Report 98-002

Note on the
Knapsack Markov Chain

M. Löwe
C. Meise

NOTE ON THE KNAPSACK MARKOV CHAIN

MATTHIAS LÖWE AND CHRISTIAN MEISE

ABSTRACT. We show that for sufficiently large knapsacks the associated Markov chain on the state space of the admissible packings of the knapsack is rapidly mixing. Our condition basically states that at least half of all items should fit into the knapsack. This is much weaker than the condition assumed by Saloff-Coste in [11].

1. Introduction and Result

In this note we consider the following counting problem. For given $a \in \mathbb{N}^n$, $b \in \mathbb{N}$ count the number of combinations $x \in \{0,1\}^n$ such that $\sum_{i=1}^n a_i x_i \leq b$. As this can be thought of as N items with weights $a_1, \dots, a_N$ to be packed into a knapsack of capacity b this is also known as the *knapsack*-problem. The knapsack-problem is known to be $\#P$-complete. This makes it attractive to consider approximate counting algorithms. Their basic idea is to approximate the knapsack of capacity b by a sequence of knapsacks of decreasing size b_k (up to $b_{b+1} = 0$) and to approximate the ratio of the sizes of two consecutive knapsacks by using a rapidly mixing Markov chain. Such a Markov chain Q has been proposed by Sinclair in [12]. So, on the state space $X = \{x \in \{0,1\}^n \mid \sum_{i=1}^n a_i x_i \leq b\}$, we define Q in the following way. Given that the Markov chain is in the state $x \in X$, the transition from x to a state y is given by

$$Q(x,y) = \begin{cases} \frac{1}{2} + |\{ i \, : \, x + e_i \notin X \}| / (2n) & : \quad y = x \\ 1/(2n) & : \quad x = y + e_i \in X. \end{cases} \tag{1}$$

Here e_i denotes the i'th unit vector and the addition is to be understood "modulo 2". In other words the chain with a packing x does nothing with probability 1/2 (to ensure aperiodicity) and otherwise it picks one of the items at random and puts is into the knapsack if it is outside and still fits in and puts it out if it is in.

This Markov chain is time-reversible with respect to the uniform measure π on X. The question whether it is also rapidly mixing for an arbitrary choice of the weights $a_1, \dots, a_n$ and the capacity b, i.e. whether there is a polynomial p in n and ε^{-1} such that

$$\|Q^t(x,\cdot) - \pi\|_{TV} \leq \varepsilon \quad \text{for all } t \geq p(n,\varepsilon),$$

is wide open. Here for two probability measures μ, ν on X the total variation distance is given by

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{x \in X} |\mu(x) - \nu(x)| \,.$$

As is well known from the theory of Markov chains one way of proving such a result would be to bound the spectral gap of the chain by the inverse of a polynomial in n.

Date: December 8, 1998.

1991 *Mathematics Subject Classification.* 60J10.

Key words and phrases. Markov chains, spectral estimates.

Let us explain this a little more detailed. Since Q is time-reversible with respect to the uniform measure π on X (i.e. the detailed balance equations

$$\pi(x)Q(x,y) = \pi(y)Q(y,x)$$

are fulfilled), irreducible and aperiodic we have that 1 is a simple eigenvalue of the linear operator $[Q\cdot]$ on $L^2(\pi)$ induced by Q. By time-reversibility the operator $[Q\cdot]$ is self-adjoint so that all the eigenvalues are real. We denote by

$$1 = \beta_0 > \beta_1 \geq \ldots \geq \beta_{|X|-1} \geq -1$$

the eigenvalues of Q (usually we identify Q with its associated operator $[Q\cdot]$ in $L^2(\pi)$). Our aim is to give a lower estimate for the spectral gap, i.e. the eigenvalue

$$\lambda_1 = 1 - \beta_1 \tag{2}$$

of $Id - Q$. By a result of Diaconis and Stroock [5] a polynomila lower bound on λ_1 indeed implies the polynomila speed of convergence for Q as we have

$$\|Q^t(x,\cdot) - \pi\|_{TV} \leq pi(x)\lambda_1^t.$$

In [11] Saloff-Coste proved by comparison with the walk on the n-dimensional cube $W_n = \{0,1\}^n$ that for knapsacks satisfying that any subset $J \subset \{1,\ldots,n\}$ of objects of size $|J| \leq n - \sqrt{n}$ is a valid packing, i.e. $\sum_{j\in J} a_j \leq b$, the spectral gap satisfies

$$\lambda_1 \geq 2e^{-4}\frac{1}{n}.$$

So for knapsacks where the set of valid packings looks "almost like the whole cube W_n" Q is indeed rapidly mixing.

Our main assumption will be the following.

Assumption 1.1. *There is $1 \geq \alpha > 1/2$ such that whenever $J \subset \{1,\ldots,n\}$ with $|J| \leq \alpha n$ the inequality*

$$\sum_{j\in J} a_j \leq b$$

holds. In other words any packing consisting of at most αn many objects is a valid packing.

With this assumption our result reads as follows.

Theorem 1.2. *There exists $N \in \mathbb{N}$ such that for any knapsack X satisfying assumption (1.1) the spectral gap of the Markov chain Q on X given by (1) satisfies*

$$\lambda_1 \geq c\,[n\log(n)]^{-1} \tag{3}$$

for all $n \geq N$ and a positive constant c. In particular the Markov chain Q is rapidly mixing in this situation.

Remark 1.3. *The rapidly mixing property is essential for the construction of approximate counters, so-called fpras-algorithms, cf.* [7] *and* [12]*. Using such algorithms it is possible to approximate the total number of elements of X in polynomial time. Unfortunately, the result of Theorem 1.2 is not sufficient to construct an fpras-algorithm. To this end we would also need to show the rapidly mixing property for the Markov chain Q on smaller knapsacks.*
On the other hand Assumption 1.1 drastically improves Saloff-Coste's condition cited above.

2. Proof

In this section we will prove Theorem 1.2. The proof uses a combination of two methods. On the one hand we will use a *martingale method*, as it has first been used in [10] on the other hand a *distance method*, which has been introduced to theory of Markov chains in [9] and [8] to investigate the spectral gap of algorithms from group theory. Both methods have their origin in the theory of diffusions on manifolds (cf. [2], [3]). Moreover, we will finally use a comparison technique that allows us to translate the result for knapsacks of even size to knapsacks of an odd size.

We will first give the proof for knapsacks with an even number of items. The rest will then follow easily.
Let us start by introducing some basic notations. First let us decompose our state space X in the following way.

$$L = \{ x \in X \ : \ |x| < n/2 \}$$

denotes the lower half of the cube (here and in the following, for $x \in W_n$ we write $|x|$ for the number of 1's in x - so $|x| = \sum_{i=1}^n x_i$),

$$M = \{ x \in X \ : \ |x| = n/2 \}$$

its middle level and

$$B = \{ x \in X \ : \ \exists i \ : \ x + e_i \notin X \}$$

is the upper boundary of X. Clearly for $x \in B$ we have $|x| \geq \alpha n$.

Furthermore for $x, y \in X$ such that $|x| = |y|$ define

$$\begin{aligned} C_1(x,y) &= \{ i \ : \ x_i = 1 = y_i \}, \\ C_0(x,y) &= \{ i \ : \ x_i = 0 = y_i \}, \\ D_1(x,y) &= \{ i \ : \ x_i = 1, \, y_i = 0 \}, \end{aligned}$$

and

$$D_0(x,y) = \{ i \ : \ x_i = 0, \, y_i = 1 \}.$$

Note that $|D_0(x,y)| = |D_1(x,y)| = d_H(x,y)/2$ since $|x| = |y|$.

For any two points $x, y \in X$ (or in W_n) let $d_H(x,y)$ denote their Hamming distance (i.e. the number of coordinates where they are different). As usual for a set $A \subset X$ let $d_H(v,A) = \min\{ d_H(v,a) \ : \ a \in A \}$ be the minimal Hamming distance from v to the set A. Moreover, the following distance (by which we only understand a positive function of two variables - as opposed to a metric where the triangle inequality has to be fulfilled) will be central in the proof:

$$\begin{aligned} d(v,w) &:= d_H(v,w)^{1/2} + c\frac{(d(v,L) \vee d(w,L) \vee 1)^2}{n} \qquad (4) \\ &=: d_H(v,w)^{1/2} + d'(v,w) \end{aligned}$$

where c denotes a positive constant. Note that both summands in (4) decrease whenever the Hamming distance included in their definition decreases. Also note that for $v, w \in L$ the distance $d(v,w)$ basically agrees with the square root of the Hamming distance of v and w (up to an additive constant $\frac{1}{n}$).

The proof now vastly consists of considerations of an eigenfunction f to the second-largest eigenvalue β_1 of Q.

First let us show that we can assume that there exist $x_1, y_1 \in M$ such that

$$f(x_1) \geq \max_{v \in X} f(v)\gamma \tag{5}$$

and

$$f(y_1) \leq \min_{v \in X} f(v)\gamma, \tag{6}$$

where $\gamma \in]0,1[$ is fixed. Indeed, suppose that for all $w \in M$ we would have

$$f(w) \leq \max_{v \in X} f(v)\gamma. \tag{7}$$

Observe that clearly

$$f(X_n) + \lambda_1 \sum_{i=0}^{n-1} f(X_i)$$

is a martingale under Q. So, starting the Markov chain Q in the state x satisfying $f(x) = \max_{v \in X} f(v)$ and stopping this martingale at the first hitting time T_M of the set M we obtain

$$f(x) - E_x\left(f\left(X_{T_M}\right)\right) \quad \leq \quad \lambda_1 E_x\left(T_M\right) f(x).$$

Clearly by (7) we have

$$E_x\left(f\left(X_{T_M}\right)\right) \leq f(x)\gamma$$

and thus

$$\lambda_1 E_x\left(T_M\right) \geq 1 - \gamma.$$

Finally, we can couple the Markov chain Q with a random walk on W_n in such a way that Q removes the i'th item from the knapsack whenever the random walk on W_n changes the i'th coordinate from 1 to 0 and that Q adds the i'th item to the knapsack whenever the random walk on $W_n$writes a 1 in the i'th coordinate and this results in an admissible packing of the knapsack. Since Q by this coupling will follow the random walk on W_n in $X \setminus B$ and B is by assumption "above the middle level" M we see, that by this coupling Q will always be closer to M than the random walk on W_n. As for the latter we have an expected first entrance time into M of $\mathcal{O}(N \log N)$ we obtain the bound

$$E_x\left(T_M\right) \leq cn\log(n)$$

(where c is a positive constant). Hence we arrive at

$$\lambda_1 \geq (1-\gamma)\left[cn\log(n)\right]^{-1}. \tag{8}$$

In the same way, if (6) is not fulfilled, we obtain for y satisfying $f(y) = \min_{v \in X} f(v)$

$$f(y) - E_x\left(f\left(X_{T_M}\right)\right) \quad \geq \quad \lambda_1 E_x\left(T_M\right) f(y).$$

This together with

$$E_x\left(f\left(X_{T_M}\right)\right) \geq f(y)\gamma$$

again yields

$$\lambda_1 \geq (1-\gamma)\left[cn\log(n)\right]^{-1}.$$

Hence, if one of (5) or (6) are violated we obtain a polynomial estimate for the spectral gap. More precisely we get

$$\lambda_1 \geq \mathcal{O}([n\log(n)]^{-1}).$$

Put

$$g(x,y) := (f(x) - f(y))^+$$

(where for some number x we define x^+ to be the supremum of x with 0).

The rest of the proof relies on a rather new technique, we call the *distance method*, for estimating the eigenvalues of a Markov chain, the idea of which is amazingly simple. It might be expedient to briefly illustrate this technique before bringing it into action.

Let $\beta \neq 1$ be an eigenvalue of Q with corresponding eigenfunction f. Assume that there exists a function $d : X \to \mathbb{R}_+$ such that $[Qd](x) \leq \alpha d(x)$ for all $x \in X$ and some $\alpha \leq 1$. Let

$$m := \max\left\{ \frac{|f(x)|}{d(x)} \; : \; x \in X \right\}.$$

Choose $x \in X$ such that $|f(x)|/d(x) = m$. By irreducibility we may assume that $m \neq 0$ and therefore also $f(x) \neq 0$. Then we have

$$\begin{aligned} |\beta f(x)| &= |[Qf](x)| = \left|\sum_{y \in X} Q(x,y) f(y)\right| \\ &\leq \sum_{y \in X} Q(x,y)\,|f(y)| = \sum_{y \in X} Q(x,y) \frac{|f(y)|}{d(y)} d(y) \\ &\leq \frac{|f(x)|}{d(x)} [Qd](x) \leq \alpha\, |f(x)|. \end{aligned}$$

Dividing by $|f(x)|$ we conclude $|\beta| \leq \alpha$.

A similar result can be obtained by using couplings. So, let $d : X \times X \to \mathbb{R}_+$ be a function, such that for some coupling $\widetilde{Q}$ of (Q, Q) the inequality $[\widetilde{Q}d](x, y) \leq \alpha d(x, y)$ for all $x \neq y$ and some $\alpha \leq 1$ holds true. Then any eigenvalue $\beta \neq 1$ of Q satisfies, $|\beta| \leq \alpha$. Indeed, defining

$$m := \max\left\{ \frac{(f(y) - f(x))^+}{d(x,y)} \; : \; x \neq y \in X \right\}$$

and choosing (x, y) such that $|g(x, y)|/d(x, y) = m$ (where $g(x, y) = (f(y) - f(x))^+$) we can conclude that

$$|\beta g(x,y)| = \left|\left[\widetilde{Q} g\right](x,y)\right| \leq c \left[\widetilde{Q} d\right](x,y) \leq c\alpha d(x,y).$$

Hence $|\beta| \leq \alpha$.

Of course, this idea is in now at all restricted to the situation of X being the knapsack or Q being the specific Markov chain (and indeed in [9],[8] has been applied to the analysis of other random walks).
In short the idea of the distance method can be described as comparing the landscape given by the eigenfunction f (resp. g) to an easier distance, for which the coupling still is contractive.

In order to bring these ideas into play and to continue the proof we need to consider the following coupling $\widetilde{Q}$ of Q (with itself).

$$\widetilde{Q}((x,y),(x+e_i, y+e_i)) = \frac{1}{2n} \quad \text{for } i \in C_1(x,y) \cup C_0(x,y), \tag{9}$$

$$\widetilde{Q}((x,y),(x+e_i, y+e_j)) = \frac{1}{2n\,|D_0(x,y)|} \tag{10}$$

for $(i,j) \in D_1(x,y) \times D_0(x,y) \cup D_0(x,y) \times D_1(x,y)$. For the other pairs of states $x, y \in X$ take an arbitrary coupling $\widetilde{Q}$ (e.g. the classical coupling). Intuitively speaking this coupling describes that we will only really couple random walks which contain the same number of items. In this case, if the first walk chooses an item which is contained in both knapsack, resp. is missing in both, the second walk will choose the same item. If the first walk selects another item, e.g. one which is contained in the first knapsack but not in the second, the second walk will try to do "the opposite", i.e. in this case will choose an item which is contained in the second knapsack but not in the first. This will bring the walls closer together in the Hamming distance.

Observe that the coupling $\widetilde{Q}$ respects the difference $|x|-|y|$ if $|x| = |y|$ and $\{x,y\} \cap B = \emptyset$.

As explained above we will be interested in the pair (x_1, y_1) given by

$$\frac{g(x_1,y_1)}{d(x_1,y_1)} = \max\left\{ \frac{g(v,w)}{d(v,w)} \; : \; v, w \in X, \; |v| = |w| \right\}. \tag{11}$$

First obeserve that by (5) and (6) the above maximum ist not zero, such that later on we will be able to divide by it.

The quotient $\frac{g(x_1,y_1)}{d(x_1,y_1)}$ can be bounded from below by taking the max only over the middle level M. By (5) and (6) we the arrive at

$$\begin{aligned} \frac{g(x_1,y_1)}{d(x_1,y_1)} &\geq \|g\|_\infty \gamma \frac{1}{d_H(x_1,y_1)^{1/2} + cn^{-1}} \\ &= \|g\|_\infty \gamma \frac{1}{n^{1/2} + cn^{-1}}. \end{aligned} \tag{12}$$

If one of v or w is in B (without loss in generality $v \in B$) we obtain

$$\frac{g(v,w)}{d(v,w)} \leq \|g\|_\infty \frac{1}{2^{1/2} + c(\alpha - 1/2)^2 n}. \tag{13}$$

Comparing the right hand side of (13) to the right hand side of (12) we see that

$$\gamma \frac{1}{n^{1/2} + cn^{-1}} \geq \frac{1}{2^{1/2} + c(\alpha - 1/2)^2 n}.$$

if and only if

$$\gamma \left(2^{1/2} + c(\alpha - 1/2)^2 n\right) \geq n^{1/2} + cn^{-1}. \tag{14}$$

Now obviously for large n the left side of (14) is growing faster than $n^{1/2}$ on the right side. This yields that in fact for n large enough the relation

$$\frac{g(x_1,y_1)}{d(x_1,y_1)} \geq \frac{g(v,w)}{d(v,w)},$$

holds true for every pair (v,w) with $\{v,w\} \cap B \neq \emptyset$. This means that for large n we can assume that neither x_1 nor y_1 is in B (where x_1 and y_1 are given by the characterisation (11)).

For the distance $\sqrt{d_H}$ the following inequality holds true.

$$\left[\widetilde{Q}\sqrt{d_H}\right](x,y) \tag{15}$$

$$= 1/2d_H(x,y)^{1/2} + \frac{d_H(x,y)}{2n}(d_H(x,y)-2)^{1/2} + \frac{n-d_H(x,y)}{2n}d_H(x,y)^{1/2}$$

$$= d_H(x,y)^{1/2}\left[1 + \frac{d_H(x,y)^{1/2}(d_H(x,y)-2)^{1/2}}{2n} - \frac{d_H(x,y)}{2n}\right] \tag{16}$$

Note that the expression $(\sqrt{t(t-2)}-t)/(2n)$ is smaller than $-1/(2n)$ for all $t \in [2,n]$, so that we finally obtain

$$\left[\widetilde{Q}\sqrt{d_H}\right](x,y) \leq \sqrt{d_H(x,y)}\left(1-\frac{1}{2n}\right). \tag{17}$$

Now we are ready to apply the distance method. The points x_1, y_1 satisfying (11) can either be both in L or in M or in $X \setminus (L \cup M \cup B)$.

Case a) Suppose that both points x_1, y_1 satisfying (11) are in L. Then (17) immediately yields that also

$$\left[\widetilde{Q}d\right](x_1,y_1) \leq d(x_1,y_1)\left(1-\frac{1}{2n}\right). \tag{18}$$

(since d' is constant on $L \cup B$ and points in L can "at most" reach B). Therefore, following the ideas explained above, in case a) we get a lower bound on the spectral gap of Q of the form

$$\lambda_1 \geq \frac{1}{2n}.$$

Case b) Suppose that both points x_1, y_1 satisfying (11) are in M.
For the Hamming distance we have

$$\left[\widetilde{Q}d_H^{1/2}\right](x_1,y_1) = d_H(x_1,y_1)^{1/2}\left(1-\frac{1}{2n}\right). \tag{19}$$

For $d'(x_1,y_1)$ we calculate

$$\begin{aligned}\left[\widetilde{Q}d'\right](x_1,y_1) &\leq \frac{1}{2}d'(x_1,y_1) + \frac{1}{4}\frac{2^2c}{n} + \frac{1}{4}\frac{c}{n} \\ &= d'(x_1,y_1)\left(1+\frac{3}{4}\right).\end{aligned}$$

Now we want to find $\kappa > 0$ such that for sufficiently large n an inequality of the form

$$d_H(x_1,y_1)^{1/2}\left(1-\frac{1}{2n}\right) + d'(x_1,y_1)\frac{7}{4} \leq \left[d_H(x_1,y_1)^{1/2} + d'(x_1,y_1)\right]\left(1-\frac{\kappa}{n}\right) \tag{20}$$

holds.
Using $d'(x_1,y_1) = cn^{-1}$ we see that this is true if and only if

$$\frac{3c}{2} \leq (1-2\kappa)\,d_H(x_1,y_1)^{1/2} - c\kappa n^{-1}. \tag{21}$$

Choosing for example $\kappa = 1/4$ the right side become $\frac{1}{2}d_H(x_1,y_1)^{1/2} - \frac{c}{2n}$ which dominates $1/2$ for n sufficiently large, since $d_H(x_1,y_1) \geq 2$ and $c/(2n) \to 0$. Hence (21) holds true for $0 < c < 1/3$.

This shows that in case b) we obtain a bound for the spectral gap of the form

$$\lambda_1 \geq \frac{1}{4n},$$

where we have to choose $0 < c < 1/3$ in the definition of the metric d in (4).

Case c) Suppose $x_1, y_1 \in X \setminus B$ such that $|x_1| = |y_1| > n/2$. Computing $\left[\widetilde{Q}d'\right](x_1, y_1)$ gives

$$\begin{aligned} &\left[\widetilde{Q}d'\right](x_1,y_1) \\ \leq\ & \frac{1}{2}d'(x_1,y_1) + \frac{|x_1|}{2n}\frac{c\,(|x_1|-1-n/2)^2}{n} + \frac{n-|x_1|}{2n}\frac{c\,(|x_1|+1-n/2)^2}{n} && (22) \\ =\ & d'(x_1,y_1) + \frac{|x_1|}{2n}\frac{c(1-2\,|x_1|+n)}{n} + \frac{n-|x_1|}{2n}\frac{c(1+2\,|x_1|-n)}{n} \\ =\ & d'(x_1,y_1) + c\frac{n-4\,|x_1|^2+4n\,|x_1|-n^2}{2n^2} && (23) \end{aligned}$$

We want to check whether we can find an appropriate $\kappa > 0$ such that for large n the inequality

$$\left[\widetilde{Q}(d_H^{1/2}+d')\right](x_1,y_1) \leq \left(1-\frac{\kappa}{n}\right)\left(d_H(x_1,y_1)^{1/2}+d'(x_1,y_1)\right) \tag{24}$$

holds. Using (19) and (23) we obtain that this is true if

$$-\frac{d_H(x_1,y_1)^{1/2}}{2n} + c\frac{n-4\,|x_1|^2+4n\,|x_1|-n^2}{2n^2} \leq -\frac{\kappa}{n}(d_H(x_1,y_1)^{1/2}+d'(x_1,y_1)). \tag{25}$$

or equivalently

$$2\kappa d'(x_1,y_1) + c\frac{n-4\,|x_1|^2+4n\,|x_1|-n^2}{n} \leq (1-2\kappa)d_H(x_1,y_1)^{1/2}. \tag{26}$$

Now note that

$$\begin{aligned} & 2\kappa d'(x_1,y_1) + c\frac{n-4\,|x_1|^2+4n\,|x_1|-n^2}{n} \\ =\ & \frac{c}{n}(4\,|x|_1\,(n-|x|_1) - n(n-1) + 2\kappa\,|(|\,x_1)^2 - 2\kappa n\,|x|_1 + \frac{\kappa}{2}n^2) \end{aligned}$$

and that for $\kappa < 2$ the function

$$f(|x|) = (4\,|x|\,(n-|x|) - n(n-1) + 2\kappa\,|(|\,x)^2 - 2\kappa n\,|x| + \frac{\kappa}{2}n^2)$$

becomes maximal for $|x| = \frac{n}{2}$. In this case the value is

$$f\left(\frac{n}{2}\right) = n.$$

Hence we can show (26) if we can find a $\kappa < 2$ such that

$$c \leq (1-2\kappa)\,d_H(x_1,y_1)^{1/2}.$$

As $(1-2\kappa)\,d_H(x_1,y_1)^{1/2}$ is at least $(1-2\kappa)\sqrt{2}$, we obtain that (26) is for example satisfied if we choose $c = 1/4$ and $\kappa = \frac{1}{3}$ in which case we then obtain the estimate

$$\lambda_1 \geq \frac{1}{3n}$$

for the spectral gap of Q.

This finishes the proof of the case where the number n of items is even. For the case of an odd number of items one could in principle try to mimic the proof for even n by replacing the middle level by the two middle levels and then try to obtain similar estimates.

Instead of this we apply a rather elegant comparison technique, the idea of which is to be credited to Diaconis and Saloff-Coste (cf. [4], e.g.). The principle idea behind this technique is the variational characterisation of the second largest eigenvalue (resp. the spectral gap) as

$$\lambda_1 = \inf\left\{\frac{\mathcal{E}(f,f)}{Var_\pi(f)}, f \text{ non-constant}\right\}$$

(cf [6], e.g.). Here $Var_\pi(f)$ denotes the variance of f with respect to the measure π and

$$\begin{aligned}\mathcal{E}(f,f) &= \langle \Gamma f, f\rangle_\pi \\ &= \frac{1}{2}\sum_{x,y}(f(x)-f(y))^2 Q(x,y)\pi(x)\end{aligned}$$

is the discrete Dirichlet form associated with $\Gamma = Id - Q$.

The idea of the comparison technique is now to compare the Dirichlet form and the variance associated to the knapsack Markov chain with an odd number of items to that of a knapsack Markov chain with an even number of items for which we already know an appropriate bound on the spectral gap.

So, let $\varepsilon > 0$ such that $\alpha - \varepsilon > 1/2$. Choose $N = \alpha/\varepsilon - 1$ such that $(\alpha-\varepsilon)(n+1) \leq \alpha n$ for all $n \geq N$. Consider the set $X' = \{0,1\} \times X$ which is again a knapsack obtained by adding another object with weight $a_0 = 0$ (and if for X the number n of items was odd, it is even for X'). Note that X' also satisfies (1.1) for all $n \geq N$ with the parameter α replaced by $\alpha - \varepsilon$.
Given $f \in L^2(X,\pi)$ we define $\widetilde{f} \in L^2(X',\pi')$ by

$$\widetilde{f}((x_0, x_1, \dots, x_n)) = f(x_1, \dots, x_n))$$

for any choice of x_0. Using that $|X'| = 2\,|X|$ together with the fact that $\frac{Q'(x,y)}{Q(x,y)} = \frac{n+1}{n}$ we obtain

$$\begin{aligned}\mathcal{E}'(\widetilde{f},\widetilde{f}) &= \frac{1}{2}\sum_{x,y\in X'}\left(f((x_1,\dots,x_n)) - f((y_1,\dots,y_n))\right)^2 Q'(x,y)\pi'(x) \\ &= \mathcal{E}(f,f)\frac{n+1}{n}.\end{aligned}$$

We compute the variance of f with respect to π using the following variational characterisation of the variance

$$Var_\pi(f) = \min_{c\in\mathbb{R}}\sum_{x\in X}|f(x)-c|^2\,\pi(x).$$

This immediately yields

$$\begin{aligned} Var_\pi(f) &= \min_{c\in\mathbb{R}} \sum_{x\in X} |f(x)-c|^2 \pi(x) \\ &= \min_{c\in\mathbb{R}} \sum_{x\in X'} \left|\widetilde{f}(x)-c\right|^2 \pi'(x) \\ &= Var_{\pi'}(\widetilde{f}). \end{aligned}$$

Hence we obtain

$$\lambda_1 \geq \lambda_1' \frac{n+1}{n}$$

which finishes the proof of the theorem.

References

[1] Aldous D., Diaconis P. Strong Uniform Times and Finite Random Walks, Advances in Applied Mathematics **8**, 69–97 (1987).

[2] M.F. Chen, F.Y. Wang, *Applications of coupling method to the first eigenvalue on manifold*, Science in China, Series A **37** No. 1, 1-14 (1994).

[3] M.F. Chen, *Estimation of Spectral Gap for Markov Chains*, Acta Mathematica Sinica, New Series **12** No. 4, 337-360 (1996).

[4] P. Diaconis, L. Saloff-Coste, *Walks on generating sets of Abelian groups*, Probab. Theory Rel. Fields 105, 393-421 (1996).

[5] P. Diaconis, D. Stroock, *Geometric bounds for eigenvalues of Markov chains*, The Annals of Appl. Prob., No. 1, 36-61 (1991).

[6] R. Horn, C. Johnson, *Matrix Analysis*, Cambridge University Press (1985).

[7] M.R. Jerrum, L.G. Valiant, V.V. Vazirani, *Random generation of combinatorial structures from a uniform distribution*, Theor. Computer Science **43**, 169–188 (1986).

[8] M. Löwe, C. Meise, *On Spectral Gap estimates of a Markov chain via distance arguments II: Walks on $\mathbb{Z}_4$*, Preliminary Version (1998).

[9] C. Meise, *Spectral gap estimates via distance arguments* , Preprint, submitted (1998).

[10] C. Meise, *On Spectral Gap estimates of a Markov chain via hitting times and coupling*, to appear: J. of Appl. Prob., Preprint 94-090, SFB 343, Universität Bielefeld (1994).

[11] E. Giné, G.R. Grimmett, L. Saloff-Coste, *Lectures on probability theory and statistics*, Lectures from 26th Summer School on Probability Theory held in St. Flour, Lecture Notes in Mathematics, 1665, Springer-Verlag Berlin (1997).

[12] Sinclair, A. The Markov chain Monte Carlo method: an approach to approximate counting and integration, http://www.cs.berkeley.edu/ sinclair/ (1996).

(Matthias Löwe) EURANDOM, P.O. Box 513, 5600 MB Eindhoven, Netherlands
E-mail address, Matthias Löwe: `lowe@eurandom.tue.nl`

(Christian Meise) Fakultät für Mathematik, Universität Bielefeld, Postfach 100 131, D-33501 Bielefeld, Germany
E-mail address, Christian Meise: `meise@mathematik.uni-bielefeld.de`