{-1}F^{-T}\nabla u^0) + F^{-T}\nabla p^0 &= f \text{ in } \Omega_1 \\
\operatorname{div}(F^{-1}u^0) &= 0 \text{ in } \Omega_1 \\
u^0 &= 0 \text{ on } \partial\Omega_1 \setminus (\{x_1 = 0\} \cup \{x_1 = L\}) \\
u^0, p^0 \text{ are } L\text{-periodic in } x_1
\end{aligned}
$$

Porous approximation

$$
\begin{aligned}
\operatorname{div}(F^{-1}A(f - F^{-T}\nabla p^D)) &= 0 \text{ in } \Omega_2 \\
p^D &= p^0 + C_\omega^{bl} \text{ on } \Sigma \\
A(f - F^{-T}\nabla p^D)\cdot e_2 &= 0 \text{ on } \{x_2 = -K\} \\
\tilde{\pi}^0 \text{ is } L\text{-periodic in } x_1
\end{aligned}
$$

Auxiliary Problem: Boundary Layer



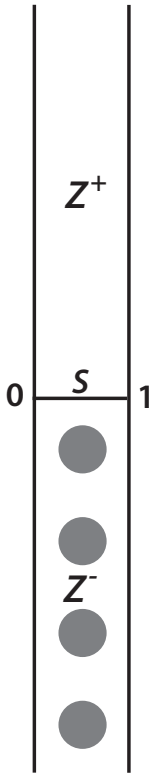
$$
\begin{aligned}
-\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\beta^{bl}(x,y)) + F^{-T}(x)\,\nabla_y\,\omega^{bl}(x,y) &= 0 \quad \text{in } \Omega\times Z\\
\operatorname{div}_y(F^{-1}(x)\beta^{bl}(x,y)) &= 0 \quad \text{in } \Omega\times Z\\
[\beta^{bl}(x)]_S(y) &= 0 \quad \text{on } \Omega\times S\\
[(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\beta^{bl}(x) - F^{-1}(x)\omega^{bl}(x))e_2]_S(y)&\\
= F^{-1}(x)F^{-T}(x)\,\nabla\,u^0(x_1,0)e_2 &\quad \text{on } \Omega\times S\\
\beta^{bl}(x,y) = 0 \quad \text{on } \Omega\times\bigcup_{k=1}^{\infty}\{\partial Y_S - \begin{pmatrix}0\\k\end{pmatrix}\}&\\
\beta^{bl}(x),\omega^{bl}(x) \text{ are 1-periodic in } y_1&
\end{aligned}
$$

Exponential stabilization

Define $C^{bl}(x_1) = \int_0^1 \beta^{bl}(x_1,y_1,0)\,\mathrm{d}y_1$, $C^{bl}_\omega(x_1) = \int_0^1 \omega^{bl}(x_1,y_1,0)\,\mathrm{d}y_1$. Then $\exists C,\gamma_0 > 0$ such that

in Z^+	in Z^-
$\lvert\beta^{bl}(x_1,y_1,y_2) - C^{bl}(x_1)\rvert \le Ce^{-\gamma_0\lvert y_2\rvert}$	$\lvert\beta^{bl}(x_1,y_1,y_2)\rvert \le Ce^{-\gamma_0\lvert y_2\rvert}$
$\lvert\omega^{bl}(x_1,y_1,y_2) - C^{bl}_\omega(x_1)\rvert \le Ce^{-\gamma_0\lvert y_2\rvert}$	$\lvert\omega^{bl}(x_1,y_1,y_2)\rvert \le Ce^{-\gamma_0\lvert y_2\rvert}$

Auxiliary Problem: Boundary Layer



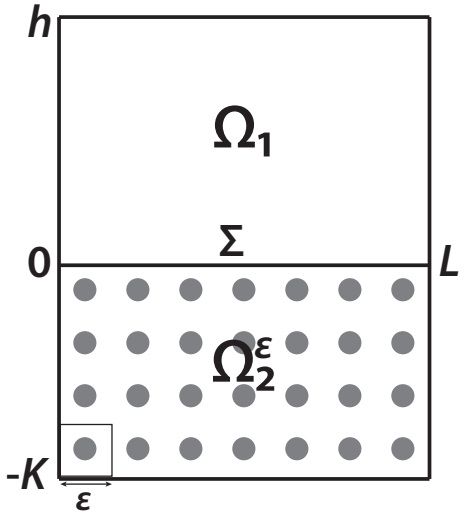
$$
\begin{aligned}
-\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\beta^{bl}(x,y)) + F^{-T}(x)\,\nabla_y\,\omega^{bl}(x,y) &= 0 \quad \text{in } \Omega\times Z\\
\operatorname{div}_y(F^{-1}(x)\beta^{bl}(x,y)) &= 0 \quad \text{in } \Omega\times Z\\
[\beta^{bl}(x)]_S(y) &= 0 \quad \text{on } \Omega\times S\\
[(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\beta^{bl}(x) - F^{-1}(x)\omega^{bl}(x))e_2]_S(y)&\\
= F^{-1}(x)F^{-T}(x)\,\nabla\,u^0(x_1,0)e_2 &\quad \text{on } \Omega\times S\\
\beta^{bl}(x,y) = 0 \quad \text{on } \Omega\times\bigcup_{k=1}^{\infty}\{\partial Y_S - \begin{pmatrix}0\\k\end{pmatrix}\}&\\
\beta^{bl}(x),\omega^{bl}(x) \text{ are 1-periodic in } y_1&
\end{aligned}
$$

Exponential stabilization

This leads to the estimates

$$
\begin{aligned}
&\|\beta^{bl}(x,\frac{x}{\varepsilon}) - C^{bl}(x_1)H(x_2)\|_{L^2(\Omega)^2} + \|\omega^{bl}(x,\frac{x}{\varepsilon}) - C^{bl}_{\omega}(x_1)H(x_2)\|_{L^2(\Omega)} + \|\nabla_y\,\beta^{bl}(x,\frac{x}{\varepsilon})\|_{L^2(\Omega)^4}\\
&+\|\nabla_x(\beta^{bl}(x,\frac{x}{\varepsilon}) - C^{bl}(x_1)H(x_2))\|_{L^2(\Omega)^4} + \|\operatorname{div}_x(F^{-1}F^{-T}\,\nabla_y\,\beta^{bl}(x,\frac{x}{\varepsilon}))\|_{L^2(\Omega)^2} \le C\varepsilon^{\frac{1}{2}}
\end{aligned}
$$

Auxiliary Problem: Counterflow



Counterflow

$$
\begin{aligned}
-\operatorname{div}(F^{-1}(x)F^{-T}(x)\,\nabla u^\sigma(x)) + F^{-T}(x)\,\nabla \pi^\sigma(x) &= 0 && \text{in } \Omega_1 \\
\operatorname{div}(F^{-1}(x)u^\sigma(x)) &= 0 && \text{in } \Omega_1 \\
u^\sigma(x_1, 0) &= C^{bl}(x_1) && \text{on } \Sigma \\
u^\sigma &= 0 && \text{on } (0, L) \times \{h\} \\
u^\sigma, \pi^\sigma \text{ are } L\text{-periodic in } x_1 &&&
\end{aligned}
$$

Idea of the proof

$$u^\varepsilon(x) \approx u^0(x) - \varepsilon\left(\beta^{bl}(x, \frac{x}{\varepsilon}) - C^{bl}H(x_2)\right) + \varepsilon u^\sigma(x) + \mathcal{O}(\varepsilon^2)$$

$$p^\varepsilon(x) \approx p^0(x)H(x_2) + p^D(x)H(-x_2) - \left(\omega^{bl}(x, \frac{x}{\varepsilon}) - H(x_2)C^{bl}_\omega\right) - \varepsilon\pi^\sigma H(x_2) + \mathcal{O}(\varepsilon)$$

1. Insert in weak formulation
2. Control the errors
3. BUT
   Divergence free test functions required

Effective Fluid Behaviour

$$-\operatorname{div}(F^{-1}F^{-T}\nabla u^{\text{eff}}) + F^{-T}\nabla p^{\text{eff}} = f \text{ in } \Omega_1$$
$$\operatorname{div}(F^{-1}u^{\text{eff}}) = 0 \text{ in } \Omega_1$$
$$\int_{\Omega_1} p^{\text{eff}}\,\mathrm{d}x = 0$$
$$u^{\text{eff}} = 0 \text{ on } (0,L)\times\{h\}$$
$$u^{\text{eff}} + \varepsilon C^{bl} = 0 \text{ on } \Sigma$$
$$u^{\text{eff}}, p^{\text{eff}} \text{ are } L\text{-periodic in } x_1$$

$$\operatorname{div}(A(f - F^{-T}\nabla\tilde{p}^{\text{eff}})) = 0 \text{ in } \Omega_2$$
$$\tilde{p}^{\text{eff}} = p^{\text{eff}} + C_\omega^{bl} \text{ on } \Sigma$$
$$A(f - F^{-T}\nabla\tilde{p}^{\text{eff}})|_{\{x_2=-K\}}\cdot e_2 = 0$$
$$\tilde{p}^{\text{eff}} \text{ is } L\text{-periodic in } x_1$$

Main estimates

$$p^\varepsilon \longrightarrow \tilde{p}^0 \text{ strongly in } L^2(\Omega_2), \quad \|u^\varepsilon - u^{\text{eff}}\|_{L^2(\Omega_1)^2} \le C\varepsilon^{\frac{3}{2}}$$
$$\|u^\varepsilon - u^{\text{eff}}\|_{H^{\frac{1}{2}}(\Omega_1)^2} + \|p^\varepsilon - p^{\text{eff}}\|_{L^1(\Omega_1)} + \|\nabla(u^\varepsilon - u^{\text{eff}})\|_{L^1(\Omega_1)} \le C\varepsilon$$
$$\frac{1}{\varepsilon^2}u^\varepsilon - A(f - F^T\nabla\tilde{p}^0) \rightharpoonup 0 \text{ in } L^2((0,L)\times(-K,-\delta))\forall\delta > 0$$
$$\frac{1}{\varepsilon}(u^\varepsilon - u^{\text{eff}}) \rightharpoonup 0 \text{ in } L^2(\Sigma)$$

Conclusions

- BC of Beavers-Joseph-Saffman for periodically curved interfaces
- Geometry of the interface influences boundary condition

Questions:

- How to obtain local information for non-periodic geometry
- Numerical simulations

Thank you for your attention!

S. Dobberschütz:
Effective behavior of a free fluid in contact with a flow in a curved porous medium.
SIAM J. Appl. Math. 75 (2015)

S. Dobberschütz:
On the Beavers-Joseph-Saffman boundary condition for curved interfaces.
arXiv 1504.05680

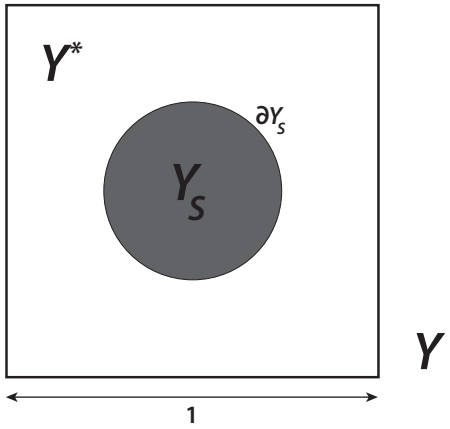
A. Marciniak-Czochra, A. Mikelic:
Effective pressure interface law for transport phenomena between an unconfined fluid and a porous medium using homogenization.
Multiscale Mod. Simul. 10 (2012)

Cell Problem and Parameter-Dependent PDEs

For fixed $x \in \Omega$, solve

$$
\begin{aligned}
-\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y w^i(x,y)) + F^{-T}(x)\,\nabla_y \pi^i(x,y) &= f(x) && \text{in } Y^* \\
\operatorname{div}_y(F^{-1}(x)w^i(x,y)) &= 0 && \text{in } Y^* \\
w^i(x,y) &= 0 && \text{in } Y_S
\end{aligned}
$$

$$w^i(x,\cdot), \pi^i(x,\cdot) \text{ are } Y\text{-periodic in } y$$



Properties with respect to x?

Cell Problem and Parameter-Dependent PDEs

Construct an operator

$$\mathcal{A} : \mathbb{R}^2 \times [H^1_{0,\#}(Y^*)^2 \times L^2_{\#}(Y^*)/\mathbb{R}] \longrightarrow (H^1_{0,\#}(Y^*)^2)' \times L^2_{0,\#}(Y^*)$$

$$\mathcal{A}(x,u,p) := \begin{pmatrix} -\operatorname{div}_y\big(F^{-1}(x)F^{-T}(x)\,\nabla_y u\big) + F^{-T}(x)\,\nabla_y p - f(x) \\ \operatorname{div}_y\big(F^{-1}(x)u\big) \end{pmatrix}$$

Lemma

$\mathcal{A}$ is continuous, the total derivative $D_{up}\mathcal{A}$ is continuous, and $(D_{up}\mathcal{A}(x,u,p))^{-1}$ exists as a continuous linear operator.

Cell Problem and Parameter-Dependent PDEs

Implicit Function Theorem for Banach spaces can be applied:

Proposition

Assume that F is m-times continuously differentiable and that $f \in \mathcal{C}^m(\Omega, L^2_\#(Y^*))$.
Then the solution (u, p) of $\mathcal{A}(x, u, p) = 0$ is in

$$\mathcal{C}^m(\Omega, [H^1_{0,\#}(Y^*)^2 \times L^2_\#(Y^*)/\mathbb{R}]),$$

i.e. $u(x, y)$ and $p(x, y)$ are m-times differentiable in x.

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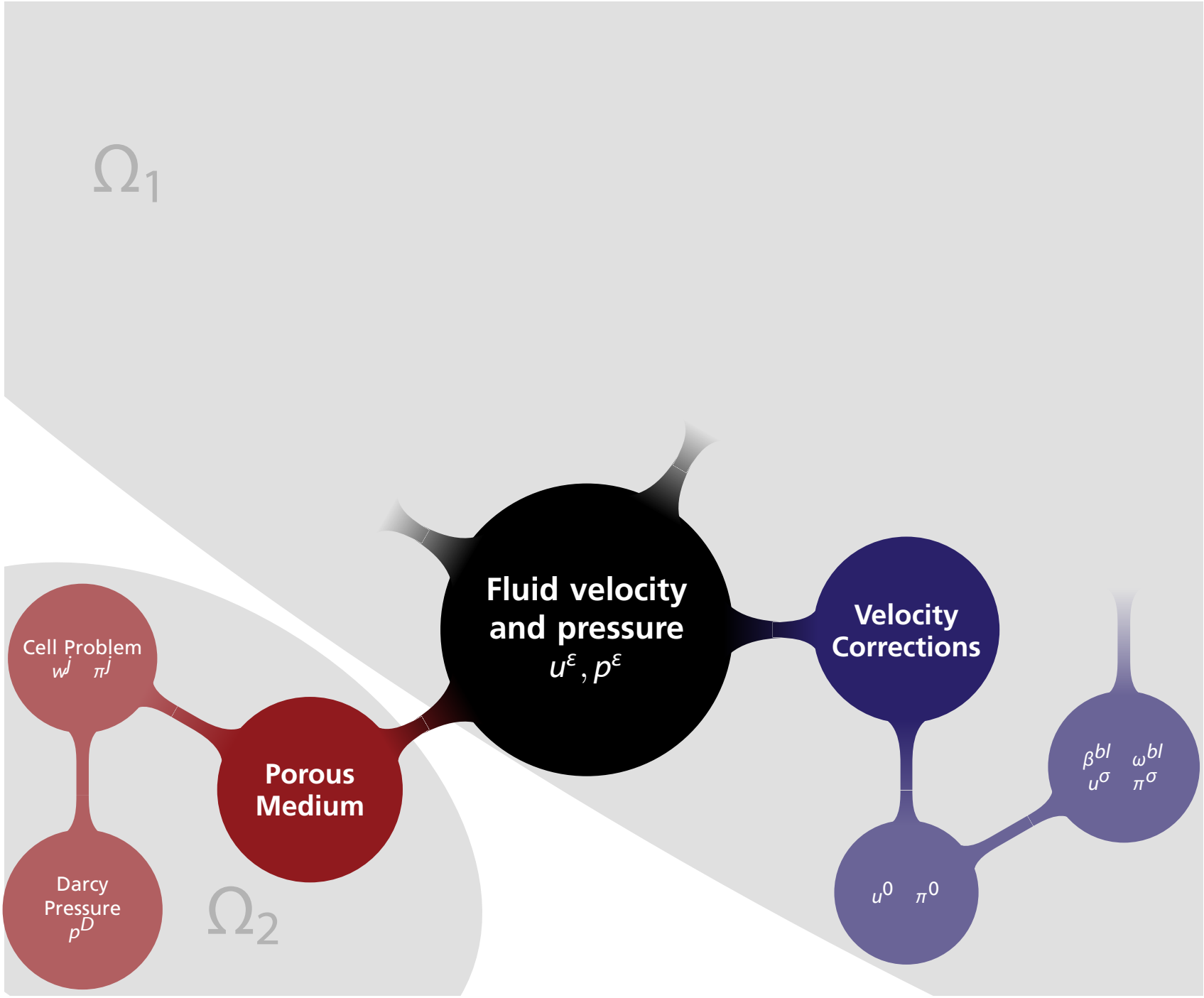
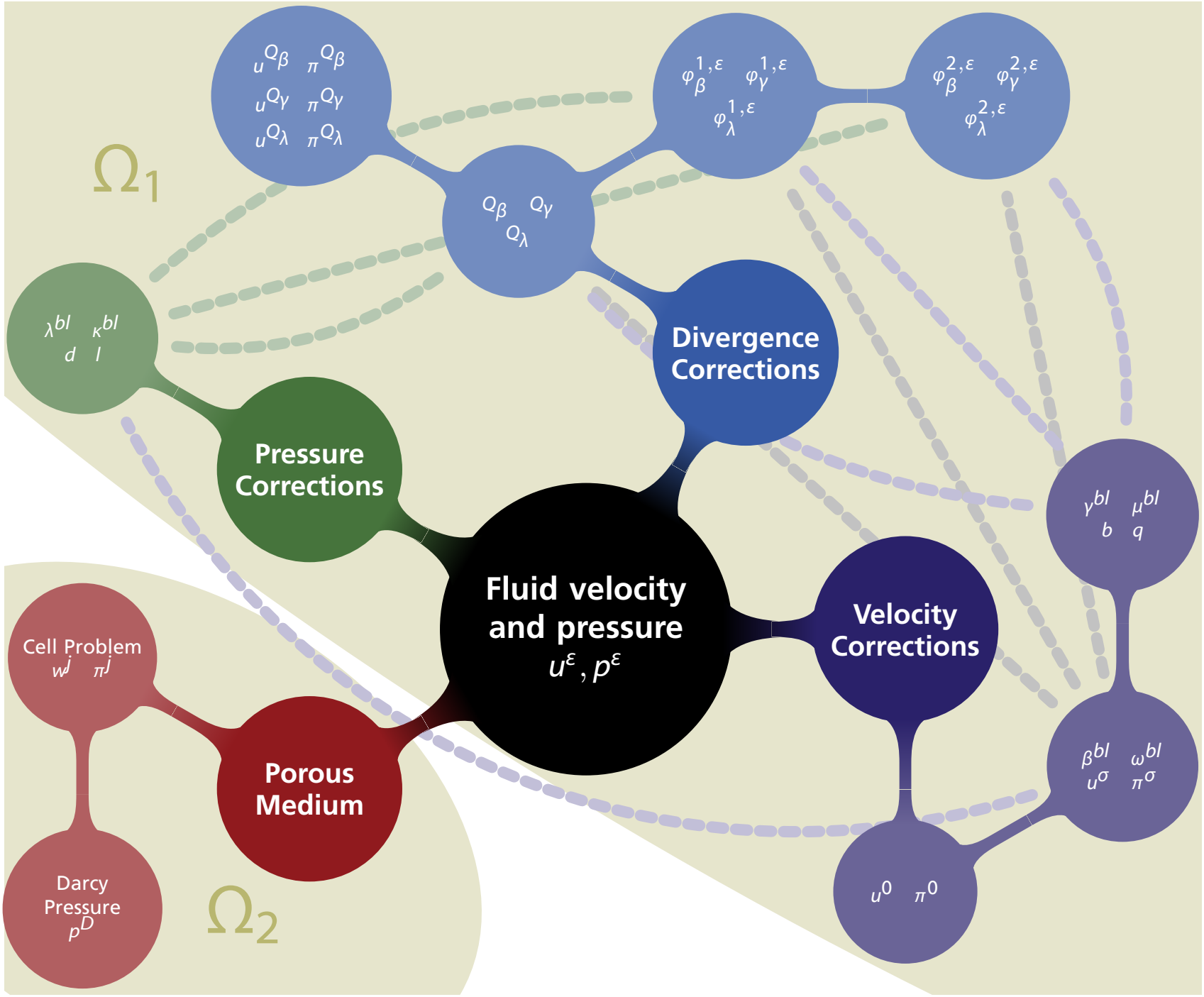
Explicit formula for derivatives

$$D_x(u,p)(x) = -D_{up}\mathcal{A}(x,u(x),p(x))^{-1} \circ D_x\mathcal{A}(x,u(x),p(x))$$

$\rightsquigarrow$ $\partial_{x_1}(u,p)$ and $\partial_{x_2}(u,p)$ fulfill transformed Stokes equation in Y^*, with adjusted right hand sides


Ω_1
Fluid velocity and pressure
$u^\varepsilon, p^\varepsilon$
Velocity Corrections
β^{bl} ω^{bl}
u^σ π^σ
u^0 π^0
Cell Problem
w^j π^j
Porous Medium
Darcy Pressure
p^D
Ω_2


u^{Q_β} π^{Q_β}
u^{Q_γ} π^{Q_γ}
u^{Q_λ} π^{Q_λ}
$\varphi_\beta^{1,\varepsilon}$ $\varphi_\gamma^{1,\varepsilon}$
$\varphi_\lambda^{1,\varepsilon}$
$\varphi_\beta^{2,\varepsilon}$ $\varphi_\gamma^{2,\varepsilon}$
$\varphi_\lambda^{2,\varepsilon}$
Ω_1
Q_β Q_γ
Q_λ
λ^{bl} κ^{bl}
d l
Divergence Corrections
Pressure Corrections
γ^{bl} μ^{bl}
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u^0 π^0