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DECAYING CORRELATIONS FOR THE SUPERCRITICAL CONTACT PROCESS CONDITIONED ON SURVIVAL

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Abstract

A d-dimensional contact process is a simplified model for the spread of a biological organism or an infection on the lattice $\mathbb{Z}^d$. At each time $t \geq 0$, every point of the lattice (or site) is either infected or healthy. As time passes, a healthy site is infected with Poisson rate λ by each of its $2d$ immediate neighbors which is itself infected; an infected site recovers and becomes healthy with Poisson rate 1. The processes involved are independent. If the process starts with a set $A \subset \mathbb{Z}^d$ of infected sites at time $t = 0$, then the infection continues forever with a positive probability iff λ exceeds a certain critical value. Such a process is called supercritical.

Consider the supercritical contact process starting with a single infected site at the origin, conditioned on surviving forever. We develop a technique for embedding this conditional process for large t in a contact process starting at a large time s with all sites of the lattice infected. This allows us to show that the covariances for the conditional process fall off faster than any negative power of the distance, provided that this distance is at most of the order t.

The results obtained in this paper will enable us to study the statistical problem of estimating the parameter λ. This will be the subject of a companion paper Fiocco and van Zwet (1999).

1 Introduction

The contact process was introduced and studied by Harris (1974). It is a simple model for the spread of an infection or –more generally– a biological population on the lattice $\mathbb{Z}^d$. At each time $t \geq 0$, each site can be in one of two possible states: infected or healthy. The state of the site $x \in \mathbb{Z}^d$ at time t will be indicated by a random variable $\xi_t(x)$,where

$$\xi_t(x) = \begin{cases} 1 & \textit{if } x \textit{ is infected} \\ 0 & \textit{if } x \textit{ is healthy} \end{cases} \tag{1.1}$$

The function $\xi_t : \mathbb{Z}^d \to \{0,1\}$ gives the state of the process at time t. It is a $\{0,1\}$-valued random field over $\mathbb{Z}^d$.

The evolution of this random field in time is described by the following dynamics. A healthy site is infected with rate λ by each of its $2d$ immediate neighbors which is itself infected; an infected site recovers with rate 1. Given the configuration ξ_t at time t, the processes involved are independent until a change occurs.

It is sometimes convenient to represent the state of the contact process at time t by the set of infected sites rather than by the function $\xi_t : \mathbb{Z}^d \to \{0,1\}$. Usually this set is also denoted by ξ_t. Thus, by an abuse of notation,

$$\xi_t = \{x \in \mathbb{Z}^d : \xi_t(x) = 1\} .$$

It remains to specify the initial state of the process at time $t = 0$. If this is deterministic it will be given by the set $A \subset \mathbb{Z}^d$ of infected sites at time $t = 0$ and we denote this contact process by $\{\xi_t^A : t \geq 0\}$. For example, $\{\xi_t^{\mathbb{Z}^d} : t \geq 0\}$ or $\{\xi_t^{\{0\}} : t \geq 0\}$ will denote the process starting with every site infected, or with infection only at the origin. Obviously $\xi_0^A = A$ for any A. The initial set of infected sites may also be chosen at random according to a probability measure α, and in this case we indicate the contact process by $\{\xi_t^\alpha : t \geq 0\}$. If we do not want to specify the initial state of the process at all, we simply write $\{\xi_t : t \geq 0\}$.

The probability distribution of the state of the processes ξ_t^A and ξ_t^α at time t will be denoted by μ_t^A and μ_t^α respectively. Obviously, $\mu_0^\alpha = \alpha$. Probability measures on the state space $H = \{0,1\}^{\mathbb{Z}^d}$, such as μ_t^A and μ_t^α, are defined on the σ-algebra $\mathcal{B}$ generated by the 'rectangles' $\{\eta \in H : \eta(x) = 1\}$. This is also the σ-algebra of Borel sets if we equip the state space $H = \{0,1\}^{\mathbb{Z}^d}$ with the product topology. For a rigorous construction of the contact process we refer the reader to Liggett (1985).

When considering the contact process, the first question that comes to mind is whether the distribution μ_t^A of ξ_t^A will converge weakly to a limit measure μ^A as $t \to \infty$. Since we employ the product topology on the state space H,

$$\mu_t^A \xrightarrow{w} \mu^A \text{ iff } \mu_t^A\{B \subset \mathbb{Z}^d : B \supset F\} \xrightarrow{w} \mu^A\{B \subset \mathbb{Z}^d : B \supset F\}$$

for every finite set $F \subset \mathbb{Z}^d$. In terms of functions $\eta = I_B$, the set $\{B \subset \mathbb{Z}^d : B \supset F\}$ corresponds to the cylinder set $\{\eta \in \{0,1\}^{\mathbb{Z}^d} : \eta(x) = 1, x \in F\}$. Thus weak convergence is equivalent to convergence in distribution of the finite dimensional projections $\{\xi_t^A(x) : x \in F\}$.

To address the convergence of μ_t^A, we appeal to Liggett (1985) for the case $d = 1$ and to Bezuidenhout & Grimmett (1990) and Durrett & Griffeath (1982) for $d \geq 2$. First of all there exists a critical value λ_d such that for $\lambda \leq \lambda_d$, the contact process dies out with probability 1, regardless of its initial state at time $t = 0$ (subcritical case). If

$\delta_\emptyset$ denotes the distribution on $\{0,1\}^{\mathbb{Z}^d}$ that assigns probability 1 to the empty set, we have for every $A \subset \mathbb{Z}^d$ in the subcritical case

$$\mu_t^A \xrightarrow{w} \delta_\emptyset \quad \text{if } \lambda \leq \lambda_d.$$

Note that $\emptyset =$ "all healthy" is a trap .

In the supercritical case when $\lambda > \lambda_d$, the contact process ξ_t^A survives forever with positive probability for every non-empty set $A \subset \mathbb{Z}^d$. It survives forever with probability 1 if A is infinite. It is easy to show that the distribution $\mu_t^{\mathbb{Z}^d}$ of the process $\xi_t^{\mathbb{Z}^d}$ that starts with all sites infected, converges weakly to the so-called upper invariant measure $\nu = \nu_\lambda$. Here 'invariant' refers to the fact that the contact process $\{\xi_t^\nu : t \geq 0\}$ with ν as initial measure is stationary; in particular, the distribution μ_t^ν of ξ_t^ν is independent of t. Also, both $\{\xi_t^{\mathbb{Z}^d} : t \geq 0\}$ and $\{\xi_t^\nu : t \geq 0\}$ are spatially translation invariant in the sense that the distribution of $\{c \oplus \xi_t : t \geq 0\}$ is independent of the shift $c \in \mathbb{Z}^d$. Here $\{c\} \oplus \xi_t = \{c + x : x \in \xi_t\}$ is the Minkovski sum. Finally, for $\lambda > \lambda_d$, ν_λ assigns probability 0 to the empty set.

For a general non empty initial state A the convergence issue is decided by the *complete convergence theorem.* Define the random hitting time

$$\tau^A = \inf\{t : \xi_t^A = \emptyset\}\ , \quad A \subset \mathbb{Z}^d \ , \tag{1.2}$$

with the convention that $\tau^A = \infty$ if $\xi_t^A \neq \emptyset$ for all $t \geq 0$.

Theorem 1.1 *Let $A \subset \mathbb{Z}^d$ and $\lambda > \lambda_d$. Then, as $t \to \infty$*

$$\mu_t^A \xrightarrow{w} \mathbb{P}(\tau^A < \infty)\delta_\emptyset + \mathbb{P}(\tau^A = \infty)\nu_\lambda \ . \tag{1.3}$$

Thus, given that the process ξ_t^A survives, it tends in distribution to $\nu = \nu_\lambda$, the weight assigned to ν being the probability of survival starting from A. For a proof for $d = 1$ see Liggett (1985), Chapter VI, Theorem 2.28; for $d > 1$ see Durrett & Griffeath (1982), Bezuidenhout & Grimmett (1990), Theorem 4, and Durrett (1991).

If $\lambda > \lambda_d$ and $A = \mathbb{Z}^d$, the process $\xi_t^{\mathbb{Z}^d}$ survives forever with probability 1 and converges exponentially to the limit process, i.e. for positive C and γ and all $t \geq 0$,

$$0 \leq \mathbb{P}(\xi_t^{\mathbb{Z}^d}(x) = 1) - \mathbb{P}(\xi^\nu(x) = 1) \leq Ce^{-\gamma t} \quad . \tag{1.4}$$

(Durrett (1991)).

A second major result concerning the contact process is the so-called *shape theorem.* To formulate this result we first have to describe the graphical representation of contact processes due to Harris (1978). This is a particular coupling of all contact processes of a given dimension d and with a given value of λ, but with every possible initial state

A or initial distribution α. Consider space-time $\mathbb{Z}^d \times [0,\infty)$. For every site $x \in \mathbb{Z}^d$ we define on the line $x \times [0,\infty)$ a Poisson process with rate 1; for every ordered pair (x,y) of neighboring sites in $\mathbb{Z}^d$ we define a Poisson process with rate λ. All of these Poisson processes are independent.

We now draw a picture of $\mathbb{Z}^d \times [0,\infty)$ where for each site $x \in \mathbb{Z}^d$ we remove the points of the corresponding Poisson process with rate 1 from the line $x \times [0,\infty)$; for each ordered pair of neighboring sites (x,y) we draw an arrow going perpendicularly from the line $x \times [0,\infty)$ to the line $y \times [0,\infty)$ at the points of the Poisson process with rate λ corresponding to the pair (x,y). For any set $A \subset \mathbb{Z}^d$, define ξ_t^A to be the set of sites that can be reached by starting at time 0 at some site in A and travelling to time t along unbroken segments of lines $x \times [0,\infty)$ in the direction of increasing time, as well as arrows. Clearly, $\{\xi_t^A : t \geq 0\}$ is distributed as a contact process with initial set A. By choosing the initial set at random with distribution α, we define $\{\xi_t^\alpha : t \geq 0\}$. The obvious beauty of this construction is that for two initial sets of infected points $A \subset B$, we have $\xi_t^A \subset \xi_t^B$ for all t. **Whenever needed we shall assume that all contact processes are coupled according to the graphical construction.**

The contact process has the property of reversibility or self-duality. If, in the graphical representation, time is run backwards and all arrows representing infection of one site by another, are reversed, then the new graphical representation has precisely the same probabilistic structure as the original one. In particular

(1.5) $$\mathbb{P}(\xi_t^A \cap B \neq \emptyset) = P(\xi_t^B \cap A \neq \emptyset), \quad \textit{for all } A, B \subset \mathbb{Z}^d \textit{ and } t \geq 0 \ .$$

With $A = \{0\}$ and $B = \mathbb{Z}^d$ this yields

$$\mathbb{P}(\tau^{\{0\}} > t) = \mathbb{P}(\xi_t^{\mathbb{Z}^d}(0) = 1)$$

and letting $t \to \infty$ in the supercritical case, this reduces to

$$\mathbb{P}(\tau^{\{0\}} = \infty) = \mathbb{P}(\xi_t^{\nu}(0) = 1) \ .$$

Combining this with (1.4) we see that if $\lambda > \lambda_d$,

(1.6) $$\mathbb{P}(t < \tau^{\{0\}} < \infty) \leq Ce^{-\gamma t}$$

(cf. Durrett (1991)).

Let us write

(1.7) $$t(x) = \inf\{t : \xi_t^{\{0\}}(x) = 1\}$$

for the first infection time of the site x when starting the process with a single infected site at the origin. Let $\|\cdot\|$ denote the L^∞ norm on $\mathbb{R}^d$ and define

(1.8) $$H_t = \{y \in \mathbb{R}^d : \exists\, x \in \mathbb{Z}^d \text{ with } \|x-y\| \leq 1/2 \text{ and } t(x) \leq t\}$$

and

$$(1.9) \qquad K_t = \{y \in \mathbb{R}^d : \exists\, x \in \mathbb{Z}^d \text{ with } \|x - y\| \leq 1/2 \text{ and } \xi_t^{\{0\}}(x) = \xi_t^{\mathbb{Z}^d}(x)\}$$

H_t and K_t are the unions of the unit cubes centered respectively at sites that were infected at some time prior to t, or where the two processes $\xi_t^{\{0\}}$ and $\xi_t^{\mathbb{Z}^d}$ are equal at time t. We now formulate the shape theorem, while reminding the reader that $\xi_t^{\{0\}}$ and $\xi_t^{\mathbb{Z}^d}$ are defined by the graphical construction.

Theorem 1.2 *There exists a bounded convex subset U of $\mathbb{R}^d$ with the origin as an interior point such that for any $\varepsilon \in (0,1)$,*

$$(1.10) \qquad (1-\varepsilon)U \;\subset\; t^{-1}(H_t \;\cap\; K_t) \;\;\subset t^{-1}H_t \;\subset\; (1+\varepsilon)U \textit{ eventually,}$$

almost surely on the event $\{\tau^{\{0\}} = \infty\}$ where $\xi_t^{\{0\}}$ survives forever.

For a proof for $d = 1$ we refer to Durrett (1980); for $d > 1$ one may follow Bezuidenhout and Grimmett (1990) and Durrett (1991).

The shape theorem says that the two processes $\xi_t^{\{0\}}$ and $\xi_t^{\mathbb{Z}^d}$ may be coupled together in such way that conditional on the survival of the process $\xi_t^{\{0\}}$, they agree almost surely on a region which is asymptotically convex and whose diameter increases linearly in time.

Having described these well-known facts concerning the contact process, we now list the main results of the present paper. **At this point we should stress that we shall only be concerned with the supercritical case, i. e. in the remainder of this paper we shall tacitly assume that $\lambda > \lambda_d$.** First of all we strengthen the lower inclusion in Theorem 1.2 as follows

Theorem 1.3 *For any $\epsilon \in (0,1)$ and $r > 0$, there exists a positive number $A_{r,\epsilon}$ such that for every $t > 0$,*

$$(1.11) \qquad \mathbb{P}\Big((1-\epsilon)tU \subset H_t \cap K_t \mid \tau^{\{0\}} = \infty\Big) \geq 1 - A_{r,\epsilon}t^{-r}.$$

For statistical purposes a drawback of Theorems 1.2 and 1.3 is that the set U -and sometimes also the time t- are unknown and the experimenter only observes the set $\xi_t^{\{0\}}$. It is therefore of interest to show that on $\{\tau^{\{0\}} = \infty\}$ the convex hull $\mathcal{C}(\xi_t^{\{0\}})$ of the set of infected sites has the same asymptotic shape tU as $H_t \cap K_t$ and H_t.

Theorem 1.4 *For every $\epsilon \in (0,1)$,*

$$(1.12) \qquad (1-\epsilon)tU \subset \mathcal{C}(\xi_t^{\{0\}}) \subset (1+\epsilon)tU \textit{ eventually,}$$

a.s.on the set $\{\tau^{\{0\}} = \infty\}$.

We can also obtain a probability bound for $C(\xi_t^{\{0\}})$ corresponding to Theorem 1.3 for $H_t \cap K_t$.

Theorem 1.5 *For any $0 < \epsilon < 1$ and $r > 0$, there exists a positive number $A_{r,\epsilon}$ such that for every $t > 0$*

$$\mathbb{P}\Big((1-\epsilon)tU \subset C(\xi_t^{\{0\}}) \mid \tau^{\{0\}} = \infty\Big) \geq 1 - A_{r,\epsilon} t^{-r} \quad .$$

At first sight, these theorems would seem to suggest that for large t we can approximate the conditional probability of an event concerning the process $\xi_t^{\{0\}} \cap (1-\epsilon)tU$ given $\{\tau^{\{0\}} = \infty\}$, by computing the unconditional probability of the same event for the process $\xi_t^{\mathbf{Z}^d} \cap (1-\epsilon)tU$. Unfortunately this is false. Conditional probabilities for $\xi_t^{\{0\}} \cap (1-\epsilon)tU$ given $\{\tau^{\{0\}} = \infty\}$ can be approximated by conditional probabilities for $\xi_t^{\mathbf{Z}^d} \cap (1-\epsilon)tU$ given $\{\tau^{\{0\}} = \infty\}$. However, the latter probabilities are as intractable as the former and since $0 < \mathbb{P}(\tau^{\{0\}} = \infty) < 1$, we have no guarantee a priori that they will be close to the unconditional probabilities for $\xi_t^{\mathbf{Z}^d} \cap (1-\epsilon)tU$, unless of course these converge to zero as t tends to infinity. It follows that as long as we are concerned with statements concerning convergence in probability - that is about probabilities converging to zero - for the $\xi_t^{\{0\}} \cap (1-\epsilon)tU$ process conditioned on $\{\tau^{\{0\}} = \infty\}$, we may compute unconditionally for the process $\xi_t^{\mathbf{Z}^d} \cap (1-\epsilon)tU$. However, as soon as we are after results concerning limit distributions of statistics related to the process $\xi_t^{\{0\}} \cap (1-\epsilon)tU$ conditioned on $\{\tau^{\{0\}} = \infty\}$, then results like Theorems 1.2 and 1.3 are not much help.

To remedy this situation we shall provide a different coupling. For $\lambda > \lambda_d$, let $\overline{\xi}_t^{\{0\}}$ denote the process $\xi_t^{\{0\}}$ conditioned on $\{\tau^{\{0\}} = \infty\}$. In Theorem 1.6 we couple the process $\overline{\xi}_t^{\{0\}}$ directly to a process $\xi_{t-s}^{\mathbf{Z}^d}$ which starts at time s instead of time 0. For large s and $(t-s)$, the theorem provides a probability bound for equality of the processes on the set $(1-\epsilon)tU$, as well as for each individual site in $(1-\epsilon)tU$ separately.

Theorem 1.6 *For every $\epsilon \in (0,1)$ and $r > 0$ there exist numbers A_ϵ and $A_{r,\epsilon}$ depending on ϵ and (r,ϵ) respectively, such that for $s \wedge (t-s) \geq A_\epsilon$,*

$$\begin{aligned}(1.13) \qquad \mathbb{P}\Big(\overline{\xi}_t^{\{0\}} \cap (1-\epsilon)tU \;&=\; \xi_{t-s}^{\mathbf{Z}^d} \cap (1-\epsilon)tU\Big) \\ &\geq\; 1 - A_{r,\epsilon}\Big(s^{-r} + \frac{s^d}{(t-s)^r}\Big) \ .\end{aligned}$$

Moreover for every $x \in (1-\epsilon)tU$

$$(1.14) \qquad \mathbb{P}\Big(\overline{\xi}_t^{\{0\}}(x) = \xi_{t-s}^{\mathbf{Z}^d}(x)\Big) \geq 1 - A_{r,\epsilon}\Big(s^{-r} + (t-s)^{-r}\Big) \ .$$

Before formulating our results concerning the decaying correlations we need to introduce some notation. Let $H = \{0,1\}^{\mathbb{Z}^d}$ denote the state space for the contact process. For $f : H \to \mathbb{R}$ and $x \in \mathbb{Z}^d$, define

$$\begin{aligned} \Delta_f(x) &= \sup\Big\{|f(\eta) - f(\zeta)| : \eta, \zeta \in H \text{ and } \eta(y) = \zeta(y) \text{ for all } y \neq x\Big\} \ , \\ \text{(1.15)} \qquad \|\!|f|\!\| &= \sum_{x\in\mathbb{Z}^d} \Delta_f(x) \ . \end{aligned}$$

For $R_1, R_2 \subset \mathbb{Z}^d$, let $d(R_1, R_2)$ denote the L^1-distance of R_1 and R_2, thus

$$d(R_1,R_2) = \inf_{x\in R_1, y\in R_2} |x-y| = \inf_{x\in R_1, y\in R_2} \sum_{i=1}^{d} |x_i - y_i| \quad ,$$

Let

$$\text{(1.16)} \qquad D_R = \{f : H \to \mathbb{R}, \|\!|f|\!\| < \infty, f(\eta) \textit{ depends on } \eta \textit{ only through } \eta \cap R\} \ ,$$

i.e. D_R is the class of functions f with $\|\!|f|\!\| < \infty$ such that $f(\eta)$ depends on η only through $\eta(x)$ with $x \in R$.

First we show that the correlations between the states of sites for the process $\xi_t^{\mathbb{Z}^d}$ decay exponentially fast in the distance between them. This follows easily from known results.

Theorem 1.7 *There exist positive numbers γ and C such that for every R_1, $R_2 \subset \mathbb{Z}^d$, $f \in D_{R_1}$ $g \in D_{R_2}$, and $t \geq 0$,*

$$\text{(1.17)} \qquad \left|\text{cov}\left(f(\xi_t^{\mathbb{Z}^d}), g(\xi_t^{\mathbb{Z}^d})\right)\right| \ \leq \ C\|\!|f|\!\|.\|\!|g|\!\| e^{-\gamma d(R_1,R_2)}.$$

Our final result deals with decaying correlations for the $\overline{\xi}_t^{\{0\}}$ process. The proof is based on the embedding of Theorem 1.6.

Theorem 1.8 *For every $\epsilon \in (0,1)$ and $r > 0$ there exists a positive number $A_{r,\epsilon}$ such that for all $t \geq 0$ and all f and g satisfying*

$$f \in D_{R_1} \textit{ with } R_1 \subset (1-\epsilon)tU \cap \mathbb{Z}^d \ ,$$

$$g \in D_{R_2} \textit{ with } R_2 \subset \mathbb{Z}^d \quad ,$$

$$\text{(1.18)} \qquad \left|\text{cov}\left(f(\overline{\xi}_t^{\{0\}}), g(\overline{\xi}_t^{\{0\}})\right)\right| \leq \ A_{r,\epsilon} \ \|\!|f|\!\| \cdot \|\!|g|\!\| \ (d(R_1,R_2) \wedge t)^{-r} \quad .$$

It is interesting to compare Theorem 1.8 with Theorem 1.7. The bound in Theorem 1.8 is a power bound as compared with the exponential bound in Theorem 1.7, which is due to the fact that we have been content with a moment bound in Theorem 2.1. This is a relatively unimportant difference for most purposes. More interestingly, the bound of Theorem 1.8 is in terms of $d((R_1, R_2) \wedge t)$ instead of $d(R_1, R_2)$. But this is to be expected. If $d(R_1, R_2)$ is of larger order than t, then at least one of the sets R_1 or R_2 would be far outside the set tU and there would be no correlation except if H_t would extend far beyond tU. All we know about this possibility is that it can occur with probability $\mathcal{O}(e^{-\gamma t})$ by Lemma 3.1. It is therefore hardly surprising that the covariance bound in (1.18) should depend on t rather than on $d(R_1, R_2)$ in this case.

For technical reasons these results will be proved in a different order than they are presented above. In Section 2 we begin with the proof of Theorem 1.7 which is then used to obtain the moment inequality of Theorem 2.1 which lies at the root of all probability bounds in this paper. Theorem 1.3 is proved next in Section 3. Section 4 is devoted to the results concerning $C(\xi_t^{\{0\}})$ of Theorems 1.4 and 1.5. In Section 5 we first prove the embedding of $\overline{\xi}_t^{\{0\}}$ of Theorem 1.6 with the aid of Theorem 1.3, and then obtain Theorem 1.8 on the decaying correlations for $\overline{\xi}_t^{\{0\}}$.

In a companion paper Fiocco and van Zwet (1999) these probabilistic results will be used for a study of the estimation problem for the parameter λ of the supercritical contact process $\xi_t^{\{0\}}$. Based on a single observation of $\xi_t^{\{0\}}$ at a single unknown time t, we obtain an estimator $\widehat{\lambda}_t^{\{0\}}$ of λ which is strongly consistent and asymptotically normal as $t \to \infty$. To establish these results, we have to apply a law of large numbers and a central limit theorem to $\widehat{\lambda}_t^{\{0\}}$ and the results of the present paper - in particular Theorems 1.4, 1.6 and 1.8 - play a crucial role in establishing the central limit behavior of $\widehat{\lambda}_t^{\{0\}}$.

2 A moment inequality

In this section we prove Theorem 1.7 as well as an inequality for the central moments of certain functions of $\xi_t^{\mathbf{Z}^d}$ that will play a central role in the remainder of this paper.

Proof. of Theorem 1.7. By changing one coordinate at a time, we see that for two configurations $\eta, \zeta \in H$,

$$\left| f(\eta) - f(\zeta) \right| \leq \sum_{x \in \mathbf{Z}^d} \Delta_f(x) \; I_{\eta(x) \neq \zeta(x)} \quad . \tag{2.1}$$

We wish to bound $|\mathbb{E} f(\xi_t^{\mathbf{Z}^d}) - \mathbb{E} f(\xi_t^{\nu})|$ and without loss of generality we may assume that $\xi_t^{\mathbf{Z}^d}$ and ξ_t^{ν} are coupled according to the graphical representation, so that $\xi_t^{\mathbf{Z}^d}(x) \geq$

$\xi_t^\nu(x)$ for all $x \in \mathbb{Z}^d$. Because both $\xi_t^{\mathbb{Z}^d}$ and ξ_t^ν are translation invariant, we find

$$\left|\mathbb{E}f(\xi_t^{\mathbb{Z}^d}) - \mathbb{E}f(\xi_t^\nu)\right| \le \mathbb{E}\left|f(\xi_t^{\mathbb{Z}^d}) - f(\xi_t^\nu)\right|$$

$$\le \sum_{x\in\mathbb{Z}^d} \Delta_f(x)\mathbb{P}\left(\xi_t^{\mathbb{Z}^d}(x) \neq \xi_t^\nu(x)\right) \equiv \|f\|\mathbb{P}\left(\xi_t^{\mathbb{Z}^d}(0) \neq \xi_t^\nu(0)\right)$$

$$= \|f\|\mathbb{P}\left(\xi_t^{\mathbb{Z}^d}(0) = 1, \xi_t^\nu(0) = 0\right) = \|f\|\left(\mathbb{P}(\xi_t^{\mathbb{Z}^d}(0) = 1) - \mathbb{P}(\xi_t^\nu(0) = 1)\right).$$

Inequality (1.4) yields

$$\left|\mathbb{E}f(\xi_t^{\mathbb{Z}^d}) - \mathbb{E}f(\xi_t^\nu)\right| \le C\|f\|e^{-\gamma t} \quad . \tag{2.2}$$

Now (1.17) follows by applying Theorem 4.20 in Chapter 1 of Liggett (1985) and the lemma is proved. □

For $A \subset \mathbb{Z}^d$, define the total number of infected sites in the set A at time t as

$$n_t(A) = \sum_{x\in A} \xi_t(x) \quad . \tag{2.3}$$

The cardinality of a set $A \subset \mathbb{Z}^d$ will be denoted by $|A|$.

Lemma 2.1 *For any $k = 1, 2, ...,$ there exists a number $C_k > 0$ such that for every $A \subset \mathbb{Z}^d$ and $t \ge 0$,*

$$\mu_{2k} = \mathbb{E}\left(n_t^{\mathbb{Z}^d}(A) - \mathbb{E}n_t^{\mathbb{Z}^d}(A)\right)^{2k} \le C_k|A|^k \ . \tag{2.4}$$

Proof. Let $R = \max_j d(x_j, \{x_1, .., x_{j-1}, x_{j+1}, .., x_{2k}\})$ and write μ_{2k} as

$$\mu_{2k} = \mathbb{E}\Big(\sum_{x\in A}(\xi_t^{\mathbb{Z}^d}(x) - \mathbb{E}\xi_t^{\mathbb{Z}^d}(x))\Big)^{2k} = \sum_{x_1\in A} .... \sum_{x_{2k}\in A} \mathbb{E}\prod_{i=1}^{2k}\left(\xi_t^{\mathbb{Z}^d}(x_i) - \mathbb{E}\xi_t^{\mathbb{Z}^d}(x_i)\right) .$$

We have, for the proper $j = 1, 2, ..., 2k$,

$$\left|\mathbb{E}\prod_{i=1}^{2k}\left(\xi_t^{\mathbb{Z}^d}(x_i) - \mathbb{E}\xi_t^{\mathbb{Z}^d}(x_i)\right)\right| = \left|\mathbb{E}\Big((\xi_t^{\mathbb{Z}^d}(x_j) - \mathbb{E}\xi_t^{\mathbb{Z}^d}(x_j)) \prod_{i=1, i\neq j}^{2k} (\xi_t^{\mathbb{Z}^d}(x_i) - \mathbb{E}\xi_t^{\mathbb{Z}^d}(x_i))\Big)\right|$$

$$= \left|cov\Big(\xi_t^{\mathbb{Z}^d}(x_j), \prod_{i=1, i\neq j}^{2k} (\xi_t^{\mathbb{Z}^d}(x_i) - \mathbb{E}\xi_t^{\mathbb{Z}^d}(x_i)\Big)\right|$$

$$\le 2Cke^{-\gamma R} \quad ,$$

where the inequality follows from Theorem 1.7 with

$$\left|\left|\left|\prod_{i=1,i\neq j}^{2k}\left(\xi_t^{\mathbb{Z}^d}(x_i)-\mathbb{E}\xi_t^{\mathbb{Z}^d}(x_i)\right)\right|\right|\right|\leq 2k\ ,\quad\text{and}\quad \|\xi_t^{\mathbb{Z}^d}(x_j)\|=1\quad .$$

Hence

$$\mu_{2k}\leq 2Ck\sum_{x_1\in A}....\sum_{x_{2k}\in A}e^{-\gamma R}\quad .$$

Notice that the distance $d(x_j,\{x_1,..,x_{j-1},x_{j+1},..,x_{2k}\})=0$ unless x_j occurs only once in the sequence $x_1,...,x_{2k}$.

Let m_j denote the number of sites that occur j times in the sequence of sites $x_1,...,x_{2k}$. Define, for $m_1,...,m_{2k}$ with $\sum_{i=1}^{2k}jm_j=2k$ and $r\geq 0$,

$F_{m_1,...,m_{2k}}(r)=$ *number of sequences* $x_1,...,x_{2k}\in A$ *with given* $m_1,...,m_{2k}$ *and* $R<r$.

Then $F_{m_1,...,m_{2k}}(0)=0$ and for $r>0$,

$$F_{m_1,...,m_{2k}}(r)\leq\begin{cases}C_k'\ |A|^{m_2+m_3+\cdots}|A|^{\frac{m_1}{2}}\ r^{d\frac{m_1}{2}} & \text{if } m_1 \text{ even}\\ C_k'\ |A|^{m_2+m_3+\cdots}|A|^{\frac{m_1-1}{2}}\ r^{d\frac{(m_1+1)}{2}} & \text{if } m_1 \text{ odd}\end{cases}$$

where d is the dimension of the contact process and C_k' is an appropriate positive constant. This bound for $F_{m_1,...,m_{2k}}(r)$ is computed by noting that the $(m_2+m_3+...)$ sites which occur more than once can be chosen anywhere in A without contributing to R, which accounts for the factor $|A|^{m_2+m_3+\cdots}$. The m_1 sites which occur only once, however, have to be chosen within a distance r from another member of $x_1,...,x_{2k}$, and this gives rise to a factor $|A|^{\frac{m_1}{2}}r^{d\frac{m_1}{2}}$ or $|A|^{\frac{m_1-1}{2}}r^{d\frac{m_1+1}{2}}$ for even or odd m_1. Finally the combinatorics of the situation refers to ordering $2k$ sites and can therefore be bounded by C_k'.

If m_1 is odd, then $(m_3+m_4+...)>0$ as $\sum_j jm_j=2k$ and as a result,

$$\begin{aligned}\frac{m_1-1}{2}+m_2+m_3+... &= -\frac{1}{2}+\frac{m_1+2m_2+2(m_3+...)}{2}\\ &\leq -1+\frac{m_1+2m_2+3(m_3+...)}{2}\leq k-1\quad ,\end{aligned}$$

while if m_1 is even,

$$\frac{m_1}{2}+m_2+m_3+...\leq k\ .$$

Hence, if we define

$F_m(r)=$ *number of sequences* $x_1,...,x_{2k}\in A$ *with* $m_1=m$ *and* $R<r$,

then

$$F_m(r) \le \begin{cases} C_k'' \; |A|^k \; r^{d\frac{m}{2}} & \text{if } m \text{ even} \\ C_k'' \; |A|^{k-1} \; r^{d\frac{(m+1)}{2}} & \text{if } m \text{ odd} \end{cases} .$$

For $r > 0$, let $F(r)$ be the number of sequences $x_1, ..., x_{2k} \in A$ with $R < r$, so that

$$F(r) = \sum_{m=0}^{2k} F_m(r).$$

Summing the terms with even or odd values of m separately we obtain for $r > 0$,

$$\begin{aligned} F(r) &= \sum_{s=0}^{k} F_{2s}(r) + \sum_{s=1}^{k} F_{2s-1}(r) \\ &\le C_k''|A|^k \sum_{s=0}^{k} r^{ds} + C_k''|A|^{k-1} \sum_{s=1}^{k} r^{ds} \\ &\le 2C_k''|A|^k \sum_{s=0}^{k} r^{ds} \\ &\le C_k'''|A|^k \; r^{dk} \quad , \end{aligned}$$

where C_k''' is an appropriately chosen constant. As $F(0) = 0$,

$$\begin{aligned} \mu_{2k} &\le 2Ck \sum_{x_1 \in A} .... \sum_{x_{2k} \in A} e^{-\gamma R} \\ &\le 2Ck \sum_{r=0}^{\infty} e^{-\gamma r}(F(r+1) - F(r)) \\ &= 2Ck \Big(\sum_{r=1}^{\infty} e^{-\gamma(r-1)} F(r) - \sum_{r=1}^{\infty} e^{-\gamma r} F(r) \Big) \\ &\le C_k''''|A|^k \sum_{r=1}^{\infty} (e^{-\gamma(r-1)} - e^{-\gamma r}) r^{dk} \\ &= C_k''''|A|^k (e^{\gamma} - 1) \sum_{r=1}^{\infty} e^{-\gamma r} r^{dk} \\ &\le C_k |A|^k \quad . \end{aligned}$$

for an appropriate $C_k > 0$. □

A more general version of Lemma 2.1 may be formulated as follows. Let $g : H \to \mathbb{R}$ satisfy $g \in D_{B_{\{0,r\}}}$, where $B_{\{0,r\}}$ is an L^∞-ball, i.e. a hypercube centered at the origin with sides $2r$. Hence $g(\eta)$ depends on η only through $\eta(x)$ with x in a fixed hypercube $B_{\{0,r\}} \subset \mathbb{R}^d$. For $a \in \mathbb{Z}^d$, let ${}_a\eta$ denote a shifted version of η with ${}_a\eta(x) = \eta(a + x)$ for all $x \in \mathbb{Z}^d$ and define $g_a : H \to R$ by $g_a(\eta) = g({}_a\eta)$ for $\eta \in H$. Note that $g_a \in D_{B_{\{a,r\}}}$. Lemma 2.1 can now easily be generalized as follows

Theorem 2.1 *For any $k = 1, 2, \ldots$ and $r > 0$ there exists a number $C_{k,r} > 0$ such that for every $A \subset \mathbb{Z}^d$, $g \in D_{B_{\{0,r\}}}$ and $t \geq 0$,*

$$\mathbb{E}\Big(\sum_{a \in A}\Big(g_a(\xi_t^{\mathbb{Z}^d}) - \mathbb{E}g_a(\xi_t^{\mathbb{Z}^d})\Big)\Big)^{2k} \leq C_{k,r}|\!|\!|g|\!|\!|^{2k}\ |A|^k \quad . \tag{2.5}$$

Proof. The proof goes through the same counting argument we have employed for proving Lemma 2.1 and uses the fact that $d(B_{a,r}, B_{b,r}) \geq d(a,b) - 4r$ as well as the inequality $|\!|\!|f \cdot g|\!|\!| \leq 2|\!|\!|f|\!|\!| \cdot |\!|\!|g|\!|\!|$ (cf. Liggett (1985), page 41). □

Recall the definition of the random hitting time for the process ξ_t^A,

$$\tau^A = \inf\{t : \xi_t^A = \emptyset\} \quad , \quad A \subset \mathbb{Z}^d \quad .$$

We shall need the following corollary.

Corollary 2.1 *For any $r > 0$ there exists a number $C_r > 0$ such that for every $A \subset \mathbb{Z}^d$ and $t \geq 0$,*

$$\mathbb{P}(n_t^{\mathbb{Z}^d}(A) \leq 1/2\mathbb{E}n_t^{\mathbb{Z}^d}(A)) \leq C_r|A|^{-r} \quad , \tag{2.6}$$

$$\mathbb{P}(\tau^A < \infty) \leq C_r|A|^{-r} \quad . \tag{2.7}$$

Proof. It is obviously enough to prove the corollary for integer $k = 1, 2 \ldots$, instead of real $r > 0$. Applying the self-duality property (1.5) for $B = \mathbb{Z}^d$, we find

$$\mathbb{P}(\tau^A < t) = \mathbb{P}(\xi_t^{\mathbb{Z}^d} \cap A = \emptyset) = \mathbb{P}(n_t^{\mathbb{Z}^d}(A) = 0) \quad .$$

Since the process $\xi_t^{\mathbb{Z}^d}$ is translation invariant, the graphical construction yields

$$\mathbb{E}n_t^{\mathbb{Z}^d}(A) = \sum_{x \in A}\mathbb{E}\xi_t^{\mathbb{Z}^d}(0) = |A|\mathbb{E}\xi_t^{\mathbb{Z}^d}(0) \geq |A|\ \mathbb{E}\xi^\nu(0)$$

where the right-hand side is independent of t. Therefore for every $t \geq 0$ and $k = 1, 2, \ldots$

$$\begin{aligned}
\mathbb{P}(\tau^A \leq t) &= \mathbb{P}(n_t^{\mathbb{Z}^d}(A) = 0) \\
&\leq \mathbb{P}(n_t^{\mathbb{Z}^d}(A) \leq 1/2\mathbb{E}n_t^{\mathbb{Z}^d}(A)) \\
&\leq \mathbb{P}\Big(\Big|n_t^{\mathbb{Z}^d}(A) - \mathbb{E}n_t^{\mathbb{Z}^d}(A)\Big| \geq 1/2\mathbb{E}n_t^{\mathbb{Z}^d}(A)\Big) \\
&\leq \frac{2^{2k}\mu_{2k}}{[\mathbb{E}n_t^{\mathbb{Z}^d}(A)]^{2k}} \leq \frac{2^{2k}\mu_{2k}}{[\mathbb{E}\xi^\nu(0)]^{2k}|A|^{2k}} \leq C_k|A|^{-k}
\end{aligned}$$

by Lemma 2.1. This implies

$$\mathbb{P}(\tau^A < \infty) \leq C_k|A|^{-k}$$

because the bound does not depend on t . □

3 A probability bound

In this section we shall prove Theorem 1.3. We proceed as follows. We first need to bound the probability that H_t is not contained in the ball $B_{\{0,ct\}}$ for large c. As a second step we establish a probability bound for the first infection time $t(x)$. Together, these results allow us to prove Theorem 1.3.

Lemma 3.1 *Let $B_{\{x,r\}} = \{y : \|y - x\| \leq r\}$ be the hypercube with side $2r$ centered at x. There exist positive numbers c, C and γ such that for all $t \geq 0$*

$$\mathbb{P}(H_t \subset B_{\{0,ct\}}) \geq 1 - Ce^{-\gamma t} \quad . \tag{3.1}$$

Proof. Compare the contact process with Richardson's growth process and apply the result in Durrett (1988), Chapter 1. □.

Let $\bar{\xi}_t^{\{0\}} = (\xi_t^{\{0\}} | \tau^{\{0\}} = \infty)$ denote the conditional process $\xi_t^{\{0\}}$ given that $\tau^{\{0\}} = \inf\{t : \xi_t^{\{0\}} = \emptyset\} = \infty$. Denote by $\overline{\mathbb{P}}$ and $\overline{\mathbb{E}}$ the conditional probability and expectation respectively, given that $\xi_t^{\{0\}}$ survives forever, i.e. $\overline{\mathbb{P}}(\cdot) = \mathbb{P}(\cdot | \tau^{\{0\}} = \infty)$ and $\overline{\mathbb{E}}(\cdot) = \mathbb{E}(\cdot | \tau^{\{0\}} = \infty)$.

We shall also have to extend the definition of $t(x)$ in (1.7) to $x \in \mathbb{R}^d$ by letting $t(x)$ be the first infection time of the point in $\mathbb{Z}^d$ closest to x. Thus for $x \in \mathbb{R}^d$,

$$t(x) = \min\{t(y) : y \in \mathbb{Z}^d, \|y - x\|t \leq \frac{1}{2}\} \quad , \tag{3.2}$$

or equivalently

$$t(x) = \inf\{t : x \in H_t\} \quad . \tag{3.3}$$

Note that this definition of $t(x)$ for $x \in \mathbb{R}^d$ implies that

$$H_t = \{x \in \mathbb{R}^d : t(x) \leq t\} \quad . \tag{3.4}$$

Moreover by (3.4), inequality (3.1) can be inverted to yield

$$\mathbb{P}\left(t(x) > \frac{\|x\|}{c}\right) \geq 1 - Ce^{-\gamma\|x\|/c} \quad . \tag{3.5}$$

Before formulating the next lemma we need to mention some results that have been proved by Durrett & Griffeath (1982). They showed that positive c, C and γ exist such that

$$\overline{\mathbb{P}}(B_{\{x,ct\}} \not\subset H_{t(x)+t+l^2} \text{ for some } \; t \geq 0) \leq Ce^{-\gamma l} \, , \tag{3.6}$$

$$\overline{\mathbb{P}}(B_{\{x,ct\}} \not\subset K_{t(x)+t+l^2} \text{ for some } \; t \geq 0) \leq Ce^{-\gamma l} \, , \tag{3.7}$$

for every $x \in \mathbb{R}^d$ and positive l. Notice that by choosing $x = 0$ and $t = \|y\|/c$ for $y \in \mathbb{Z}^d$ in (3.6) we obtain

$$\overline{\mathbb{P}}\Big(t(y) \le \frac{\|y\|}{c} + l^2 \quad for\ all\ y \in \mathbb{R}^d\Big) \ge 1 - Ce^{-\gamma t} \quad . \tag{3.8}$$

Moreover there exists a finite function $\mu : \mathbb{R}^d \to \mathbb{R}$ such that for every $x \in \mathbb{R}^d$,

$$\lim_{n\to\infty} \frac{t(nx)}{n} = \mu(x) \quad \overline{\mathbb{P}} \qquad a.\ s. \tag{3.9}$$

The set U in the shape theorem is defined by

$$U = \{x \in \mathbb{R}^d : \mu(x) \le 1\} \quad . \tag{3.10}$$

Lemma 3.2 *For every $x \in \mathbb{R}^d$ and every positive r and ϵ, there exists a positive number $A_{x,r,\epsilon}$ such that for all $n = 1, 2, ..,$*

$$\overline{\mathbb{P}}\Big(\frac{t(nx)}{n} \ge \mu(x) + \epsilon\Big) \le A_{x,r,\epsilon} \quad n^{-r} \tag{3.11}$$

Proof. As $t(0) = 0$ and $\mu(0) = 0$, the lemma is trivial for $x = 0$. Assume therefore that $x \ne 0$. Durrett & Griffeath (1982) proved that under $\overline{\mathbb{P}}$ we have for all x, $y \in \mathbb{R}^d$,

$$t(x + y) \le t(x) + s(y) + v(x, y) \quad , \tag{3.12}$$

where $s(y)$ is an appropriately chosen copy of $t(y)$, independent of $t(x)$, and $v(x, y)$ is an error term such that

$$\overline{\mathbb{E}}(v^2(x, y)) \le C(\|x\| + 1) \tag{3.13}$$

and

$$\overline{\mathbb{P}}(v(x, y) \ge C\|x\|^{\frac{1}{2}}) \le Ae^{-\alpha\|x\|^{\frac{1}{4}}} \quad , \tag{3.14}$$

with C, A and α positive constants. In (3.12) we substitute mx and $(n - m)x$ for x and y respectively, where $x \in \mathbb{R}^d$, $x \ne 0$ and $1 \le m < n$ are integers. This yields

$$t(nx) \le t(mx) + s((n - m)x) + v(mx, (n - m)x) \quad . \tag{3.15}$$

Fix $x \ne 0$ in $\mathbb{R}^d$ and define

$$a_n = a_n(x) = \overline{\mathbb{E}}(t(nx)) \quad . \tag{3.16}$$

Clearly (3.13) ensures that for $C_x = C^{1/2}(\|x\|^{1/2} + 1)$,

$$\overline{\mathbb{E}}\Big|v(mx, (n - m)x)\Big| \le \Big[C\Big(\|mx\| + 1\Big)\Big]^{\frac{1}{2}} \le C^{\frac{1}{2}}\Big(\|mx\|^{\frac{1}{2}} + 1\Big) \le C_x m^{\frac{1}{2}} \quad . \tag{3.17}$$

It follows from (3.15) that for $n > m \geq 1$,

$$a_n \leq a_m + a_{n-m} + C_x m^{\frac{1}{2}} \quad . \tag{3.18}$$

Write $n = km + l$ with $k = 1, 2, ..$ and $0 \leq l \leq m-1$. Iterate (3.18) to obtain

$$a_n \leq k a_m + a_l + C_x k m^{\frac{1}{2}} \quad . \tag{3.19}$$

Notice that for $m = 1$, (3.19) yields $a_n \leq n a_1 + C_x n$ so that a_n/n is bounded by $(a_1 + C_x)$ for all n. Also from (3.19),

$$\begin{aligned} \frac{a_n}{n} &\leq \frac{km}{n}\frac{a_m}{m} + \frac{\max_{\{1 \leq l \leq m-1\}} a_l}{n} + C_x \frac{km^{\frac{1}{2}}}{n} \\ &\leq \frac{a_m}{m} + (a_1 + C_x)\frac{m}{n} + C_x \frac{km^{\frac{1}{2}}}{n} \\ &\leq \frac{a_m}{m} + C'_x\left(\frac{m}{n} + m^{-1/2}\right) \quad . \end{aligned} \tag{3.20}$$

Let $m_j \to \infty$ and $n_j \to \infty$ be sequences such that $\frac{a_{m_j}}{m_j} \to \liminf_n \frac{a_n}{n}$ and $\frac{a_{n_j}}{n_j} \to \limsup_n \frac{a_n}{n}$. Moreover, suppose that $m_j \leq n_j$ for all j and that $\frac{m_j}{n_j} \to 0$ as $j \to \infty$. This is always possible because we can "thin out " n_j as much as we want. Replacing m and n in (3.20) by m_j and n_j and letting $j \to \infty$, we find

$$\limsup_n \frac{a_n}{n} \leq \liminf_n \frac{a_n}{n}$$

and hence

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{\mathbb{E}t(nx)}{n} = \alpha = \alpha(x)$$

exists. Since a_n/n is bounded, α is finite and obviously $\alpha(0) = 0$. Moreover (3.5) implies that

$$\mathbb{P}\left(\frac{t(nx)}{n} > \frac{\|x\|}{c}\right) \geq 1 - Ce^{-\gamma n\|x\|/c}$$

and hence $\alpha(x) > 0$ for $x \neq 0$.

Having proved that $\mathbb{E}t(nx)/n \to \alpha = \alpha(x)$ for all $x \in \mathbb{R}^d$, we return to (3.15). Fix $x \neq 0$, $\epsilon > 0$ and take m_0 sufficiently large to ensure that

$$\left|\frac{a_m}{m} - \alpha\right| \leq \frac{\epsilon}{8} \qquad \text{for all } m \geq m_0 = m_0(x, \epsilon) \quad .$$

For $1 \leq m < n$ we write $v_{m,n-m}$ for $v(mx, (n-m)x)$ and rewrite (3.15) as

$$t(nx) = t(mx) + s((n-m)x) + Y_{m,n-m} \ , \qquad Y_{m,n-m} \leq v_{m,n-m} \quad . \tag{3.21}$$

Notice that $Y_{m,n-m}$ is bounded above by $v_{m,n-m}$ but may have large negative values. However (3.21) does imply that

$$\mathbb{E}Y_{m,n-m} = a_n - a_m - a_{n-m} \tag{3.22}$$

which allows us to deal with this problem. For $n = 1, 2, \ldots$, write Z_n for $t(nx)$, $(Z_1^{(i)}, Z_2^{(i)}, \ldots)$ for independent copies of $(Z_1, Z_2, \ldots)$ and F_n for the conditional distribution function of $Z_n - \mathbb{E}Z_n$ given $\{\tau^{\{0\}} = \infty\}$. Take $n = km + l$ with $0 \le l \le m-1$. By (3.21)

$$\begin{aligned}
1 - F_n(\epsilon n) &= \mathbb{P}(Z_n - \mathbb{E}Z_n \ge \epsilon n) \\
&= \mathbb{P}((Z_m^{(1)} - \mathbb{E}Z_m) + (Z_{n-m}^{(2)} - \mathbb{E}Z_{n-m}) + (Y_{m,n-m} - \mathbb{E}Y_{m,n-m}) \ge \epsilon n) \\
&\le \mathbb{P}((Z_m^{(1)} - \mathbb{E}Z_m) + (Z_{n-m}^{(2)} - \mathbb{E}Z_{n-m}) + v_{m,n-m} \ge \epsilon n + a_n - a_m - a_{n-m}) \quad .
\end{aligned}$$

Now (3.14) yields

$$\mathbb{P}\left(v_{m,n-im} \ge D_x m^{\frac{1}{2}}\right) \le Ae^{-\alpha'_x m^{1/4}}$$

for positive $D_x = C\|x\|^{1/2}$, $\alpha'_x = \alpha\|x\|^{1/4}$ and $i = 1, 2, \ldots, k$. Hence

$$1 - F_n(\epsilon n) \le 1 - F_m * F_{n-m}(\epsilon n + a_n - a_m - a_{n-m} - D_x m^{1/2}) + Ae^{-\alpha'_x m^{1/4}}$$

where $*$ denotes the convolution of the distribution functions F_m and F_{n-m}. Applying this argument to F_{n-m} instead of F_n, we find for every $z \in \mathbb{R}$,

$$\begin{aligned}
&1 - F_{n-m}(\epsilon n + a_n - a_m - a_{n-m} - D_x m^{1/2} - z) \\
&\quad \le 1 - F_m * F_{n-2m}(\epsilon n + a_n - 2a_m - a_{n-2m} - 2D_x m^{1/2} - z) + Ae^{-\alpha'_x m^{1/4}} \quad .
\end{aligned}$$

It follows that

$$\begin{aligned}
1 - F_n(\epsilon n) &\le \int_{-\infty}^{\infty} [1 - F_{n-m}(\epsilon n + a_n - a_m - a_{n-m} - D_x m^{1/2} - z)] dF_m(z) + Ae^{-\alpha'_x m^{1/4}} \\
&\le \int_{-\infty}^{\infty} [1 - F_m * F_{n-2m}(\epsilon n + a_n - 2a_m - a_{n-2m} - 2D_x m^{1/2} - z) + Ae^{-\alpha'_x m^{1/4}}] dF_m(z) \\
&\quad + Ae^{-\alpha'_x m^{1/4}} \\
&= 1 - F_m * F_m * F_{n-2m}(\epsilon n + a_n - 2a_m - a_{n-2m} - 2D_x m^{1/2}) + 2Ae^{-\alpha'_x m^{1/4}} \quad .
\end{aligned}$$

Repeating this procedure we arrive at

$$1 - F_n(\epsilon n) \le 1 - F_m^{*k} * F_l(\epsilon n + a_n - ka_m - a_l - kD_x m^{1/2}) + kAe^{-\alpha'_x m^{1/4}}$$

where F_m^{*k} denotes the k-fold convolution of F_m.

For $m \geq m_0(x,\epsilon)$, we have $a_m \leq (\alpha+\epsilon/8)m$ and $a_n \geq (\alpha-\epsilon/8)n$. We noted earlier that $a_n \leq (a_1 + C_x)n$ for all n. It follows that for $n \geq 4(a_1(x)+C_x)m/\epsilon$,

$$\begin{aligned} \epsilon n + a_n - ka_m - a_l &\geq \epsilon n - \epsilon/8(n+km) + \alpha(n-km) - a_l \\ &\geq (3/4)\epsilon n + \alpha l - (a_1+C_x)l \\ &\geq (3/4)\epsilon n - (a_1+C_x)m \geq \frac{\epsilon}{2}n \quad . \end{aligned}$$

Moreover, for $m \geq m_1(x,\epsilon) = (16D_x^2/\epsilon^2) \vee m_0(x,\epsilon)$, we have $kD_x m^{1/2} \leq D_x n m^{-1/2} \leq \epsilon n/4$ and hence

$$\begin{aligned} &\mathbb{P}\Big(Z_n - \mathbb{E}Z_n \geq \epsilon n\Big) \\ &\leq \mathbb{P}\Big(\sum_{i=1}^{k}(Z_m^{(i)} - \mathbb{E}Z_m) + (Z_l^{(k+1)} - \mathbb{E}Z_l) \geq \frac{\epsilon}{4}n\Big) + Ake^{-\alpha_x' m^{1/4}} \end{aligned} \tag{3.23}$$

provided $m \geq m_1(x,\epsilon)$ and $n \geq A(x,\epsilon)m$ with $A(x,\epsilon) = 1 \vee 4(a_1(x)+C_x)/\epsilon$.

Next we bound the moments of $\sum_{i=1}^k (Z_m^{(i)} - \mathbb{E}Z_m) + (Z_l^{(k+1)} - \mathbb{E}Z_l)$ of even order $2r$. We have

$$\begin{aligned} &\mathbb{E}\Big(\sum_{i=1}^{k}(Z_m^{(i)} - \mathbb{E}Z_m) + (Z_l^{(k+1)} - \mathbb{E}Z_l)\Big)^{2r} = \\ &\sum{}^{*} \frac{(2r)!}{\prod \nu_i!}\, \mathbb{E}\prod_{i=1}^{k}(Z_m^{(i)} - \mathbb{E}Z_m)^{\nu_i}(Z_l^{(k+1)} - \mathbb{E}Z_l)^{\nu_{k+1}} \ , \end{aligned}$$

where the summation $\sum^*$ runs over all non-negative integers $\nu_1, ..., \nu_{k+1}$ with sum $2r$. Now a term for which even a single ν_i equals 1 vanishes because the $(Z_1^{(i)}, Z_2^{(i)}, ...)$ are independent. For the remaining terms, the expectation may be bounded as follows:

$$\begin{aligned} &\mathbb{E}\Big|\prod_{i=1}^{k}(Z_m^{(i)} - \mathbb{E}Z_m)^{\nu_i}(Z_l^{(k+1)} - \mathbb{E}Z_l)^{\nu_{k+1}}\Big| \leq \\ &\prod_{i=1}^{k}\Big(\mathbb{E}\Big|Z_m^{(i)} - \mathbb{E}Z_m\Big|^{2r}\Big)^{\nu_i/2r}\Big(\mathbb{E}\Big|Z_l^{(k+1)} - \mathbb{E}Z_l\Big|^{2r}\Big)^{\frac{\nu_{k+1}}{2r}} \\ &= \Big(\mathbb{E}\Big|Z_m - \mathbb{E}Z_m\Big|^{2r}\Big)^{1-\nu_{k+1}/2r}\Big(\mathbb{E}\Big|Z_l - \mathbb{E}Z_l\Big|^{2r}\Big)^{\nu_{k+1}/2r} \\ &\leq 2^{2r}\Big(\mathbb{E}Z_m^{2r}\Big)^{1-\nu_{k+1}/2r}\Big(\mathbb{E}Z_l^{2r}\Big)^{\nu_{k+1}/2r} \\ &\leq 2^{2r}\Big(\mathbb{E}Z_m^{2r} \vee \mathbb{E}Z_l^{2r}\Big) \ . \end{aligned}$$

It follows that

$$\overline{\mathbb{E}}\Big(\sum_{i=1}^{k}(Z_m^{(i)} - \overline{\mathbb{E}}Z_m) + (Z_l^{(k+1)} - \overline{\mathbb{E}}Z_l)\Big)^{2r} \leq \tag{3.24}$$

$$(2r!)\ 2^{2r}(\overline{\mathbb{E}}Z_m^{2r} \vee \overline{\mathbb{E}}Z_l^{2r}) \sum{}^{**}1\ ,$$

where $\sum^{**}$ indicates summation over non-negative integers $\nu_1, ..., \nu_{k+1}$ which are not equal to 1 and sum to $2r$. The number of terms in this sum is bounded by $A_r(k+1)^r$, where A_r depends only on r. Remembering that $Z_m = t(mx)$, we find that (3.8) implies that for all m, $\overline{\mathbb{P}}(Z_m \geq \frac{\|x\|}{c}m + z^2) \leq Ce^{-\gamma z}$. Hence $\overline{\mathbb{E}}Z_m^{2r} \leq C_{x,r}m^{2r}$ and $\overline{\mathbb{E}}Z_l^{2r} \leq C_{x,r}l^{2r} \leq C_{x,r}m^{2r}$. Together with (3.24) and the Markov inequality this yields for $k \geq 1$

$$\overline{\mathbb{P}}\Big(\sum_{i=1}^{k}(Z_m^{(i)} - \overline{\mathbb{E}}Z_m) + (Z_l^{(k+1)} - \overline{\mathbb{E}}Z_l) \geq \frac{\epsilon}{4}n\Big) \leq C_{x,r,\epsilon}\frac{k^r m^{2r}}{n^{2r}} \leq C_{x,r,\epsilon}k^{-r}. \tag{3.25}$$

Combining this with (3.23) we find that for every $x \neq 0$, $r > 0$ and $\epsilon > 0$,

$$\overline{\mathbb{P}}(Z_n - \overline{\mathbb{E}}Z_n) \geq \epsilon n) \leq C_{x,r,\epsilon}k^{-r} + Ake^{-\alpha_x' m^{1/4}} \quad ,$$

provided $m \geq m_1(x, \epsilon)$ and $n \geq A(x, \epsilon)m$. Throughout we have $n = km + l$, $0 \leq l \leq m - 1$ and as $n \geq m$, we have $k \geq 1$. Choosing $m = [n^\delta]$ for some small δ and and $k \sim n^{1-\delta}$ we find that for every $x \neq 0$ in $\mathbb{R}^d$, $r > 0$ and $\epsilon > 0$, there exists $A_{x,r,\epsilon} > 0$ such that

$$\overline{\mathbb{P}}(t(nx) \geq \overline{\mathbb{E}}t(nx) + \epsilon n) \leq A_{x,r,\epsilon}n^{-r} \quad .$$

As $\overline{\mathbb{E}}t(nx)/n \to \alpha(x)$ the proof of the lemma is completed if we show that $\alpha(x) = \mu(x)$ for all $x \neq 0$ in $\mathbb{R}^d$.

Once more we use (3.8) to obtain

$$\overline{\mathbb{P}}\Big(t(nx) \geq \frac{\|x\|}{c}n + z^2\Big) \leq Ce^{-\gamma z} \quad .$$

This implies that

$$\begin{aligned}
\overline{\mathbb{E}}\Big(\frac{t(nx)}{n} - \frac{\|x\|}{c}\Big)^+ &= \int_{\|x\|/c} \overline{\mathbb{P}}\Big(\frac{t(nx)}{n} > s\Big)ds \\
&= \int_0^\infty \overline{\mathbb{P}}\Big(\frac{t(nx)}{n} \geq \frac{\|x\|}{c} + v\Big)dv \\
&\leq \int_0^\infty Ce^{-\gamma(nv)^{1/2}}dv \to 0 \quad as \quad n \to \infty \quad .
\end{aligned}$$

As $t(nx)/n \to \mu(x)$ $\overline{\mathbb{P}}$ –a.s. (cf. Durrett 1988), this ensures that $\mu(x) \leq \|x\|/c$ for all x. Hence by the dominated convergence theorem,

$$\overline{\mathbb{E}}\Big(\frac{t(nx)}{n} \wedge \frac{\|x\|}{c}\Big) \to \mu(x) \wedge \frac{\|x\|}{c} = \mu(x) \ ,$$

so

$$\overline{\mathbb{E}}\frac{t(nx)}{n} = \overline{\mathbb{E}}\Big(\frac{t(nx)}{n} - \frac{\|x\|}{c}\Big)^+ + \overline{\mathbb{E}}\Big(\frac{t(nx)}{n} \wedge \frac{\|x\|}{c}\Big) \to \mu(x) \ .$$

It follows that $\alpha(x) = \mu(x)$ for all x, which completes the proof. □

Remark 3.1 We have considered in Lemma 3.2 the limiting behavior of $t(nx)/n$ as n runs through the integers 1, 2... The same results will hold for $t(sx)/s$ for real $s \to \infty$, to wit for $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$,

(3.26)
$$\lim_{s\to\infty} \frac{\overline{\mathbb{E}}t(sx)}{s} = \mu(x) \ ,$$

$$\mathbb{P}\Big(\frac{t(sx)}{s} \geq \mu(x) + \epsilon\Big) \leq A_{x,r,\epsilon}\ s^{-r} \ .$$

The proof is straightforward. One simply replaces the integers $1 \leq m \leq n$ with $n = km + l$, $0 \leq l \leq m-1$ by the real numbers $1 \leq u \leq w$ with $w = ku + v$ for integer k and $0 \leq v < u$.

Proof of Theorem 1.3. From the obvious linearity of μ on lines through the origin and the definition of U in (3.10), it follows that

$$x \in (1-\epsilon)U \Rightarrow \mu(x) \leq 1 - \epsilon \ . \tag{3.27}$$

Because U is compact we can cover $(1-\epsilon)U$ by a finite number of balls $B_i = B_{\{x_i, c(1-\mu(x_i))/2\}}$ $i = 1, 2, .., N_\epsilon$ where all $x_i \in (1-\epsilon)U$, so that $\mu(x_i) \leq (1-\epsilon)$ by (3.27). Invoking (3.11) with ϵ replaced by $\epsilon/4$, we see that for every $i = 1, 2.., N_\epsilon$ we have with $\overline{\mathbb{P}} - probability \geq 1 - A_{x_i,r,\epsilon/4}\ s^{-r}$,

$$\begin{aligned} t(sx_i) + \frac{1-\mu(x_i)}{2}s + l^2 &\leq s(\mu(x_i) + \frac{\epsilon}{4}) + \frac{1-\mu(x_i)}{2}s + l^2 \\ &= \frac{s}{2}\mu(x_i) + \frac{s}{2} + \frac{\epsilon}{4}s + l^2 \\ &\leq s - \frac{\epsilon}{4}s + l^2 \ . \end{aligned} \tag{3.28}$$

By taking $x = sx_i$ and $t = s(1-\mu(x_i))/2$ in (3.6) we find that for every $i = 1, 2, ..., N(\epsilon)$,

$$\overline{\mathbb{P}}(sB_i \subset H_{t(sx_i)+\frac{1-\mu(x_i)}{2}s+l^2} \textit{ for all } \quad s \geq 0) \geq 1 - Ce^{-\gamma l}$$

where $sB_i = B_{\{sx_i, cs(1-\mu(x_i))/2\}}$. Hence for $l^2 = (\epsilon/4)s$ inequality (3.28) and the monotonicity of H_t in t imply that

$$\overline{\mathbb{P}}\Big(\bigcup_{i=1}^{N_\epsilon} sB_i \subset H_s\Big) \geq 1 - CN(\epsilon)\ e^{-\gamma(\epsilon s)^{1/2}/2} - \sum_{i=1}^{N(\epsilon)} A_{x_i,r,\epsilon/4}\ s^{-r} \geq 1 - A_{r,\epsilon} s^{-r} \quad .$$

As $\bigcup_{i=1}^{N_\epsilon} sB_i \supset (1-\epsilon)sU$, we arrive at

$$\overline{\mathbb{P}}((1-\epsilon)sU \subset H_s) \geq 1 - A_{r,\epsilon}\ s^{-r} \tag{3.29}$$

for every positive ϵ, r and s.

Things are a bit more complicated for K_s because it is not monotone in s. Recall inequality (3.7)

$$\overline{\mathbb{P}}(B_{\{x,ct\}} \subset K_{t(x)+t+l^2} \textit{ for all } \quad t \geq 0) \geq 1 - Ce^{-\gamma l}. \tag{3.30}$$

Since the event involved is that $B_{\{x,ct\}} \subset K_{t(x)+t+l^2}$ for all $t \geq 0$, this also includes random times $T \geq 0$. For this random time we choose

$$T = (s(1 - \frac{\epsilon}{4}) - t(sx_i)) \vee 0 \quad ,$$

and obtain

$$\overline{\mathbb{P}}(B_{\{sx_i,cT\}} \subset K_{t(sx_i)+T+l^2}) \geq 1 - Ce^{-\gamma l}. \tag{3.31}$$

According to (3.11) with ϵ replaced by $\epsilon/4$ we know that

$$\frac{t(sx_i)}{s} \leq \mu(x_i) + \frac{\epsilon}{4} \tag{3.32}$$

on a set of $\overline{\mathbb{P}}$-probability at least $1 - A_{x_i,r,\epsilon/4}\ s^{-r}$. Since $\mu(x_i) \leq 1 - \epsilon$, (3.32) yields

$$s(1-\frac{\epsilon}{4}) - t(sx_i) \geq s(1 - \frac{\epsilon}{4} - \mu(x_i) - \frac{\epsilon}{4}) \geq \frac{\epsilon s}{2} \geq 0 \quad ,$$

so that $T \geq 0$ and hence

$$T = s(1-\epsilon/4) - t(sx_i) \quad . \tag{3.33}$$

But this implies

$$\begin{aligned} cT = c(s(1-\frac{\epsilon}{4}) - t(sx_i)) \quad &\geq \quad cs(1 - \frac{\epsilon}{4} - \mu(x_i) - \frac{\epsilon}{4}) \\ &\geq \quad cs\frac{1-\mu(x_i)}{2} \quad , \end{aligned} \tag{3.34}$$

again because $\mu(x_i) \leq 1 - \epsilon$. This, in turn, ensures that

$$B_{\{sx_i, cT\}} \supset B_{\{sx_i, cs\frac{1-\mu(x_i)}{2}\}} = sB_i \quad . \tag{3.35}$$

Notice that we are now using the monotonicity of $B_{\{a,r\}}$ in r. Finally (3.33) implies that for $l^2 = (\epsilon/4)s$,

$$t(sx_i) + T + l^2 = s(1 - \frac{\epsilon}{4}) + l^2 = s. \tag{3.36}$$

Combining these matters we see that

$$\overline{\mathbb{P}}(sB_i \subset K_s) \geq 1 - Ce^{-\gamma(\epsilon s)^{1/2}/2} - A_{x_i, r, \epsilon/4} s^{-r}$$

and hence for every $\epsilon > 0$, $s > 0$ and $r > 0$,

$$\overline{\mathbb{P}}((1 - \epsilon)sU \subset K_s) \geq 1 - A_{r,\epsilon}\ s^{-r} \ . \tag{3.37}$$

Together (3.29) and (3.37) prove Theorem 1.3. □

4 The asymptotic shape of the convex hull

In this section we prove a shape theorem for the convex hull $\mathcal{C}(\xi_t^{\{0\}})$ of the set of infected sites $\xi_t^{\{0\}}$ (cf. Theorem 1.4).

Definition 4.1 *A convex polytope is a set which is the convex hull of a finite number of points.*

Lemma 4.1 *For every $0 < \epsilon < 1/2$, there exists a convex polytope $P \subset \mathbb{R}^d$ such that*

$$(1 - 2\epsilon)U \subset P \subset (1 - \frac{3\epsilon}{2})U. \tag{4.1}$$

Proof. By Theorem 33 in Chapter 4 of Eggleston (1958) we have, for every $\delta > 0$, a convex polytope P containing $(1 - 2\epsilon)U$ and contained in a δ-neighborhood $\{x : d(x, (1 - 2\epsilon)U) \leq \delta\}$ of $(1 - 2\epsilon)U$. Here d is L^1 distance. Since 0 is an interior point of U, this δ-neighborhood of $(1 - 2\epsilon)U$ is contained in $(1 - 3\epsilon/2)U$ for sufficiently small δ. □

Let $x_1, x_2, ..., x_k$ be the extreme points of a convex polytope P satisfying (4.1). For each of these points x_i we define a set

$$A_i = \{x : \exists\ \ \eta > 0, \ \ x_i - \eta(x - x_i) \in P\} \cap (1 - \epsilon)U \ . \tag{4.2}$$

The set A_i is the intersection of $(1 - \epsilon)U$ and the exterior cone of P at x_i, and as $P \subset (1 - 3\epsilon/2)U$, we see that A_i contains an open set in $\mathbb{R}^d$. For any $B \subset \mathbb{R}^d$, let $\mathcal{C}(B)$ denote the convex hull of B. We have

Lemma 4.2 *If $x_i' \in A_i$, for $i = 1, ..., k$, then*

$$P \subset \mathcal{C}(\{x_1', ..., x_k'\}). \tag{4.3}$$

Proof. If $x_1' \in A_1$, then $x_1 - \eta(x_1' - x_1) \in P$, or

$$(1+\eta)x_1 - \eta x_1' = \sum_{j=1}^{k} \lambda_j x_j$$

for some $\eta > 0$, $\lambda_j \geq 0$ for each j, and $\sum_{j=1}^{k} \lambda_j = 1$. Hence

$$x_1 = \frac{\eta}{(1+\eta-\lambda_1)} x_1' + \sum_{j=2}^{k} \frac{\lambda_j}{1+\eta-\lambda_1} x_j,$$

and because $\eta + \sum_{j=2}^{k} \lambda_j = (1+\eta-\lambda_1)$, this implies that $x_1 \in \mathcal{C}(\{x_1', x_2, ..., x_k\})$, and as a result $P \subset \mathcal{C}(\{x_1', x_2, ..., x_k\})$.

Suppose that $P \subset \mathcal{C}(\{x_1', .., x_{m-1}', x_m, ..., x_k\})$, so that

$$x_m - \tilde{\eta}(x_m' - x_m) \in P \subset \mathcal{C}(\{x_1', .., x_{m-1}', x_m, .., x_k\}).$$

Then

$$x_m = \frac{\tilde{\eta}}{(1+\tilde{\eta}-\tilde{\lambda}_m)} x_m' + \sum_{j=1}^{m-1} \frac{\tilde{\lambda}_j}{1+\tilde{\eta}-\tilde{\lambda}_m} x_j' + \sum_{j=m+1}^{k} \frac{\tilde{\lambda}_j}{1+\tilde{\eta}-\tilde{\lambda}_m} x_j$$

for some $\tilde{\eta} > 0$, $\tilde{\lambda}_j \geq 0$ for each j, and $\sum_{j=1}^{k} \tilde{\lambda}_j = 1$. This implies that $x_m \in \mathcal{C}(\{x_1', ..., x_m', x_{m+1}, ..., x_k\})$ and as a result $P \subset \mathcal{C}(\{x_1', ..., x_m', x_{m+1}, ..., x_k\})$. Induction yields $P \subset \mathcal{C}(\{x_1', ..., x_k'\})$. □

Proof of Theorem 1.4. On the set where $\xi_t^{\{0\}}$ survives forever, $H_t \subset (1+\epsilon)tU$ eventually a.s. by Theorem 1.2. Since U is convex, this implies that $\mathcal{C}(H_t) \subset (1+\epsilon)tU$. In view of the definition of H_t in (1.7)-(1.8), $\xi_t^{\{0\}} \subset H_t$ and hence $\mathcal{C}(\xi_t^{\{0\}}) \subset \mathcal{C}(H_t)$. Combining this we arrive at

$$\mathcal{C}(\xi_t^{\{0\}}) \subset \mathcal{C}(H_t) \subset (1+\epsilon)tU \tag{4.4}$$

eventually a.s. on the set where $\xi_t^{\{0\}}$ survives forever. This establishes the almost sure upper bound for $\mathcal{C}(\xi_t^{\{0\}})$ in (1.12).

To obtain the lower bound in (1.12) we begin by noting that (2.6) ensures that for every $r > 0$ and $i = 1, 2, ..., k$,

$$\mathbb{P}\Big(n_t^{\mathbb{Z}^d}(tA_i) \leq \frac{1}{2}\mathbb{E}n_t^{\mathbb{Z}^d}(tA_i)\Big) \leq C_r |tA_i|_D^{-r} \leq C_{r,\epsilon} t^{-dr} .$$

Here

$$|A|_D = |A \cap \mathbb{Z}^d|$$

denotes the discrete cardinality of a set $A \subset \mathbb{R}^d$ and the final inequality follows from the fact that for fixed ϵ, $A_1, ..., A_k$ are fixed subsets of $\mathbb{R}^d$ with non-empty interiors. In view of the graphical representation we have for $i = 1, ..., k$ and $t \geq m_\epsilon$,

$$\frac{1}{2}\ \mathbb{E}n_t^{\mathbb{Z}^d}(tA_i) \geq \frac{1}{2}\ \mathbb{E}n_t^{\nu}(tA_i) = \frac{1}{2}\ |tA_i|_D\ \mathbb{E}\xi^{\nu}(0) \geq c_\epsilon t^d\ .$$

It follows that for $t \geq m_\epsilon$,

(4.5) $$\mathbb{P}(n_t^{\mathbb{Z}^d}(tA_i) \leq c_\epsilon t^d) \leq C_{r,\epsilon} t^{-dr}\ ,$$

for appropriately chosen positive c_ϵ and $C_{r,\epsilon}$ and integer $m_\epsilon > 0$. Hence for $m \geq m_\epsilon$,

$$\mathbb{P}(n_t^{\mathbb{Z}^d}(tA_i) = 0 \textit{ for some } t \in [m, m+1)) \leq$$
$$C_{r,\epsilon} m^{-dr} + \mathbb{P}(n_t^{\mathbb{Z}^d}(tA_i) = 0 \textit{ for some } t \in [m, m+1) \mid n_m^{\mathbb{Z}^d}(mA_i) > c_\epsilon m^d)\ .$$

The latter conditional probability is bounded by the probability that the maximum of $[c_\epsilon m^d] + 1$ independent standard exponential waiting times is smaller than 1, i.e.

$$\mathbb{P}(n_t^{\mathbb{Z}^d}(tA_i) = 0 \textit{ for some } t \in [m, m+1)) \leq C_{r,\epsilon}\ m^{-dr} + (1 - e^{-1})^{c_\epsilon m^d}$$

and choosing $dr \geq 2$ we see that the Borel–Cantelli lemma implies that for $i = 1, .., k$,

(4.6) $$n_t^{\mathbb{Z}^d}(tA_i) \neq 0 \quad \textit{eventually a.s.}\ .$$

Obviously this also holds for $i = 1, 2, .., k$ simultaneously.

By (4.2), $tA_i \subset (1-\epsilon)tU$ for $i = 1, .., k$, and hence, Theorem 1.2 implies that on the set $\{\tau^{\{0\}} = \infty\}$, $\xi_t^{\{0\}}(x) = \xi_t^{\mathbb{Z}^d}(x)$ eventually a.s. for all $x \in tA_i$ for $i = 1, .., k$. Hence, (4.6) ensures that on the set where $\xi_t^{\{0\}}$ survives forever,

(4.7) $$n_t^{\{0\}}(tA_i) \neq 0 \quad \textit{for } i = 1, ..., k \ \ \textit{eventually a.s.}\ .$$

If $n_t^{\{0\}}(tA_i) \neq 0$ for $i = 1, .., k$, then each of the sets tA_i contains a point of $\xi_t^{\{0\}}$ and by (4.1)–(4.3) this implies that

(4.8) $$(1 - 2\epsilon)tU \subset tP \subset \mathcal{C}(\xi_t^{\{0\}})\ .$$

In view of (4.7), (4.8) holds eventually a.s. on the set where $\xi_t^{\{0\}}$ survives forever. Since ϵ is arbitrary the lemma is proved. □

Proof of Theorem 1.5. In the proof of Theorem 1.4 we note that (4.5) implies that

$$\mathbb{P}(n_t^{\mathbb{Z}^d}(tA_i) = 0 \mid \tau^{\{0\}} = \infty) \leq \frac{1}{\mathbb{P}(\tau^{\{0\}} = \infty)} \mathbb{P}(n_t^{\mathbb{Z}^d}(tA_i) = 0)$$
$$\leq \frac{1}{\mathbb{P}(\tau^{\{0\}} = \infty)} C_{r,\epsilon} t^{-dr} = C'_{r,\epsilon} t^{-dr}$$

for an appropriate $C'_{r,\epsilon} > 0$ as the process is supercritical. Invoking Theorem 1.3, we arrive at

$$\mathbb{P}(n_t^{\{0\}}(tA_i) \neq 0 \; for \; i = 1, ..., k \mid \tau^{\{0\}} = \infty) \geq 1 - C'_{r,\epsilon} t^{-dr} - A_{r,\epsilon} t^{-r}$$

Copying the remainder of the proof of Theorem 1.4 we obtain instead of (4.8) a probability bound for $\mathcal{C}(\xi_t^{\{0\}})$, which is the statement of Theorem 1.5. □

5 Spatially decaying correlations for the conditional contact process $\overline{\xi}_t^{\{0\}}$

In this section we deal with a coupling of the process $\overline{\xi}_t^{\{0\}}$ and $\xi_{t-s}^{\mathbb{Z}^d}$ for which they are equal on $(1-\epsilon)tU$ with overwhelming probability for large s and $t-s$ (cf. Theorem 1.6). Then we use this coupling results to show that for the conditional process far away sites develop almost independently (cf. Theorem 1.8).

For the proof of Theorems 1.6 and 1.8 we need some auxiliary results. Choose $\epsilon \in (0,1)$ and $0 < s < t$. The set U is compact and we can cover $(1-\epsilon)sU$ by a finite number of sets of the form $U_{x_i,\rho} = \{x \in \mathbb{R}^d : \frac{x-x_i}{\rho} \in U\}$, where $x_i \in (1-\epsilon)sU$ and $\rho = (t-s)\epsilon/4$. Let N be the smallest number of such sets needed to cover $(1-\epsilon)sU$ and let $\bigcup_{i=1}^{N} U_i$ with $U_i = U_{x_i,\rho}$ be such a minimal covering of $(1-\epsilon)sU$. If $(t-s)\epsilon/4 \geq (1-\epsilon)s$, we have $N = 1$ and $U_1 = U_{0,\rho} = \rho U$. If $(t-s)\epsilon/4 < (1-\epsilon)s$, then N is of the order of $(s/(t-s))^d$. Hence in general

$$N \leq C_\epsilon \Big(\frac{s}{t-s}\Big)^d + 1 \tag{5.1}$$

for an appropriate $C_\epsilon > 0$ depending only on ϵ.

Define $\widetilde{U}_i = U_i \cap (1-\epsilon)sU$, so that

$$\bigcup_{i=1}^{N} \widetilde{U}_i = (1-\epsilon)sU \; . \tag{5.2}$$

If $(t-s)\epsilon/4 \geq (1-\epsilon)s$ and hence $N = 1$ and $U_1 = (t-s)(\epsilon/4)U$, then $\widetilde{U}_1 = (1-\epsilon)sU$. For large $s \geq A'_\epsilon$ the number of lattice points in $\widetilde{U}_1$ will be proportional to s^d. For

$(t-s)\epsilon/4 < (1-\epsilon)s$, the $|U_i|_D$ will be proportional to $(t-s)^d$ for $t-s \geq A''_\epsilon$. It is not hard to see that this is also true for $|\widetilde{U}_i|_D$. Combining all of this we find that there exists $A_\epsilon > 0$ and $c_\epsilon > 0$ such that for $i = 1, .., N$,

$$(5.3) \qquad |\widetilde{U}_i|_D > c_\epsilon(s \wedge (t-s))^d \quad \text{if} \quad s \wedge (t-s) \geq A_\epsilon \quad .$$

For $s > 0$ we now define three related processes. To define the first process we begin by choosing any version of $\{\bar{\xi}_t^{\{0\}} : t \leq s\}$ where $\bar{\xi}_t^{\{0\}}$ is distributed as $\xi_t^{\{0\}}$ conditional on $\{\tau^{\{0\}} = \infty\}$. Define

$$(5.4) \qquad \{\tilde{\xi}_t, t \geq 0\} = \begin{cases} \bar{\xi}_t^{\{0\}} & \text{if } t \leq s \\ \xi_{t-s}^{\bar{\xi}_s^{\{0\}}} & \text{if } t > s \end{cases}$$

Here $\xi_t^{\bar{\xi}_s^{\{0\}}}$ denotes a contact process starting at time $t = 0$ with a set of infected sites $\bar{\xi}_s^{\{0\}}$. This process is constructed according to a graphical representation. Thus $\tilde{\xi}_t$ is distributed up to time s as a process $\xi_t^{\{0\}}$ conditioned on surviving forever, but after time s this conditioning is dropped.

Next we extend the definition of $\bar{\xi}_t^{\{0\}}$ to times $t > s$. Let $\tilde{\tau} = \inf\{t : \tilde{\xi}_t = \emptyset\}$. On the set $\{\tilde{\tau} = \infty\}$ where $\tilde{\xi}_t$ survives forever we take $\bar{\xi}_t^{\{0\}} = \tilde{\xi}_t$ for $t > s$. On the set $\{\tilde{\tau} < \infty\}$ we may define $\{\bar{\xi}_t^{\{0\}} : t > s\}$ in any way we like, provided that the conditional distribution of $\{\bar{\xi}_t^{\{0\}} : t > s\}$ given $\{\tilde{\tau} < \infty\}$ is the same as the conditional distribution of $\{\bar{\xi}_t^{\{0\}} : t > s\}$ given $\{\tilde{\tau} = \infty\}$. Obviously the process $\{\bar{\xi}_t^{\{0\}} : t \geq 0\}$ is distributed as $\{\xi_t^{\{0\}} | \tau^{\{0\}} = \infty\}$ as the notation suggests. Moreover, our construction implies that for an appropriate constant $C' > 0$,

$$\begin{aligned}
\mathbb{P}(\bar{\xi}_t^{\{0\}} \neq \tilde{\xi}_t \textit{ for some } t \geq 0) &\leq \sum_{A \neq \emptyset} \mathbb{P}(\bar{\xi}_s^{\{0\}} = A)\mathbb{P}(\tau^A < \infty) \\
&= \sum_{A \neq \emptyset} \frac{\mathbb{P}(\xi_s^{\{0\}} = A \wedge \tau^{\{0\}} = \infty)}{\mathbb{P}(\tau^{\{0\}} = \infty)} \mathbb{P}(\tau^A < \infty) \\
&\leq \frac{1}{\mathbb{P}(\tau^0 = \infty)} \sum_{A \neq \emptyset} \mathbb{P}(\xi_s^{\{0\}} = A)\mathbb{P}(\tau^A < \infty) \\
&= \frac{\mathbb{P}(s < \tau^{\{0\}} < \infty)}{\mathbb{P}(\tau^{\{0\}} = \infty)} \leq C' e^{-\gamma s}
\end{aligned}$$

because $\xi_t^{\{0\}}$ is supercritical and (1.6) holds. Hence

$$(5.5) \qquad \mathbb{P}(\bar{\xi}_t^{\{0\}} = \tilde{\xi}_t \textit{ for all } t \geq 0) \geq 1 - C' e^{-\gamma s} \quad .$$

The third process is a process $\{\xi_t' : t \geq s\}$ defined by

(5.6) $$\xi_t' = \xi_{t-s}^{\mathbb{Z}^d} \qquad for\ \ t \geq s \quad .$$

Here $\{\xi_t^{\mathbb{Z}^d} : t \geq 0\}$ is defined by the same graphical construction as $\{\xi_t^{\overline{\xi}_s^{\{0\}}} : t \geq 0)\}$ and hence

$$\xi_t^{\overline{\xi}_s^{\{0\}}} \subset \xi_t^{\mathbb{Z}^d} \qquad or$$

(5.7) $$\widetilde{\xi}_t \subset \xi_t' \qquad for\ all\ \ t \geq s \quad .$$

We now show that at time s, $\widetilde{\xi}_s = \overline{\xi}_s^{\{0\}}$ contains many infected sites in the set $\widetilde{U}_i$. Let

$$\overline{n}_s^{\{0\}}(\widetilde{U}_i) = \sum_{x \in \widetilde{U}_i} \overline{\xi}_s^{\{0\}}(x)$$

be the number of infected sites in $\widetilde{U}_i$ for the $\overline{\xi}^{\{0\}}$–process at time s. Because $\widetilde{U}_i \subset (1-\epsilon)sU$, Theorem 1.3 and (2.6) imply that for any $r > 0$ and $i = 1, 2, ..., N$,

$$\begin{aligned}
\mathbb{P}\Big(\overline{n}_s^{\{0\}}(\widetilde{U}_i) \leq \frac{1}{2}\mathbb{E}n_s^{\mathbb{Z}^d}(\widetilde{U}_i)\Big) &= \mathbb{P}\Big(n_s^{\{0\}}(\widetilde{U}_i) \leq \frac{1}{2}\mathbb{E}n_s^{\mathbb{Z}^d}(\widetilde{U}_i) \mid \tau^{\{0\}} = \infty\Big) \\
&\leq \mathbb{P}\Big(n_s^{\mathbb{Z}^d}(\widetilde{U}_i) \leq \frac{1}{2}\mathbb{E}n_s^{\mathbb{Z}^d}(\widetilde{U}_i) \mid \tau^{\{0\}} = \infty\Big) + A_{r,\epsilon}s^{-r} \\
&\leq \frac{1}{\mathbb{P}(\tau^{\{0\}} = \infty)}\mathbb{P}\Big(n_s^{\mathbb{Z}^d}(\widetilde{U}_i) \leq \frac{1}{2}\mathbb{E}n_s^{\mathbb{Z}^d}(\widetilde{U}_i)\Big) + A_{r,\epsilon}\ s^{-r} \\
&\leq C_r'\ |\widetilde{U}_i|_D^{-r} + A_{r,\epsilon}\ s^{-r} \quad .
\end{aligned}$$

In view of the graphical representation, $\mathbb{E}\xi_s^{\mathbb{Z}^d}(0) \geq \mathbb{E}\xi^\nu(0)$ which is independent of s, and hence

$$\begin{aligned}
\frac{1}{2}\ \mathbb{E}n_s^{\mathbb{Z}^d}(\widetilde{U}_i) &= \frac{1}{2}\ |\widetilde{U}_i|_D \mathbb{E}\xi_s^{\mathbb{Z}^d}(0) \\
&\geq \frac{1}{2}\ |\widetilde{U}_i|_D\ \mathbb{E}\xi^\nu(0) \geq a\ |\widetilde{U}_i|_D
\end{aligned}$$

for some $a > 0$. It follows that for any $r > 0$,

(5.8) $$\mathbb{P}(\overline{n}_s^{\{0\}}(\widetilde{U}_i) \leq a\ |\widetilde{U}_i|_D) \leq C_r'\ |\widetilde{U}_i|_D^{-r} + A_{r,\epsilon}\ s^{-r} \quad .$$

Now let $\{\xi_t^{*_i}, t \geq s\}$ denote a contact process that starts at time s with $(\overline{\xi}_s^{\{0\}} \cap \widetilde{U}_i)$ as the set of infected sites. We define these processes by the same graphical construction as $\{\tilde{\xi}_t : t \geq s\}$ and $\{\xi'_t : t \geq s\}$. Since the initial sets of infected sites at time s for these three processes are ordered because

$$\xi_s^{*_i} = \overline{\xi}_s^{\{0\}} \cap \tilde{U}_i \subset \overline{\xi}_s^{\{0\}} = \tilde{\xi}_s \subset \xi'_s = \mathbb{Z}^d \quad , \tag{5.9}$$

we have

$$\xi_t^{*_i} \subset \tilde{\xi}_t \subset \xi'_t \qquad \textit{for all } t \geq s \quad . \tag{5.10}$$

Define $\tau^{*_i} = inf\{t : \xi_t^{*_i} = \emptyset\}$. In view of (5.8) and (2.7) we find that for any $r > 0$,

$$\mathbb{P}(\tau^{*_i} = \infty) \geq 1 - A_{r,\epsilon}\ s^{-r} - C'_r\ |\tilde{U}_i|_D^{-r} - C_r a^{-r}\ |\tilde{U}_i|_D^{-r} \quad . \tag{5.11}$$

If $\tau^{*_i} = \infty$, i.e. if $\xi_t^{*_i}$ survives forever, then one of the processes that starts at time s with a single infected site at some $y \in \overline{\xi}_s^{\{0\}} \cap \tilde{U}_i$ and is defined by the same graphical construction, must survive forever. For this particular random y Theorem 1.3 ensures that for any $t > s$ and $r > 0$,

$$\begin{aligned}\mathbb{P}\Big\{\xi_t^{\{y,s\}} \cap \Big(\{y\} \oplus (1-\epsilon/2)(t-s)U\Big) &= \xi'_t \cap \Big(\{y\} \oplus (1-\epsilon/2)(t-s)U\Big) \mid \tau^{*_i} = \infty\Big\} \\ &\geq 1 - A_{r,\epsilon/2}\ (t-s)^{-r}\end{aligned}$$

where $\xi_t^{\{y,s\}}$ is the process starting at y at time s. Because $\xi_t^{\{y,s\}} \subset \xi_t^{*_i}$, (5.10)–(5.11) imply that unconditionally

$$\begin{aligned}\mathbb{P}\Big\{\tilde{\xi}_t \cap \Big(\{y\} \oplus (1-\epsilon/2)(t-s)U\Big) &= \xi'_t \cap \Big(\{y\} \oplus (1-\epsilon/2)(t-s)U\Big)\Big\} \\ &\geq 1 - A_{r,\epsilon/2}\ (s^{-r} + (t-s)^{-r}) - C''_r\ |\tilde{U}_i|_D^{-r} \quad .\end{aligned} \tag{5.12}$$

However, this is true for some unknown random $y \in \tilde{U}_i$. We therefore need the following lemma

Lemma 5.1 *For every* $y \in \tilde{U}_i$ *and* $\epsilon \in (0,1)$,

$$\{y\} \oplus (1-\epsilon/2)(t-s)U \supset \tilde{U}_i \oplus (1-\epsilon)(t-s)U \ . \tag{5.13}$$

Proof. Choose a point $x \in \tilde{U}_i \oplus (1-\epsilon)(t-s)U$. As $\tilde{U}_i \subset U_i = U_{x_i,\rho}$ with $x_i \in (1-\epsilon)sU$ and $\rho = (\epsilon/4)(t-s)$, we can write x as $x = x_i + (\epsilon/4)(t-s)u' + (1-\epsilon)(t-s)v$ with $u', v \in U$. Similarly $y \in \tilde{U}_i$ can be written as $y = x_i + (\epsilon/4)(t-s)u''$ with $u'' \in U$. Hence $x - y = (\epsilon/4)(t-s)(u' - u'') + (1-\epsilon)(t-s)v$. As the set U is symmetric about the origin, $u'' \in U$ implies $-u'' \in U$ and in view of the fact that U is convex, $u' - u'' \in U \oplus U = 2U$, so $u' - u'' = 2u$ with $u \in U$. It follows that $x - y = (\epsilon/2)(t-s)u + (1-\epsilon)(t-s)v \in (\epsilon/2)(t-s)U \oplus (1-\epsilon)(t-s)U = (1-\epsilon/2)(t-s)U$, again because of the convexity of U. □

Combining (5.12) and Lemma 5.1 we find

Lemma 5.2 *For every $\epsilon \in (0,1)$ and $r > 0$ there exist positive numbers $A_{r,\epsilon}$ and C_r such that for every $i = 1, 2, ..., N$ and $0 < s < t$,*

$$(5.14) \qquad \mathbb{P}\left\{\tilde{\xi}_t \cap \left(\tilde{U}_i \oplus (1-\epsilon)(t-s)U\right) = \xi'_t \cap \left(\tilde{U}_i \oplus (1-\epsilon)(t-s)U\right)\right\} \geq 1 - A_{r,\epsilon}(s^{-r} + (t-s)^{-r}) - C_r\, |\tilde{U}_i|_D^{-r} \quad .$$

By inequality (5.5) and as $e^{-\gamma s}$ is of smaller order than s^{-r} for any $r > 0$, we may replace $\tilde{\xi}_t$ by $\overline{\xi}_t^{\{0\}}$ in (5.14) if we also replace $A_{r,\epsilon}$ by a larger constant. By (5.6) we may also replace ξ'_t by $\xi_{t-s}^{\mathbf{Z}^d}$ for $t > s$. Combining this with (5.3), we have proved the following theorem.

Theorem 5.1 *For every $\epsilon \in (0,1)$ and $r > 0$ there exist positive numbers A_ϵ and $A_{r,\epsilon}$ depending on ϵ and (r,ϵ) respectively, such that for $s \wedge (t-s) \geq A_\epsilon$,*

$$(5.15) \quad \mathbb{P}\left\{\overline{\xi}_t^{\{0\}} \cap \left(\tilde{U}_i \oplus (1-\epsilon)(t-s)U\right) \; = \; \xi_{t-s}^{\mathbf{Z}^d} \cap \left(\tilde{U}_i \oplus (1-\epsilon)(t-s)U\right)\right\} \geq \; 1 - A_{r,\epsilon}(s^{-r} + (t-s)^{-r}) \quad ,$$

for $i = 1, 2, ..N$.

Now we have all the tools we need for proving Theorems 1.6 and 1.8.

Proof of Theorem 1.6 The probability that $\overline{\xi}_t^{\{0\}}$ equals $\xi_{t-s}^{\mathbf{Z}^d}$ on every $\tilde{U}_i \oplus (1-\epsilon)(t-s)U$ for $i = 1, 2, ..N$ is at least $1 - NA_{r,\epsilon}(s^{-r} + (t-s)^{-r})$. With a somewhat changed value of r and $A_{r,\epsilon}$, this is at least $1 - A_{r,\epsilon}(s^{-r} + s^d(t-s)^{-r})$ because (5.1) holds and $(t-s) \geq A_\epsilon$. But if $\overline{\xi}_t^{\{0\}} = \xi_{t-s}^{\mathbf{Z}^d}$ on every $\tilde{U}_i \oplus (1-\epsilon)(t-s)U$, then $\overline{\xi}_t^{\{0\}} = \xi_{t-s}^{\mathbf{Z}^d}$ on

$$\bigcup_{i=1}^{N} [\tilde{U}_i \oplus (1-\epsilon)(t-s)U] = (\bigcup_{i=1}^{N} \tilde{U}_i) \oplus (1-\epsilon)(t-s)U \; .$$

In view of (5.2) and the convexity of U this set equals

$$(1-\epsilon)sU \oplus (1-\epsilon)(t-s)U = (1-\epsilon)tU \; ,$$

which proves (1.13). The proof of (1.14) is immediate because $\bigcup_{i=1}^{N} [\tilde{U}_i \oplus (1-\epsilon)(t-s)U] = (1-\epsilon)tU$, so $x \in (1-\epsilon)tU$ implies that x must be in $\tilde{U}_i \oplus (1-\epsilon)(t-s)U$ for some i. □

Proof of Theorem 1.8 Without loss of generality we replace $f(\cdot)$ by $f(\cdot) - f(\eta)$ for a fixed $\eta \in H$. The effect of this is to ensure that $\|f\| \leq |\!|\!|f|\!|\!|$. Let $\tilde{\xi}_t$, $\overline{\xi}_t^{\{0\}}$ and ξ'_t denote the processes defined above. In this proof we write d^* for $d(R_1, R_2)$.

By (5.5) we have

$$\left|\mathbb{E}f(\bar{\xi}_t^{\{0\}})g(\bar{\xi}_t^{\{0\}}) - \mathbb{E}f(\tilde{\xi}_t)g(\tilde{\xi}_t)\right| \leq 2C'\|f\| \cdot \|g\|e^{-\gamma s} \quad . \tag{5.16}$$

Fix $\epsilon \in (0,1)$ and $r > 0$. Choose $s \in (0,t)$ with

$$t - s \leq \frac{d^*}{3c} \quad , \tag{5.17}$$

where c is the constant in Lemma 3.1 if the L^∞-ball $B_{\{0,ct\}}$ is replaced by the L^1-ball $\tilde{B}_{\{0,ct\}}$ which is obviously possible. Hence

$$\mathbb{P}(H_{t-s} \subset \tilde{B}_{\{0,d^*/3\}}) \geq \mathbb{P}(H_{d^*/3c} \subset \tilde{B}_{\{0,d^*/3\}}) \geq 1 - Ce^{-(\gamma/3c)d^*} \quad , \tag{5.18}$$

where $\tilde{B}_{\{0,\rho\}} = \{x \in \mathbb{R}^d : |x| \leq \rho\}$ denotes an L^1-ball in $\mathbb{R}^d$.

In the graphical representation we have $\tilde{\xi}_t(x) = 1$ if a site in $\tilde{\xi}_s$ at time s is connected to a site x at time t by a chain of infection. We now construct a process ξ_t^* by defining $\xi_t^*(x)$ for each $x \in \mathbb{Z}^d$ in the same way as $\tilde{\xi}_t(x)$, but now ignoring chains of infection passing through any site $y \notin \tilde{B}_{\{x,d^*/3\}}$ at any time in the interval $[s,t]$. Thus

$$\xi_t^*(x) = \begin{cases} 1 & \textit{if there is a chain of infection from } (\tilde{\xi}_s, s) \textit{ to } (x,t) \\ & \textit{passing only through sites in } \tilde{B}_{\{x,d^*/3\}} \textit{ during the} \\ & \textit{time interval } [s,t], \\ & \\ 0 & \textit{otherwise.} \end{cases}$$

Thus in defining $\xi_t^*(x)$ we ignore the influence of infected sites outside $\tilde{B}_{\{x,d^*/3\}}$ between times s and t. Let $\mathcal{E}_x$ denote the event that all chains of infection starting at any site in $\mathbb{Z}^d$ at time s end ending at x at time t, passes only through sites in $\tilde{B}_{\{x,d^*/3\}}$ during the time interval $[s,t]$. Reversing time and the direction of the infection arrows, the self-duality of the graphical construction yields

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_x) &\geq \mathbb{P}(\tau^{\{x\}} < t-s) + \mathbb{P}(\tau^{\{x\}} \geq t-s, \bigcup_{0\leq u\leq t-s} \xi_u^{\{x\}} \subset \tilde{B}_{\{x,d^*/3\}}) \\
&= 1 - \mathbb{P}(\tau^{\{x\}} \geq t-s, \bigcup_{0\leq u\leq t-s} \xi_u^{\{x\}} \not\subset \tilde{B}_{\{x,d^*/3\}}) \\
&\geq 1 - \mathbb{P}(\bigcup_{0\leq u\leq t-s} \xi_u^{\{0\}} \not\subset \tilde{B}_{\{0,d^*/3\}}) \\
&\geq 1 - \mathbb{P}(H_{t-s} \not\subset \tilde{B}_{\{x,d^*/3\}}) \geq 1 - Ce^{-(\gamma/3c)d^*}
\end{aligned}$$

by (5.18). Obviously, $\xi_t^*(x) = \tilde{\xi}_t(x)$ on $\mathcal{E}_x$ and hence

$$\mathbb{P}(\xi_t^*(x) = \tilde{\xi}_t(x)) \geq 1 - Ce^{-(\gamma/3c)d^*} \quad \textit{for all } x \in \mathbb{Z}^d \quad . \tag{5.19}$$

As $f \in D_{R_1}$ and $g \in D_{R_2}$, it follows from (2.1) that

$$\begin{aligned}
&\left|\mathbb{E}f(\tilde{\xi}_t)g(\tilde{\xi}_t) - \mathbb{E}f(\xi_t^*)g(\xi_t^*)\right| \\
&\leq \|g\|\ \mathbb{E}\left|f(\tilde{\xi}_t) - f(\xi_t^*)\right| + \|f\|\ \mathbb{E}\left|g(\tilde{\xi}_t) - g(\xi_t^*)\right| \\
&\leq \|g\| \sum_{x \in R_1} \Delta_f(x)\ \mathbb{P}(\tilde{\xi}_t(x) \neq \xi_t^*(x)) + \|f\| \sum_{x \in R_2} \Delta_g(x)\ \mathbb{P}(\tilde{\xi}_t(x) \neq \xi_t^*(x))
\end{aligned}$$

and by (5.19)

$$\left|\mathbb{E}f(\tilde{\xi}_t)g(\tilde{\xi}_t) - \mathbb{E}f(\xi_t^*)g(\xi_t^*)\right| \leq C\ \left(\|g\| \cdot |||f||| + \|f\| \cdot |||g|||\right)\ e^{-(\gamma/3c)d^*} \quad . \tag{5.20}$$

For the ξ^* -process, chains of infection determining $\xi_t^* \cap R_1$ and $\xi_t^* \cap R_2$ pass only through sites in $R_1 \oplus \tilde{B}_{\{0,d^*/3\}}$ and $R_2 \oplus \tilde{B}_{\{0,d^*/3\}}$ during $[s,t]$. Because $d(R_1, R_2) = d^*$, these sets of sites are disjoint. This implies that given $\tilde{\xi}_s$, the random sets $\xi_t^* \cap R_1$ and $\xi_t^* \cap R_2$ are conditionally independent because they depend on two disjoint subsets of a collection of independent Poisson processes. As $f \in D_{R_1}$ and $g \in D_{R_2}$, we have shown that

$$\mathbb{E}[f(\xi_t^*)g(\xi_t^*)|\tilde{\xi}_s] = \mathbb{E}[f(\xi_t^*)|\tilde{\xi}_s].\mathbb{E}[g(\xi_t^*)|\tilde{\xi}_s] \quad . \tag{5.21}$$

Continuing with the right-hand side of (5.21), we may write

$$\begin{aligned}
\mathbb{E}\Big(\mathbb{E}[f(\xi_t^*)|\tilde{\xi}_s].\mathbb{E}[g(\xi_t^*)|\tilde{\xi}_s]\Big) &= \mathbb{E}\Big(\mathbb{E}[f(\xi_t')|\tilde{\xi}_s]\ .\ \mathbb{E}[g(\xi_t^*)|\tilde{\xi}_s]\Big) \\
&+ \mathbb{E}\Big(\mathbb{E}[f(\xi_t^*) - f(\xi_t')|\tilde{\xi}_s]\ .\ \mathbb{E}[g(\xi_t^*)|\tilde{\xi}_s]\Big)
\end{aligned}$$

Because ξ_t' is clearly independent of $\tilde{\xi}_s = \overline{\xi}_s^{\{0\}}$, the first term on the right equals

$$\mathbb{E}f(\xi_t')\mathbb{E}(\mathbb{E}[g(\xi_t^*)|\tilde{\xi}_s]) = \mathbb{E}f(\xi_t')\ \mathbb{E}g(\xi_t^*) \quad ,$$

and the second term is bounded in absolute value by

$$\begin{aligned}
\|g\|\mathbb{E}\left|f(\xi_t^*) - f(\xi_t')\right| &\leq \|g\| \sum_{x \in R_1} \Delta_f(x)\ \mathbb{P}(\xi_t^*(x)) \neq \xi_t'(x)) \\
&\leq \|g\| \cdot |||f|||\Big(Ce^{(-\gamma/3c)d^*} + C'e^{-\gamma s} + A_{r,\epsilon}(s^{-r} + (t-s)^{-r})\Big)
\end{aligned}$$

by (5.19), (5.5), (5.6), (1.14) and because $R_1 \subset (1-\epsilon)tU$. Note that by (5.17) we may absorb the first two remainder terms in the third one with $A_{r,\epsilon}$ replaced by a larger $A'_{r,\epsilon}$. Combining all of this with (5.21) we obtain

$$\left|\mathbb{E}f(\xi_t^*)g(\xi_t^*) - \mathbb{E}f(\xi_t')\mathbb{E}g(\xi_t^*)\right| \leq A'_{r,\epsilon} \cdot \|g\| \cdot |||f|||(s^{-r} + (t-s)^{-r}) \quad . \tag{5.22}$$

Again by (5.6) and (1.14) and the fact that $R_1 \subset (1-\epsilon)tU$,

$$\left|\mathbb{E}f(\xi_t') - \mathbb{E}f(\overline{\xi}_t^{\{0\}})\right| \le A_{r,\epsilon}\cdot |||f|||(s^{-r} + (t-s)^{-r})$$

and by (5.19) and (5.5),

$$\left|\mathbb{E}g(\xi_t^*) - \mathbb{E}g(\overline{\xi}_t^{\{0\}})\right| \le |||g|||\cdot\left(Ce^{-(\gamma/3c)d^*} + C'e^{-\gamma s}\right) \quad .$$

It follows that

$$\left|\mathbb{E}f(\xi_t')\mathbb{E}g(\xi_t^*) - \mathbb{E}f(\overline{\xi}_t^{\{0\}})\mathbb{E}g(\overline{\xi}_t^{\{0\}})\right| \le$$

(5.23)

$$A_{r,\epsilon}\cdot |||f|||\cdot \|g\|(s^{-r} + (t-s)^{-r}) + \|f\|\cdot |||g|||\left(Ce^{-(\gamma/3c)d^*} + C'e^{-\gamma s}\right) \quad .$$

Combining (5.16), (5.20), (5.22) and (5.23) we find after some simplification of remainder terms

$$\left|\mathrm{cov}\left(f(\overline{\xi}_t^{\{0\}}), g(\overline{\xi}_t^{\{0\}})\right)\right| \le A''_{r,\epsilon}\cdot \|f\|\cdot |||f|||\cdot (s^{-r} + (t-s)^{-r}) +$$
$$C''\cdot \|f\|\cdot |||g|||(e^{-(\gamma/3c)d^*} + e^{-\gamma s}) \ .$$

It remains to choose $s \in (0,t)$ subject to $t - s \le d^*/3c$ as required by (5.17). Taking

$$t - s = \frac{d^*}{3c} \wedge \frac{t}{2} \quad ,$$

and writing $d(R_1, R_2)$ for d^* again, we obtain

$$\left|\mathrm{cov}\left(f(\overline{\xi}_t^{\{0\}}), g(\overline{\xi}_t^{\{0\}})\right)\right| \quad \le \quad A'''_{r,\epsilon}\cdot |||f|||\cdot \|g\| \ (d(R_1,R_2) \wedge t)^{-r} +$$

(5.24)

$$A\cdot \|f\|\cdot |||g||| \ e^{-\alpha(d(R_1,R_2)\ \wedge\ t)}$$

for appropriate positive $A'''_{r,\epsilon}$, A and α. We may simplify (5.24) further by replacing $\|f\|$ and $\|g\|$ by $|||f|||$ and $|||g|||$ to arrive at

$$\left|\mathrm{cov}\left(f(\overline{\xi}_t^{\{0\}}), g(\overline{\xi}_t^{\{0\}})\right)\right| \le A''''_{r,\epsilon}\cdot |||f|||\cdot |||g||| \ (d(R_1,R_2) \wedge t)^{-r} \quad ,$$

which is the statement of Theorem 1.8. □

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