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RECONSTRUCTING A 3-COLOR SCENERY BY OBSERVING IT ALONG A SIMPLE RANDOM WALK PATH

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Abstract

Let $\{\xi(n)\}_{n\in\mathbb{Z}}$ be a 3-color random scenery, that is a random coloration of $\mathbb{Z}$ in three colors, such that the colors of the different points in $\mathbb{Z}$ are i.i.d. Let $\{S(n)\}_{n\in\mathbb{N}}$ be a symmetric random walk starting at 0. Our main result shows that a.s., $\xi \circ S$ (the composition of ξ and S) determines ξ up to translation and reflection. In other words, by observing the scenery along the random walk path S, we can a.s. reconstruct ξ up to translation and reflection. This result allows us to give a positive answer to the question of H. Kesten of whether one can a.s. detect a single defect in a 3-color random scenery by observing it only along a random walk path.

1 INTRODUCTION

Let $\xi, \eta : \mathbb{Z} \longrightarrow \{0,1\}$ and let $\{S(n)\}_{n\in\mathbb{N}}$ be a symmetric random walk on $\mathbb{Z}$. Let the process $\{\chi(n)\}_{n\in\mathbb{N}}$ be equal to either $\{\xi(S(n))\}_{n\in\mathbb{N}}$ or $\{\eta(S(n))\}_{n\in\mathbb{N}}$. Is it possible by observing only one path realization of $\{\chi(n)\}_{n\in\mathbb{N}}$ to say to which one of the two $\{\xi(S(n))\}_{n\in\mathbb{N}}$ or $\{\eta(S(n))\}_{n\in\mathbb{N}}$, $\{\chi(n)\}_{n\in\mathbb{N}}$ is equal to? If yes, we say that it is possible to distinguish between the sceneries ξ and η by observing them along a path of $\{S(n)\}_{n\in\mathbb{N}}$. Otherwise, when it is not possible to figure out almost surely by observing $\{\chi(n)\}_{n\in\mathbb{N}}$ alone whether $\{\chi(n)\}_{n\in\mathbb{N}}$ is generated on ξ or on η, we say that

ξ and η are indistinguishable. The problem of distinguishing two sceneries was raised independently by I. Benjamini and by den Hollander and Keane. The motivation came from problems in ergodic theory, such as the T, T^{-1}problem (see Kalikow [6]) and from the study of various aspects of $\{\xi(n)\}_{n\in\mathbb{N}}$, where $\{\xi(S(n))\}_{n\in\mathbb{N}}$ is random. (See Kesten and Spitzer in [8], Keane and den Hollander in [7], den Hollander in [2]). Benjamini and Kesten showed in [1] that one can distinguish almost any two random sceneries even when the random walk is in $\mathbb{Z}^2$. (They assumed the sceneries to be random themselves, so that the $\xi(n)$'s and the $\eta(n)$'s are i.i.d. Bernoulli.) Kesten in [9] proved that when the random sceneries are i.i.d. and have four colors, i.e., ξ and $\eta : \mathbb{Z} \longrightarrow \{0,1,2,3\}$, and differ only in one point, they can be a.s. distinguished. He asked whether this result might still hold with fewer colors. This paper provides a positive answer to that question in the 3-color case, where $\{S(n)\}_{n\in\mathbb{N}}$ is a simple random walk. The author has also solved the 2-color case. However, the reconstruction in the two-color case is much more complicated, so it is well worth to publish the 3-color case separately.

Recently, Lindenstrauss [11] exhibited two sceneries on $\mathbb{Z}$ which he proved to be indistinguishable. Before that, Howard in [3], [4] and [5] proved that any two periodical sceneries of $\mathbb{Z}$ which are not equivalent modulo translation and reflection are distinguishable and that one can a.s distinguish single defects in periodical sceneries. Kesten in [10] asked whether this result would still hold when the random walk would be allowed to jump. In our opinion, this is a central open problem at present.

Basic notations and definitions.

Let from now on, a random scenery $\{\xi(n)\}_{n\in\mathbb{Z}}$ be a random coloring of $\mathbb{Z}$ in three colors: 0,1 and 2. We assume throughout that the $\xi(n)$'s are i.i.d. random variables such that if

$$p_0 = P(\xi(0) = 0),\ p_1 = P(\xi(0) = 1) \text{ and } p_2 = P(\xi(0) = 2)$$

we have $p_0, p_1, p_2 > 0$ and $p_{01} + p_1 + p_2 = 1$. Let $\{S(n)\}_{n\in\mathbb{N}}$ be a simple random walk starting at the origin. We will assume throughout that $\{\xi(n)\}_{n\in\mathbb{Z}}$ and $\{S(n)\}_{n\in\mathbb{N}}$ both live on the same probability space and are independent of each other. Whenever we will use a.s., it will mean almost surely with respect to the probability measure on that underlying probability space.

When we move on the scenery following the path of $\{S(n)\}_{n\in\mathbb{N}}$, we see a sequence denoted by $\{(\xi \circ S)(n)\}_{n\in\mathbb{N}}$, which we call the scenery seen along the random walk path or the observations generated by S on ξ. (Here

$(\xi \circ S)(n) = \xi(S(n))$.) In general we will use the following notation: if $\{X(n)\}_{n \epsilon B}$ is a process where B is any numerable set, then X will denote the path of $\{X(n)\}_{n \epsilon B}$. In order to state the main result of our paper, we need two definitions.

Definition 1 *Two sceneries ξ, $\chi : \mathbb{Z} \longrightarrow \{0,1,2\}$ are said to be equivalent modulo translation and reflection iff there exists $a \epsilon \mathbb{Z}$ such that $\xi(z+a) = \chi(z)$ for all z in $\mathbb{Z}$ or $\xi(z+a) = \chi(-z)$ for all z in $\mathbb{Z}$. Henceforth we will simply say that they are equivalent and write $\xi \approx \chi$.*

Definition 2 *Let A, B be two countable sets. We say that a function $f : \{0,1,2\}^A \longrightarrow \{0,1,2\}^B$ is measurable iff it is measurable with respect to σ_A and σ_B, where σ_A is the $\sigma-$ algebra induced by the canonical projections of $\{0,1,2\}^A$ onto its coordinates and σ_B is the $\sigma-$ algebra induced by the projections of $\{0,1,2\}^B$ onto its coordinates.*

The main result of this paper states that if we know only $\xi \circ S$, we can a.s. reconstruct ξ up to translation and reflection. In a more formal way, the main result reads as follows:

Theorem 3 ***The Main Result.*** *Let $\{\xi(n)\}_{n \epsilon \mathbb{Z}}$, $\{S(n)\}_{n \in \mathbb{N}}$, ξ, S, $\xi \circ S$ be as described above. Then, there exists a measurable function $\mathcal{A} : \{0,1,2\}^{\mathbb{N}} \longrightarrow \{0,1,2\}^{\mathbb{Z}}$ such that $\mathcal{A}(\xi \circ S) \approx \xi$ a.s.*

To prove our main result, we will actually give a method for reconstructing ξ up to equivalence which uses only the information χ. Let us give a simple, informal description of this reconstruction method before giving a precise description in chapter 3. We will use a three regular tree $T = (E_T, V_T)$, whose vertices we will color with the three colors $\{0,1,2\}$ in such a way that each vertex of T has three neighbors of different colors. We will then "represent" the scenery ξ as a nearest neighbor walk $R : \mathbb{Z} \longrightarrow V_T$ on T. (By nearest neighborwalk on T, we mean that R has to go over an edge of T at each time unit so that for each n$\in \mathbb{Z}$ we have: $\{R(n), R(n+1)\} \in E_T$.) We define the nearest neighborewalk R, by requesting for each time $n \epsilon \mathbb{Z}$ that R, should be at a vertex of T of color $\xi(n)$. It is clear that once we know $R(0)$, R is uniquely determined by ξ and vice versa. The trick thus is to try to reconstruct R using the observations χ. This is achieved by representing the color-sequence χ itself as a nearest neighbor walk on the tree T. Again, we choose the only (up to the origin) nearest neighbor walk such that for each time, $n \in \mathbb{N}$, that neighbor walk at time n is on a vertex of T of color $\chi(n)$. Then, it is easy to see that the thus defined nearest neighbor walk (which represents the color sequence χ as a nearest neighbor walk on the tree T) is nothing other than $R \circ S$, i.e. the "random walk S on the path of R. " Let now x and y be two vertices of T which get visited by R. When the nearest neighbor walk $R \circ S$ will go from points x to y in T in a shortest possible time, it will reveal the shape of the shortest possible path of R between x and y. As we will see, R is transient and so if x and y are far away from each other, with big probability, there will be only one passage of R from x to y. Thus, we will be able to know the shape of the passage of R from x to y. We will then repeat the same operation for couples x and y which lie further and further away from each other. Eventually, we will assemble the thus reconstructed pieces in order to retrieve all of R.

2 DEFINITIONS

We will designate graphs by (E, V) where E is the set of edges and V is the set of vertices.

A three-color scenery on a graph (E, V) is a map from V to $\{0,1,2\}$.

A nearest-neighbor walk W on a graph (E, V) is a map from a integer interval $D = I \cap \mathbb{Z}$, (where I is an interval) to V such that at each time unit W moves over an edge of (E, V). More precisely, this means that $W : D \longrightarrow V$ is a nearest neighbor walk iff every time $x, y \in D$ with $|x - y| = 1$ we get $\{W(x), W(y)\} \in E$. Let φ be a three color scenery on the graph (E, V). Let $W : D \longrightarrow V$ be a nearest neighborwalk on (E, V). Let $\psi : D \longrightarrow \{0, 1, 2\}$ be a three-color scenery on D then we say that W generates ψ on φ iff $\psi = \varphi \circ W$

Let $W : D \longrightarrow V$ be a nearest neighbor walk on the graph (E, V) and let $a, b \in V$ with a $\neq$b. Then $(x, y) \in D \times D$ is called a crossing by W of (a,b) iff $W(z)$ is different from $W(x) = a$ and $W(y) = b$, for every z in $\mathbb{Z}$ strictly between x and y. The crossing (x, y) is called positive, respectively negative iff $x < y$ respectively, $x > y$ For two crossings by W, (x, y) and (r, s) we say that (r, s) happens during (x, y) iff min{x,y}<r,s<max{x,y}. We say that a crossing (x, y) by W of (a, b) is straight iff it occurs in the minimum possible time, that is iff, $|x - y| = d(a, b)$, where $d(a, b)$ designates the length of the shortest possible path from a to b.

For the graph $\{U, \mathbb{Z}\}$ where $U = \{\{x, y\} \subset \mathbb{Z} : |x - y| = 1\}$ we will omit the set of edges and just speak of the graph $\mathbb{Z}$. Thus $W : D \longrightarrow \mathbb{Z}$

is a nearest neighbor walk iff for all $x, y \in D$ with $|x-y| = 1$ we have $|W(x) - W(y)| = 1$. (x,y) is a straight crossing by $W : D \longrightarrow \mathbb{Z}$ of (a,b) iff it is a crossing by W of (a,b) and $|a-b| = |x-y|$

In what follows, $T = (E_T, V_T)$ is a three regular tree. We pick one vertex in T, call it the origin and write 0 for it.

In what follows, φ^0,φ^1,$\varphi^2 : V \longrightarrow \{0,1,2\}$ will be three non-random 3-colorings of V_T choosen such that:

2.1 $\varphi^0(0) = 0$, $\varphi^1(0) = 1$, $\varphi^2(0) = 2$ and

2.2 for $i \in \{0,1,2\}$ we have that for any v in $\mathrm{V_T}$, there is exactly one adjacent vertex of v having color $j \in \{0,1,2\}$ in the coloring φ^i. That is, if v_1, v_2, v_3 are the three adjacent vertices to v, then $\{\varphi^i(v_1), \varphi^i(v_2), \varphi^i(v_3)\} = \{0,1,2\}$.

Note that 2.2 has two important consequences.

First, if $W : D = I \cap \mathbb{Z} \longrightarrow V_T$ is a nearest neighbor walk, such that $0 \in D$ and I is an interval, then,W(0) and $\varphi^i \circ W$ together uniquely determine W. (We assume that we know φ^i.)

Secondly, for every scenery on $D = I \cap \mathbb{Z}$ and for every $i \in \{0,1,2\}$ we can find a nearest neighbor walk $W : D \longrightarrow V_T$ which generates that scenery on φ^i.

φ will designate the three-color random scenery such that for all $v \in V_T$, $\varphi(v) = \varphi^{\xi(0)}(v)$. Thus, φ is only random as "far as $\xi(0)$ is. " Once we know the value of $\xi(0)$, φ is no longer random. Furthermore, φ always equals φ^0,φ^1or φ^2 and thus 2.2 applies to φ.

We designate by $\{R(n)\}_{n \epsilon \mathbb{Z}}$ the unique process, which is the nearest neighbor walk on T starting at the origin and generating ξ on φ. If R designates the path of $\{R(n)\}_{n \epsilon \mathbb{Z}}$, then R is the only nearest neighbor walk on T such that $\varphi \circ R = \xi$ and $R(0) = 0$.

If $p_0 = p_1 = p_2 = 1/3$, (see 1) then $\{R(n)\}_{n \epsilon \mathbb{Z}}$ is a symmetric random walk on T starting at 0. Otherwise, we still have that $\{R(n)\}_{n \epsilon \mathrm{N}}$ and$\{R(-n)\}_{n \epsilon \mathrm{N}}$ are two i.i.d. Markov processes with constant transition probabilities, as soon as $\xi(0)$ is given and φ is non-random.

3 MAIN IDEAS

Let us assume that we would be able by using only the observations χ to determine all the crossings by S of $(x,y) \in \mathbb{Z} \times \mathbb{Z}$. Then, we could reconstruct, up to translation and reflection, the restriction of ξ to the integer interval $\mathbb{Z} \cap [min\{x,y\}, max\{x,y\}]$. As a matter of fact, we could just pick among those crossings a positive one (s,t) which lasts a minimum amount of time. Such a crossing would then have to be straight and thus the restriction of χ to the interval $[s,t] \cap \mathbb{Z}$ would be equivalent to ξ restricted to $[min\{x,y\}, max\{x,y\}] \cap \mathbb{Z}$.

How can we locate sets C of crossings by S, for which there exists $x, y \in \mathbb{Z}$ such that C is precisely the set of all crossings by S of (x,y)? The idea is to use R, the representation of ξ as a nearest neighbor walk. (That is, R is defined to be the unique one-step walk on T starting at the origine and generating ξ on $\varphi^{\xi(0)}$.$\{R(n)\}_{n\in\mathbb{Z}}$ will designate the random process which has as path R). Of course, we do not know R. However, we can easily figure out $R \circ S$, since

$$\varphi \circ (R \circ S) = (\varphi \circ R) \circ S = \xi \circ S = \chi.$$

Thus, $R \circ S$ generates χ on φ and thus we can reconstruct $R \circ S$ knowing χ, since $R \circ S$ is a nearest neighbor walk. Furthermore, as we will see in Lemma 4.1, (s,t) is a crossing by $R \circ S$ of (a,b) iff (s,t) is a crossing by S on a crossing by R of (a,b) Thus, in case R crosses (a,b) only once (i.e. when there exists only one crossing (x,y) by R of (a,b)), there exists an easy way to determine all the crossing by S of (x,y): the crossings by S of (x,y) are exactly the crossing by $R \circ S$ of (a,b).

But how can we make sure that for $(a,b) \in T \times T$ there exists only one crossing by R of (a,b)? Well, we cannot, but we know that if a and b are far away from each other the probability that R crosses (a,b) more than once (i.e. the probability that there exist more than one crossing by R of (a,b)) tends to be very small.

At this stage we give a precise description of our reconstruction method for ξ.

3.1 **Figure out $R \circ S$** (We suppose that we know φ^0, φ^1 and φ^2. Then, since $\varphi = \varphi^{\xi(0)}$ and since we know $\xi(0)$ because $\xi(0) = \chi(0)$, we also know φ) $R \circ S$ is the only nearest neighbor walk on T which generates χ on φ and starts at the origin. Thus, we are able to reconstruct $R \circ S$. If imf denotes the image of the map f, then we get a.s. $im(R \circ S) = imR$. Thus, we also know imR.

3.2 **Choose a set $\mathbf{U} \subset \mathbf{T}$ of finite cardinality such that $\mathbf{imR} - \mathbf{U}$ consists of two connected components of infinite cardinality.** (By two connected components, we mean that the two components are dijoint with respect to each other). This is always possible (see 5.3).

3.3 **Choose in $\mathbf{imR} - \mathbf{U}$ an increasing sequence of intervals $\{(\mathbf{a}_i, \mathbf{b}_i)\}$.** The (a_i, b_i)'s must be chosen such that the a_i 's are all on one connected component of $imR - U$, the b_i's are on the other and such that for every $i \geq 2$, the shortest path in T from a_i to b_i passes first by a_{i-1} and then by b_{i-1} . (It is always possible to find a sequence $\{(a_i, b_i)\}$ satisfying the above conditions when $imR - U$ consists of two infinite components. As a matter of fact, proceed as follows: take a_0 to be in the first component and b_0 to be in the second one such that $d(a_0, b_0)$ equals the minimal distance between the two components. Take R_a, resp. R_b, to be an injective nearest neighbor walk from $\mathbb{N}$ into the first, resp. second component and such that $R_a(0) = a_0$, resp. $R_b(0) = b_0$. It is easy to see, that if we define $(a_i, b_i) := (R_a(i), R_b(i))$ for all $i \in \mathbb{N}$, then the thus defined collection of pairs of vertices satisfies 3.3.)

3.4 **Choose for every $\mathbf{i \geq 1}$ one positive shortest crossing by $\mathbf{R \circ S}$ of (a_i, b_i) and call it (s_i, t_i).** We know that since (s_i, t_i) is shortest it must be straight and thus the restriction of χ to $[s_i, t_i] \cap \mathbb{Z}$ is equivalent to a piece of ξ. We will write ξ^i for the restriction of $\chi | [s, t] \cap \mathbb{Z}$.

3.5 **Assemble the different pieces ξ^i so as to get a scenery equivalent to ξ.** For this purpose call (s'_i, t'_i) the first crossing of (a_i, b_i) by $R \circ S$ during the time $[s_{i+1}, t_{i+1}]$. Define $r_i := s'_{i-1} - s_i$ for i≥ 2 and $r_1 := 0$. Translate ξ^i to the left by $s_i + \sum_{j=1}^{i} r_j$ to get $\hat{\xi}^i$. More precisely,

$$\hat{\xi}^i(s) = \xi^i(s_i + \sum_{j=1}^{i} r_j + s), \quad \text{if} \quad s \in [-\sum_{j=1}^{i} r_j, -\sum_{j=1}^{i} r_j + t_i - s_i]$$

$$\text{and} \quad \hat{\xi}^i(s) = 0 \quad \text{otherwise.}$$

Put $lim_{i \to \infty} \hat{\xi}^i(s) := \hat{\xi}(s)$ for all s in $\mathbb{Z}$. (Where, as we will see later that the limit exists.)

Now, $\hat{\xi} : \mathbb{Z} \longrightarrow \{0, 1, 2\}$ is our reconstruction of ξ . We will prove later that $\xi \approx \hat{\xi}$.

Why does the reconstruction work?

All the $\hat{\xi}^i$'s are pieces of ξ. In order to assemble them correctly and to get finally a scenery equivalent to ξ we need to know their relative position to each other. How can we do that? Assume that (a_i, b_i) and (a_{i+1}, b_{i+1}) satisfy the condition in 3.3 and that there exists only one crossing by R of (a_i, b_i), call it (x_i, y_i) and only on of (a_{i+1}, b_{i+1}), call it (x_{i+1}, y_{i+1}). Then (s_i', t_i') must also be a crossing of (x_i, y_i) by S and since the crossing by S (s_{i+1}, t_{i+1}) is straight, we get that $|x_{i+1} - x_i| = s_i' - s_{i+1}$. Furthermore, (x_i, y_i) and (x_{i+1}, y_{i+1}) must have same orientation and thus we know their relative position to each other. This implies that if for every (a_i, b_i) there would be only one crossing (x_i, y_i) by R of (a_i, b_i), we would know the relative positions of the (x_i, y_i) and thus $\hat{\xi} \approx \xi$. However, in general we will be able to show only that the condition that there exists only one crossing by R of (a_i, b_i) holds for all but a finite number of i's (see 5.4). This finite number of possible errors does not bother us, since $\hat{\xi}$ is defined as a limit; i.e. $\hat{\xi} = lim_{i \to \infty} \hat{\xi}^i$ and thus if only a finite number of $\hat{\xi}^i$'s are wrongly positioned with respect to the other $\hat{\xi}^i$'s, this has no influence on $\hat{\xi}$.

4 DETAILS

Lemma 4 *Let $R : \mathbb{Z} \longrightarrow V_T$ be a nearest neighbor walk and let $S : \mathbb{N} \longrightarrow \mathbb{Z}$ be another nearest neighbor walk. Then,*

4.1 (s,t) is a crossing by $R \circ S$ iff it is a crossing by S of a crossing by R, that is iff $(S(s), S(t))$ is a crossing by R whilst (s,t) is a crossing by S.

4.2 if (x, y) is the only crossing by R of (a, b), then (s, t) is a crossing by S of (x, y) iff (s, t) is also a crossing by $R \circ S$ of (a, b).

4.3 let $a_1, a_2, b_1, b_2 \in imR$ such that the shortest path from a_2 to b_2 first passes by a_1 and then by b_1 before arriving at b_2. Let us furthermore assume that there exists only one crossing (x_1, y_1) by R of (a_1, b_1) and only one crossing (x_2, y_2) by R of (a_2, b_2). Then, $(x_1 - y_1)(x_2 - y_2) > 0$ and if (s, t) is a straight crossing by S of (x_2, y_2) then the time between s and the first passage by $R \circ S$ at a_1 after s equals $|x_1 - x_2|$. Futhermore, $min(x_2, y_2) \leqslant x_1, y_1 \leqslant max(x_2, y_2)$; that is, (x_1, y_1) lies within (x_2, y_2). Thus, if we know $|x_1 - x_2|$ in this case we know the relative position of (x_1, y_1) and (x_2, y_2) to one another.

The proof of Lemma 4 requires the 'nearest neighbor 'walk property of R and S. The properties used to prove lemma 4 are similar to the properties of continuous function in $\mathbb{R}$, which in order to get from $x \in \mathbb{R}$ to $y \in \mathbb{R}$

have to pass at every point between x and y. The skipfreeness property of nearest neighbor walks immediately implies 4.1 and 4.3 . On the other hand 4.2 is an immediate consequence of 4.1. The skipfreeness property means that in a tree (E, V) if for $a, b, v \in V$, a and b lie in different components of $V - \{v\}$ then a nearest neighbor walk going from a to b must meet sometime in between the vertex v.

Lemma 5 *5.1 A.s. every vertex of T is visited only a finite number of times by R.*

5.2 A.s. $R(\mathbb{N}) \cap R(\mathbb{Z}_-)$ has a finite cardinality.

5.3 There exists a random set $U \subset V_T$ of finite cardinality, such that $imR - U$ consists of two infinite connected components.

5.4 For every set $U \subset T$ with finite cardinality (depending on ω) such that $imR - U$ consists of two infinite connected components, there exists n (depending on ω and U) such that for all a, b lying in different connected components of $imR - U$ with $d(a,0), d(b,0) \geq n$ there exists at most one crossing by R of (a, b).

When we use a distance for vertices on any graph $G = (E, V)$ it will always be the length of the shortest path between the two vertices. For $v, w \in V$ we will write $d(v, w)$ for that distance. We will assume that φ is non-random. We are allowed to do this, since when $\xi(0)$ is given, φ is non-random and since we know $\xi(0)$ because $\chi(0) = \xi(0)$. Since we assume φ to be non-random, $\{R(n)\}_{n \in \mathbb{Z}}$ is a Markov chain with stationary transition probability such that $\{R(n)\}_{n \in \mathbb{N}}$ and $\{R(-n)\}_{n \in \mathbb{N}}$ are i.i.d.

We are first going to prove 5.1, i.e. that. $\{R(n)\}_{n \in \mathbb{Z}}$ is transient. To prove that $\{R_{(n)}\}_{n \in \mathbb{Z}}$ is transient we need the following: **5.5** Let (E,V) be a 3-regular tree and 0 designate a vertex in V. Let $h : E \longrightarrow \{q_1, q_2, q_3\}$where $q_1 \geq q_2 \geq q_3 \geq 0$ and $q_1 + q_2 + q_3 = 1$ such that the sum of the images by h of the edges incident to any vertex is 1. Let $\{Y(n)\}_{n \in \mathbb{N}}$ be a Markov process starting at 0 with state space V and stationary transition probabilities defined by: for $v, w \in V, P_{vw} = h(\{v, w\})$ if $\{v, w\} \in E$ and $= 0$ otherwise. Then $\{Y(n)\}_{n \in \mathbb{N}}$ is transient.

Proof. of 5.5: We will show that $\{d(Y(2n), 0\}_{n \in \mathbb{N}}$ is stochastically bounded below by a random walk in $\mathbb{Z}$ converging towards infinity. Let μ be the probability measure such that:

$$\mu(\{2\}) = q_1 q_2 + q_2 q_3 + q_1 q_3$$

$$\mu(\{0\}) = q_1^2 + q_2^2 + q_3^2 + q_1 q_3 + q_2 q_3$$

and

$$\mu(\{-2\}) = q_1 q_2.$$

It is easy to see the first moment of μ is strictly positive. On the other hand, let us now assume that at time $2n$ the process $\{Y(n)\}_{n\in\mathbb{N}}$ is at $v \in V$, where $d(v,0) \geq 2$. Then there exists a permutation $\pi : (1,2,3) \longrightarrow (1,2,3)$ such that

$$P(\ d(Y(2n),0) = d(Y(2n+2),0)\ \ |Y(2n) = v) = q_1^2 + q_2^2 + q_3^2 + q_{\pi(1)}q_{\pi(3)}$$

and

$$P(\ d(Y(2n+2),0) - d(Y(2n),0)\ \ = -2\ |Y(2n) = v) = q_{\pi(1)}q_{\pi(2)}.$$

the law of $d(Y(2n+2),0) - d(Y_{(2n)},0)$ is stochastically bounded below by μ and thus the process $\{d(Y(2n),0)\}_{n\in\mathbb{N}}$ is bounded below by the random walk on $\mathbb{Z}$ starting at 0 whose increments are distributed according to μ. Since the first moment of μ is strictly positive, that random walk a.s. converges to infinity when n $\rightarrow \infty$ and thus $lim_{n\rightarrow\infty} d(Y(2n),0) = +\infty$. We have just proved that $\{Y(n)\}_{n\in\mathbb{N}}$ is transient. ■

Let us now get back to the **proof** of 5.1. Since$\{R(n)\}_{n\in\mathbb{N}}$ and $\{R(-n)\}_{n\in\mathbb{N}}$ are i.i.d. it is enough to prove that $\{R(n)\}_{n\in\mathbb{N}}$ is transient. We prove it by contradiction. So let us assume on the contrary that $\{R(n)\}_{n\in\mathbb{N}}$ is recurrent. We first need some definitions and assumptions: we assume that $\xi(0) = 1$ so that $\varphi(0) = 1$. (The other cases being similar are left to the reader.) Furthermore V1 will designate those vertices $v \in V_T$ for which all of the following two conditions hold: $\varphi(v) = 1$ and there exists a nearest neighbor walk $W : \mathbb{N} \longrightarrow V_T$ starting at the origin and visiting v ($v \in W(\mathbb{N})$) such that $\varphi \circ W$ is of the form $x_1,...,x_n,...$ where $x_i \in$ {finite words consisting only of 1's}, when i is odd and $x_i \in \{00, 22\}$ otherwise. (Here, for example, if $x_1 = 1$, $x_2 = 22$, $x_3 = 111$ and $x_4 = 00$, $x_5 = 1$ and $x_6 = 22$. We get that the sequence $\varphi \circ W$ starts like 12211100122.)

We define the set of edges $E1$ by: $\{v, w\} \in E1$ iff $v, w \in V1$ where $v \neq w$ and there is a path from v to w in T, not passing through any point of $V1 - \{v, w\}$.

It is easy to verify that $(E1, V1)$ is a 3-regular tree. Because $\{R_{(n)}\}_{n\in\mathbb{N}}$ is by assumption recurrent, we can define the process $\{Z(n)\}_{n\in\mathbb{N}}$ of the "visits by R to $V1$. " As a matter of fact, because $\{R(n)\}_{n\in\mathbb{N}}$ is by assumption recurrent it will always come back to $V1$. Let the times of the visits to $V1$,v_0,v_1,...,v_n,... be defined as follows:

$v_0 := 0$, $v_i :=$ first visit by $\{R(n)\}_{n\in\mathbb{N}}$ to a point of $V1 - \{R(v_{i-1})\}$ after time v_{i-1} for $i \geq 1$. Then, we define $Z(n) := R(v_n)$. $\{Z(n)\}_{n\in\mathbb{N}}$'s path is a nearest neighbor walk on (E1,V1). The process $\{Z(n)\}_{n\in\mathbb{N}}$ is also a Markov process with stationary transition probabilities. Finally, for any v, w such that $\{v,w\} \in E1$, because of symmetry,

$$P(Z(t+1) = w|Z(t) = v) = P(Z(t+1) = v|Z(t) = w).$$

Thus $\{Z(n)\}_{n\in\mathbb{N}}$ satisfies all the conditions in 5.5 and is transient. This implies that Z visits 0 only finitely many times which in terms, implies that $\{R(n)\}_{n\in\mathbb{N}}$ visits 0 only a finite number of times contradicting the assumption that R is recurrent. Thus $\{R(n)\}_{n\in\mathbb{N}}$ is transient. ■

Proof. of 5.2. For n≥1, let $v_n \in V_T$ designate the last vertex at distance n of 0, visited by $\{R(n)\}_{n\in\mathbb{N}}$ before time ∞. Let A be the event that $R(\mathbb{N}) \cap R(\mathbb{Z}_-)$ has infinite cardinality. We want to prove that $P(A) = 0$. Now, A can be expressed in the following way: $A = \{\omega \in \Omega$ such that all the vertices v_1,...,v_n,... are visited at least once by $\{R(-n)\}_{n\in\mathbb{N}}\}$. Let A_n be the event A_n=$\{\omega \in \Omega$ such that v_n is visited at least once by$\{R(-n)\}_{n\in\mathbb{N}}$, that is $v_n \in R(\mathbb{Z}_-)\}$. Then, $A = \cap_{n=1}^{\infty} A_n$. However, it is easy to see that a nearest neighbor walk coming from 0, has to pass at v_{n-1} in order to visit v_n. This implies that

$$A_1 \supset A_2 \supset ... \supset A_n \supset ...$$

Let w_1, w_2, w_3 be the three vertices adjacent to 0 in T. Let q be the maximum over all $i,j \in \{0,1,2\}$ of $P(\{R(n)\}_{n\in\mathbb{N}}$ visits w_i at least once after time zero$|\xi(0) = j$; $R(0) = 0)$. By 5.1 $\{R(n)\}_{n\in\mathbb{N}}$ is transient, and thus $q < 1$.As a matter of fact, if $\{R(n)\}_{n\in\mathbb{N}}$ is transcient, then $\{R(n)\}_{n\in\mathbb{N}}$ could, with positive probability, never visit w_i. Since $\{R(-n)\}_{n\in\mathbb{N}}$ and $\{R(n)\}_{n\in\mathbb{N}}$ are independent of each other, we have $P(A_n) \leqslant q^n$. Thus, by continuity of probabilities, we have

$$P(A) = \lim_{n\to\infty} P(A_n) = 0$$

and we are done with the proof of 5.2. ■

Proof. of 5.3. Let s and t be the first and last visit by R to $R(\mathbb{N}) \cap R(\mathbb{Z}_-)$. (s and t exist and are finite by 5.2). Take

$$U = imR - (R((t,\infty) \cap \mathbb{Z}) \cup R((-\infty, s) \cap \mathbb{Z})).$$

Then

$$imR - U = R((t, \infty) \cap \mathbb{Z}) \cup R((-\infty, s) \cap \mathbb{Z}).$$

Also $U \subset (R[s,t] \cap \mathbb{Z})$ and thus U has finite cardinality. Furthermore, since R is also a nearest neighbor walk, $R((t, \infty) \cap \mathbb{Z})$ and $R((-\infty, s) \cap \mathbb{Z})$ are two connected components. These components have infinite cardinality, since R is transcient. Since it is clear that the connected components are disjoint, U satisfies all the conditions of 5.3 and we are done. ■

Proof. of 5.4. Let r be the radius of U, that is $r := max_{u \in U} d(0, u)$.. 5.4 is a direct consequence of 5.1. As a matter of fact, let us prove this by contradiction. Let us assume on the contrary that 5.4 does not hold. Then, for any $m \geq 1$ we can find two vertices a and b lying in different components of $imR - U$ with $d(a, 0), d(b, 0) \geq m + r + 1$ and such that there exists at least two different crossings by R of (a, b). Let us call these crossings (x, y) and (z, w). It is easy to see that, since these crossings are different from each other, they must be disjoint, i.e.

$$[\min[x, y\}, \max\{x, y\}] \cap [\min\{z, w\}, \max\{z, w\}] = \emptyset.$$

It is also easy to see that within time m of x, y, z and w, R does not visit U. But during both crossings $(x, y)$and (z, w), R must visit U. We have just proved that there exist two visits to U by R separated in time by at least $2m$. Since we can take m as large as we want, this proves that there would exist infinitely many visits by R to the finite set U, which contradicts 5.1 and we are done. ■

Corollary 6 *Let $\psi : \mathbb{Z} \longrightarrow \{0, 1, 2\}$ be a (non-random) scenery and let $\{S_{(n)}\}_{n \in \mathbb{N}}$ designate as usual a simple random walk on $\mathbb{Z}$ starting at the origin. Then it is possible to reconstruct a.s. ψ up to translation and refection by observing ψ along the path of $\{S_{(n)}\}_{n \in \mathbb{N}}$ provided R satisfies all the conditions 5.1, 5.2 and 5.3. Here, R designates the only nearest neighbor walk on V_T starting at 0 and generating ψ on $\varphi^{(\psi(0))}$. Since ψ is non-random, so is R. Thus, in case 5.2 holds for R, $\psi \circ S$ determines ψ up to translation and reflection.*

Proof. When we showed in part 4 that our reconstruction must a.s. work, we only used the results of lemma 1 and lemma 2. However, 4.1, 4.2 and 4.3 are always true for nearest neighbor walks. Thus, if R also satisfies all the conditions in 5.1, 5.2 and 5.3, our reconstruction will a.s. work. ■

Moreover, it is interesting to notice that in order to prove that the reconstruction which we propose succeeds a.s., we only used the following property of $\{S_{(n)}\}_{n\in\mathbb{N}}$: S walks directly across every interval (a,b) (of integers) at least once. Thus the corollary above still holds even if $\{S_{(n)}\}_{n\in\mathbb{N}}$ is not a simple random walk, as long as $\{S_{(n)}\}_{n\in\mathbb{N}}$ is a nearest neighbor walk which crosses directly every interval at least once. Also, Mathias Löwe and the author have been able to prove that even when the ξ scenery is not i.i.d., 5.1, 5.2 , 5.3 and 5.4 still hold a.s. as long as the distribution of the scenery is a given by a hidden Markov process with finite many underlying states and stationary transition probabilities, and if the image of the representation of the scenery as a nearest neighbore walk is a.s. a tree having at least three disjoint infinite branches. As is obvious from our proof, theorem 1 will also hold if $\{\xi_{(n)}\}_{n\in\mathbb{Z}}$ is not i.i.d. but satisfies only 5.1, 5.2, 5.3 and 5.4. For example, if the scenery is a Markov chain of a finite collection of finite words with the letters 0,1 and 2, such that a.s. the representation of the scenery as a nearest neighbor walk contains at least three infinite connex disjoint components then one can a.s. reconstruct that scenery.

5 DETECTING A SINGLE DEFECT.

The main purpose of this paper was to investigate the question of H. Kesten, of whether one can detect a single defect in a one-dimensional scenery with less than 4 colors. Thanks to our main result, we are able to give a positive answer in the 3-color case.

Let $\bar{\xi}$: $\mathbb{Z} \longrightarrow \{0,1,2\}$ be a random scenery, which is defined on the same probability space as ξ and S. Furthermore, let ξ and $\bar{\xi}$ be equal to each other at all but a finite number of points. This means that there exists a possibly random set $J \subset \mathbb{Z}, J \neq \emptyset$ with finite cardinality such that $j \notin J \Longleftrightarrow \xi(j) = \bar{\xi}(j)$. Thus $\xi \neq \bar{\xi}$ a.s. Let $\chi = \xi \circ S$ and let $\bar{\chi} = \bar{\xi} \circ S$ be a scenery which is equal to either χ or $\bar{\chi}$ a.s.. The problem of recognizing a finite defect in a one-dimensional, two-color scenery can now be phrased as follows:

If we know $\xi, \bar{\xi}$ and $\hat{\chi}$ can we a.s. figure out whether $\hat{\chi}$ is equal to χ or $\bar{\chi}$?

Lemma 7 *ξ and $\bar{\xi}$ are a.s. not equivalent modulo translation and reflection.*

Proof. Let $F \subset \mathcal{P}(\mathbb{Z})$, (where $\mathcal{P}(\mathbb{Z})$ is the set of all bijections from $\mathbb{Z}$ to $\mathbb{Z}$), be the group generated by the reflection about the origin of $\mathbb{Z}$ and the translations in $\mathcal{P}(\mathbb{Z})$. Then, for every $f \neq id$ in F we can find a sequence

$$x_1 < x_2 < ... < x_i < x_{i+1} < ...$$

such that

$$f(x_i) \neq x_i \quad \text{for } i \in \mathbb{N} - \{0\}$$

and

$$f(\{x_1, x_2, ...\}) \cap \{x_1, x_2, ...\} = \emptyset.$$

Let A_i be the event that $\xi(f(x_i)) = \xi(x_i)$. We have

$$P(A_i) = p_0^2 + p_1^2 + p_2^2 < 1$$

Furthermore, the A_i's are independent events. From this it follows that there are a.s. an infinite number of x_i's such that $\xi(x_i) \neq \xi(f(x_i))$. Since ξ and $\bar{\xi}$ differ only at a finite number of points, there must exist a.s. an infinite number of x_i's such that $\xi(x_i) \neq \bar{\xi}(f(x_i))$. Thus a.s. $\xi \neq \bar{\xi} \circ f$. Also, we assumed that $\xi \neq \bar{\xi}$ and thus $\xi \neq \bar{\xi} \circ id$. Since there exists only countably many elements in F, it follows that

$$P(\xi \neq \bar{\xi} \circ f \quad \text{for all } f \in F) = 1.$$

Thus ξ and $\bar{\xi}$ are a.s. not equivalent. ■

Theorem 8 *If one is given ξ, $\bar{\xi}$ and $\hat{\chi}$ one can recognize whether $\hat{\chi} = \chi$ or $\hat{\chi} = \bar{\chi}$. In other words there exists a measurable function (depending on $\xi$and $\bar{\xi}$).*

$$\mathcal{B} : \{0,1,2\}^{\mathbb{Z}} \times \{0,1,2\}^{\mathbb{Z}} \times \{0,1,2\}^{\mathbb{N}} \longrightarrow \{0,1\}$$

such that a.s. $\mathcal{B}(\xi, \bar{\xi}, \hat{\chi}) = 0$ if $\hat{\chi} = \chi$ and $\mathcal{B}(\xi, \bar{\xi}, \hat{\chi}) = 1$ if $\hat{\chi} = \bar{\chi}$ (Here, $\mathcal{B}$ represents a statistical test capable of telling us, whether $\hat{\chi}$ was generated by $\{S(n)\}_{n \in \mathbb{N}}$ on ξ or $\bar{\xi}$ with 100% accuracy. This can also be formulated as follows: let μ,resp.$\bar{\mu}$, be the conditional distribution of χ, resp.$\bar{\chi}$, given ξ, resp. $\bar{\xi}$. Then, the distributions μ and $\bar{\mu}$ are random measures which are a.s. orthogonal to each other.)

Proof. As stated in corollary 6, our reconstruction works a.s. with any scenery such that its representation as a nearest neighbor walk "satisfies 5.1, 5.2 and 5.3. However, the event that these conditions hold is in the tail-field of the scenery. Thus, they still hold if we change the color for only a finite number of points in our scenery. Among others, this implies that the reconstruction described in 3.1 to 3.5 works for ξ as well as for $\bar{\xi}$. This clearly shows how one can recognize ξ from $\bar{\xi}$ by observing $\hat{\chi}$. As a matter of fact, apply the reconstruction 3.1-3.5 to $\hat{\chi}$ to get $\hat{\xi}$ modulo translation and reflection. If $\hat{\chi}$ equals χ, $\hat{\xi}$ will be equivalent to ξ, otherwise $\hat{\xi}$ will be equivalent to $\bar{\xi}$. Since ξ and $\bar{\xi}$ are a.s. not equivalent, we are finished. ■

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