Gibbsianness related to minimisers of a large deviation rate function

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Motivation: Gibbs-non-Gibbs transitions on the lattice

μ_0 Gibbs measure
μ_t evolved measure by dynamics over $t > 0$, ($\mu_t = \mu_0 S_t$)

E.g. μ_0 Ising model (on $\{-1, 1\}^{\mathbb{Z}^2}$)
E.g. spin flips $-1 \Rightarrow 1$ & $1 \Rightarrow -1$ with certain rate

+	+	+	+	+	−
+	+	−	+	+	+
+	−	−	−	+	−
+	−	−	+	+	−
−	−	−	+	+	−
−	+	+	+	+	+

$\Longrightarrow$

+	+	+	+	+	−
−	+	−	+	−	+
+	−	−	−	+	−
+	+	−	+	+	+
−	−	+	+	+	−
−	+	+	−	+	+

$\Longrightarrow$

+	+	+	−	+	+
−	+	−	+	−	+
+	−	+	−	+	−
+	−	−	+	+	+
+	−	+	+	−	−
−	+	+	−	+	+

Question Is μ_t Gibbs? For which t?

Mean-field systems (level-1)

A mean-field system describes countably many spins

- no spatial structure
- equal interaction between spins

As initial system we consider the mean field system $(\mu_{n,0})_{n\in\mathbb{N}}$

$$\mu_{n,0} \;\propto\; e^{-nV\circ m_n(x)}\, \mathrm{d}\lambda^n(x) \quad \text{(on } \mathbb{R}^n\text{)},$$

where $m_n(x) = \frac{x_1+\cdots+x_n}{n}$ called magnetisation of x,
$\lambda \sim \mathcal{N}(0,1)$

How to describe a “probability” for the infinite number of spins?

Mean-field Gibbsianness

For all $x_2, \ldots, x_n \in \mathbb{R}$ with $m_{n-1}(x_2, \ldots, x_n) = \alpha$:

$$\mu_{n,0}(\,\cdot\,|x_2, \ldots, x_n) \;\propto\; e^{-nV(\frac{x}{n} + \frac{n-1}{n}\alpha)}\, \mathrm{d}\mu_{\mathcal{N}(0,1)}(x).$$

sequentially Gibbs $\approx$ asymptotic version of this:

Definition

$(\mu_{n,0})_{n\in\mathbb{N}}$ is sequentially Gibbs if $\forall\, \alpha \in \mathbb{R}\ \exists$ probability γ_α s.t.

$$m_{n-1}(x_2^n, \ldots, x_n^n) \to \alpha \quad \Longrightarrow \quad \mu_{n,0}(\,\cdot\,|x_2^n, \ldots, x_n^n) \to \gamma_\alpha.$$

Theorem

If $V \in C^1(\mathbb{R}, [0,\infty))$, then $(\mu_{n,0})_{n\in\mathbb{N}}$ is sequentially Gibbs with

$$\gamma_\alpha \;\propto\; e^{-x_1 V'(\alpha)}\, \mathrm{d}\mu_{\mathcal{N}(0,1)}(x_1),$$

Brownian motions for the mean-field system

Evolve $(\mu_{n,0})_{n\in\mathbb{N}}$ by independent Brownian motions to $(\mu_{n,t})_{n\in\mathbb{N}}$.

Theorem

If $V \in C^1(\mathbb{R}, [0,\infty))$, $t > 0$, then $(\mu_{n,t})_{n\in\mathbb{N}}$ is sequentially Gibbs $\iff$ $\Psi_{t,\alpha} : r \mapsto V(r) + \frac{r^2}{2} + \frac{(r-\alpha)^2}{2t}$ has a unique global minimiser for all α.

$X_1, X_2, \dots$: initial coordinates
$Y_1, Y_2, \dots$: final/evolved coordinates

$$\mathbb{P}(m_n(X_1,\dots,X_n) \approx x \,|m_n(Y_1,\dots,Y_n) = \alpha) \approx e^{-n(\Psi_{t,\alpha}(x) - C_{t,\alpha})}$$

$\Psi_{t,\alpha}$ (up to constant) is the LDP-rate function of the magnetisation ($m_n(x_1,\dots,x_n) = \frac{x_1+\cdots+x_n}{n}$) at time 0 given the magnetisation at time t equals α.

Second difference quotient and global minimisers

The second difference quotient determines whether $\Psi_{t,\alpha}$ has unique global minimisers for all α.

$$\Phi_2 V(x,y,z) = \frac{1}{z-x}\left(\frac{V(z)-V(y)}{z-y} - \frac{V(y)-V(x)}{y-x}\right) \qquad (x<y<z).$$

Theorem (Summary)

Let $V \in C^1(\mathbb{R},[0,\infty))$. Then $(\mu_{n,0})_{n\in\mathbb{N}}$ is sequentially Gibbs and for $t>0$ TFAE:

(a) $(\mu_{n,t})_{n\in\mathbb{N}}$ is sequentially Gibbs.

(b) $\Psi_{t,\alpha} : r \mapsto V(r) + \frac{r^2}{2} + \frac{(r-\alpha)^2}{2t}$ has a unique global minimiser for all α.

(c) $\Phi_2 V > -\frac{1+t}{2t}$.

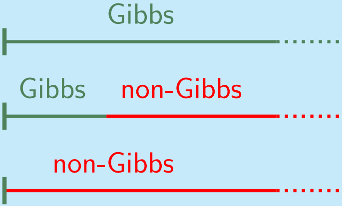
Possible scenarios for Gibbsianness $(\mu_{n,t})_{n\in\mathbb{N}}$

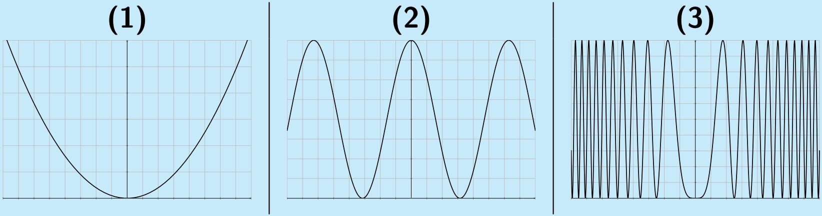
Think of $\Phi_2 V$ as V''.

(1) $\Phi_2 V \geq -\frac{1}{2}$

(2) $\Phi_2 V \geq -M$ for some $M > \frac{1}{2}$

(3) $\Phi_2 V$ is not bounded from below





Mean-field systems (level-2)

Level-2 mean-field system: $(\rho_n)_{n\in\mathbb{N}}$ with ρ_n a probability measure on $\mathcal{X}^n$,

$$\rho_n \propto e^{-nF_n\circ L_n(x)}\,\mathrm{d}\lambda^n(x)$$

$L_n(x) = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ empirical distribution of x
Whenever $F(\zeta) = V(\int z\,\mathrm{d}\zeta(z))$, as $m_n(x) = \frac{1}{n}\sum_{i=1}^n x_i = \int z\,\mathrm{d}[L_n(x)](z)$

$$\rho_n \propto e^{-nV\circ m_n(x)}\,\mathrm{d}\lambda^n(x)$$

Definition

$(\rho_n)_{n\in\mathbb{N}}$ is sequentially Gibbs if $\forall\,\zeta\in\mathcal{P}_c(\mathcal{X})\ \exists$ probability γ_ζ s.t.

$$L_{n-1}(x_2^n,\dots,x_n^n) \xrightarrow{*} \zeta \quad\Longrightarrow\quad \mu_{n,0}(\,\cdot\,|x_2^n,\dots,x_n^n) \to \gamma_\zeta.$$

Initial Gibbsianness in level-2

With $F : \mathcal{P}_c(\mathcal{X}) \to [0, \infty)$ and

$$\mu_{n,0} \;\propto\; e^{-nF\circ L_n(x)}\, \mathrm{d}\lambda^n(x)$$

We obtain an analogues statement for sequentially Gibbs.

Theorem

If F is "C^1" then $(\mu_{n,0})_{n\in\mathbb{N}}$ is sequentially Gibbs with

$$\gamma_\zeta \;\propto\; e^{-\delta V(\zeta,\delta_{x_1})}\, \mathrm{d}\lambda(x_1).$$

$\delta V(\zeta, \delta_x)$ sort of directional derivative of V at ζ in the direction of δ_x.

Unique minimiser implies Gibbs

$P : \mathcal{X} \times \mathcal{B}(\mathcal{Y}) \to [0,1]$ transformation kernel from $\mathcal{X}$ to $\mathcal{Y}$,
$\mu_{n,t} = \mu_{n,0} P^n$, transformed measure with independent transformations.
$X_1, X_2, \ldots$: initial coordinates
$Y_1, Y_2, \ldots$: final/evolved coordinates

Theorem

Suppose that for all $\zeta \in \mathcal{P}_c(\mathcal{Y})$ there exists a rate function I_ζ such that for all $\zeta_n \xrightarrow{*} \zeta$ we have the large deviation principle

$$\mathbb{P}(L_n(X_1, \ldots, X_n) \approx \xi \,|L_n(Y_1, \ldots, Y_n) = \zeta_n) \approx e^{-nI_\zeta(\xi)}.$$

Then (a) implies (b):

(a) I_ζ *has a unique global minimiser for all* $\zeta \in \mathcal{P}_c(\mathcal{Y})$.
(b) $(\mu_{n,t})_{n \in \mathbb{N}}$ *is sequentially Gibbs.*

Such LDP exists in case $\mathcal{X}$ and $\mathcal{Y}$ are finite.