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Abstract

A d-dimensional contact process is a simplified model for the spread of an infection on the lattice $\mathbb{Z}^d$. The dynamics of the process can be shortly described as follows. At any given time $t \geq 0$, certain sites $x \in \mathbb{Z}^d$ are infected while the remaining once are healthy. Infected sites recover at constant rate 1, while healthy sites are infected at a rate proportional to the number of infected neighboring sites on the lattice. The model is parametrized by the proportionality constant denoted by λ. If λ is sufficiently small, infection dies out eventually (subcritical process), whereas if λ is sufficiently large infection tends to be permanent (supercritical process).

In this paper we shall study the estimation problem for the parameter λ of the supercritical contact process starting with a single infected site at the origin, given that the process survives forever. Based on an observation of this process at a single time t, we obtain an estimator for the parameter λ which is consistent and asymptotically normal as $t \to \infty$.

The probabilistic results needed to establish these facts are taken from a companion paper Fiocco and van Zwet (1998b).

1 Introduction

A d-dimensional contact process is a simplified model for the spread of a biological organism or an infection on the lattice $\mathbb{Z}^d$. At each time $t \geq 0$, every point of the lattice (or site) is either infected or healthy. As time passes, a healthy site is infected with Poisson rate λ by each of its $2d$ immediate neighbors which is itself infected; an infected site recovers and becomes healthy with Poisson rate 1. Given the set of infected sites ξ_t at time t, the processes involved are independent until a change occurs. If the process starts with a set $A \subset \mathbb{Z}^d$ of infected sites at time $t = 0$, then ξ_t^A will denote the set of infected sites at time $t \geq 0$ and $\{\xi_t^A : t \geq 0\}$ will denote the contact process. For example, $\{\xi_t^{\mathbb{Z}^d} : t \geq 0\}$ or $\{\xi_t^{\{0\}} : t \geq 0\}$ will denote the processes starting with every site infected, or with a single infected site at the origin. If the starting set is chosen at random according to a probability distribution α, then the process will be written as $\{\xi_t^\alpha : t \geq 0\}$. If we do not want to specify the initial state of the process at all, we simply write $\{\xi_t : t \geq 0\}$.

We also need a compact notation for the state of a single site $x \in \mathbb{Z}^d$ at time t. For any contact process ξ_t we write

$$\xi_t(x) = 1_{\xi_t}(x) = \begin{cases} 1 & \textit{if } x \textit{ is infected at time } t \\ 0 & \textit{if } x \textit{ is healthy at time } t \end{cases}, \tag{1.1}$$

thus using the same symbol ξ_t for both the set of infected points and its indicator function. Of course $\xi_t^A(x)$ and $\xi_t^\alpha(x)$ will refer to the processes ξ_t^A and ξ_t^α in the same manner.

The first thing to note about the contact process is that for all non empty $A \subset \mathbb{Z}^d$, the infection will continue forever with positive probability if and only if λ exceeds a certain *critical value* λ_d. Such a process is called *supercritical*. Thus, if we define the random hitting time

$$\tau^A = \inf\{t : \xi_t^A = \emptyset\} , \quad A \subset \mathbb{Z}^d , \tag{1.2}$$

with the convention that $\tau^A = \infty$ if $\xi_t^A \neq \emptyset$ for all $t \geq 0$, then for the supercritical contact process

$$\mathbb{P}(\tau^A = \infty) > 0 \tag{1.3}$$

for every non-empty $A \subset \mathbb{Z}^d$. Moreover, if A has infinite cardinality $|A| = \infty$, then

$$\mathbb{P}(\tau^A = \infty) = 1 \quad . \tag{1.4}$$

In the supercritical case, the process $\xi_t^{\mathbb{Z}^d}$ that starts with all sites infected converges in distribution to the so-called upper invariant measure $\nu = \nu_\lambda$. Here convergence in distribution means convergence of probabilities of events defined by the behavior of the process on finite subsets of $\mathbb{Z}^d$, and 'invariant' refers to the fact that the process $\{\xi_t^\nu : t \geq 0\}$ is stationary. In particular, the distribution of ξ_t^ν is equal to ν for all t. Obviously, ν is also invariant under integer valued translations of $\mathbb{Z}^d$. The long range behavior of the supercritical contact process $\{\xi_t^A : t \geq 0\}$ for arbitrary non-empty $A \subset \mathbb{Z}^d$ is described by the *complete convergence theorem.* Let μ_t^A denote the probability distribution of ξ_t^A and $\delta_\emptyset$ the distribution that assigns probability 1 to the empty set.

Theorem 1.1 *Let $A \subset \mathbb{Z}^d$ and $\lambda > \lambda_d$. Then, as $t \to \infty$*

$$\mu_t^A \xrightarrow{w} \mathbb{P}(\tau^A < \infty)\delta_\emptyset + \mathbb{P}(\tau^A = \infty)\nu_\lambda \ . \tag{1.5}$$

For a proof for $d = 1$ see Liggett (1985), Chapter VI, Theorem 2.28; for $d > 1$ see Durrett & Griffeath (1982), Bezuidenhout & Grimmett (1990), Theorem 4, and Durrett (1991).

If $\lambda > \lambda_d$ and $A = \mathbb{Z}^d$, the process $\xi_t^{\mathbb{Z}^d}$ survives forever with probability 1 by (1.4) and converges exponentially to the limit process, i.e. for positive C and γ and all $t \geq 0$,

$$0 \leq \mathbb{P}(\xi_t^{\mathbb{Z}^d}(x) = 1) - \mathbb{P}(\xi^\nu(x) = 1) \leq Ce^{-\gamma t} \quad . \tag{1.6}$$

(Durrett (1991)).

Another major result concerning the contact process is the *shape theorem*. To formulate this result we first have to describe the graphical representation of contact processes due to Harris (1978). This is a particular coupling of all contact processes of a given dimension d and with a given value of λ, but with every possible initial state A or initial distribution α. Consider space-time $\mathbb{Z}^d \times [0, \infty)$. For every site $x \in \mathbb{Z}^d$ we define on the line $x \times [0, \infty)$ a Poisson process with rate 1; for every ordered pair (x, y) of neighboring sites in $\mathbb{Z}^d$ we define a Poisson process with rate λ. All of these Poisson processes are independent.

We now draw a picture of $\mathbb{Z}^d \times [0, \infty)$ where for each site $x \in \mathbb{Z}^d$ we remove the points of the corresponding Poisson process with rate 1 from the line $x \times [0, \infty)$; for each ordered pair of neighboring sites (x, y) we draw an arrow going perpendicularly from the line $x \times [0, \infty)$ to the line $y \times [0, \infty)$ at the points of the Poisson process with rate λ corresponding to the pair (x, y).

For any set $A \subset \mathbb{Z}^d$, define ξ_t^A to be the set of sites that can be reached by starting at time 0 at some site in A and traveling to time t along unbroken segments of lines $x \times [0, \infty)$ in the direction of increasing time, as well as along arrows. Clearly, $\{\xi_t^A : t \geq 0\}$ is distributed as a contact process with initial state A. By choosing the initial set at random with distribution α, we define $\{\xi_t^\alpha : t \geq 0\}$. The obvious beauty of this coupling is that for two initial sets of infected sites $A \subset B$, we have $\xi_t^A \subset \xi_t^B$ for all $t \geq 0$.

From this point on we shall assume that all contact processes are defined according to this graphical construction. We shall also restrict attention to the supercritical case.

Before formulating the shape theorem we need to introduce some notation. Let $\|\cdot\|$ denote the L^∞ norm on $\mathbb{R}^d$ that is

$$\|x\| = \max_{1 \leq i \leq d} |x_i| \quad .$$

for $x = (x_1, ..., x_d) \in \mathbb{R}^d$, and let $Q = \{x \in \mathbb{R}^d : \|x\| \leq 1/2\}$ denote the unit hypercube centered at the origin. For $A, B \subset \mathbb{R}^d$, $A \oplus B = \{x + y : x \in A, y \in B\}$ will denote the direct sum of A and B and for real r, $rA = \{rx : x \in A\}$. Define

$$H_t = \bigcup_{s \leq t} \xi_s^{\{0\}} \oplus Q \quad , \tag{1.7}$$

(1.8) $$K_t = \{x \in \mathbb{Z}^d : \xi_t^{\{0\}}(x) = \xi_t^{\mathbb{Z}^d}(x)\} \oplus Q \quad .$$

Thus for the process $\{\xi_t^{\{0\}} : t \geq 0\}$ that starts with a single infected site at the origin, H_t is obtained by taking the union of the sites that have been infected up to or at time t, and replacing these sites by unit hypercubes centered at these sites in order to fill up the space between neighboring sites. Similarly K_t is the filled-up version of the set of sites where $\xi_t^{\{0\}}$ and $\xi_t^{\mathbb{Z}^d}$ coincide. We are now in the position to formulate the shape theorem.

Theorem 1.2 *There exists a bounded convex subset U of $\mathbb{R}^d$ with the origin as an interior point and such that for any $\varepsilon \in (0,1)$,*

(1.9) $$(1-\varepsilon)U \subset t^{-1}(H_t \cap K_t) \subset t^{-1}H_t \subset (1+\varepsilon)U \ ,$$

eventually almost surely on the event $\{\tau^{\{0\}} = \infty\}$ where $\xi_t^{\{0\}}$ survives forever.

For a proof for $d = 1$ we refer to Durrett (1980); for $d > 1$ one may follow Bezuidenhout and Grimmett (1990) and Durrett (1991). For a version of the shape theorem for the process ξ_t^A on $\{\tau^A = \infty\}$ for arbitrary $A \subset \mathbb{Z}^d$ see Fiocco & van Zwet (1998a).

The shape theorem describes the growth of the set of infected sites if the process $\xi_t^{\{0\}}$ survives forever. Roughly speaking the convex hull of the set of infected sites will grow linearly in time as $t \to \infty$ and acquire an asymptotic shape tU, where U is a fixed convex set with the origin as an interior point. Inside of this set, say in $(1-\epsilon)tU$, the smallest and the largest possible process $\xi_t^{\{0\}}$ and $\xi_t^{\mathbb{Z}^d}$ are equal eventually a.s. and this must mean that, for large t, their distribution is close to the equilibrium distribution ν. Together, the complete convergence theorem and the shape theorem describe the peculiar type of convergence of the supercritical contact process to its limiting distribution. The infection spreads at a constant speed and relatively soon after it has reached a site x, equilibrium will set in at that site. In the evocative language of Durrett & Griffeath (1982) the infection behaves like a "blob in equilibrium". For a more precise and detailed account of the facts mentioned so far, the reader is referred to e.g. Durrett & Griffeath (1982), Liggett (1985), Bezuidenhout & Grimmett (1990), Durrett (1991) and also Fiocco (1997).

A third important property of the contact process is its reversibility or self-duality. If, in the graphical representation, time is run backwards and all arrows representing infection of one site by another, are reversed, then the new graphical representation has precisely the same probabilistic structure as the original one. In particular

(1.10) $$\mathbb{P}(\xi_t^A \cap B \neq \emptyset) = P(\xi_t^B \cap A \neq \emptyset), \quad \textit{for all } A, B \subset \mathbb{Z}^d \textit{ and } t \geq 0 \quad .$$

With $A = \{0\}$ and $B = \mathbb{Z}^d$ this yields

$$\mathbb{P}(\tau^{\{0\}} > t) = \mathbb{P}(\xi_t^{\mathbb{Z}^d}(0) = 1)$$

and letting $t \to \infty$ in the supercritical case, this reduces to

$$\mathbb{P}(\tau^{\{0\}} = \infty) = \mathbb{P}(\xi_t^{\nu}(0) = 1) \quad .$$

Combining this with (1.6) we see that if $\lambda > \lambda_d$,

$$\mathbb{P}(t < \tau^{\{0\}} < \infty) \leq Ce^{-\gamma t} \tag{1.11}$$

(cf. Durrett (1991)).

In this paper we shall study the estimation problem for the parameter λ of the supercritical contact process $\xi_t^{\{0\}}$, given that it does not die out. Based on an observation of $\xi_t^{\{0\}}$ at a single time t we derive an estimator $\widehat{\lambda}_t^{\{0\}}$ and show that it is consistent and asymptotically normal as $t \to \infty$.

The informal description of the convergence of the contact process immediately suggests a way to derive an estimator of the parameter λ. If $\xi_t^{\{0\}}$ survives forever, then observing $\xi_t^{\{0\}}(x)$ for all sites x contained in $(1-\epsilon)tU$ is asymptotically the same as observing the limit process $\xi_t^{\nu}(x)$ on this set. This asymptotic "equivalence" of ξ_t^{ν} and $\xi_t^{\{0\}}$ on $(1-\epsilon)tU$ should allow us to derive an estimator of λ based on the limit process $\xi_t^{\nu}(x)$ for sites $x \in (1-\epsilon)tU$, and hope that this estimator will also work for the process $\xi_t^{\{0\}}$. The advantage of deriving the estimator under ξ_t^{ν} is that we can use the stationarity of this process to set up the estimating equation.

For $D \subset \mathbb{Z}^d$, define the total number of infected sites in the set D at time t as

$$n_t(D) = \sum_{x \in D} \xi_t(x) \quad , \tag{1.12}$$

and the total number of pairs of neighboring sites for which one site is healthy and lies in D and the other is infected as

$$k_t(D) = \sum_{x \in D} k_t(x) \quad , \tag{1.13}$$

where

$$k_t(x) = (1 - \xi_t(x)) \sum_{|x-y|=1} \xi_t(y) \quad . \tag{1.14}$$

Here $|x-y|$ denotes the L^1 distance between sites x and y. When we need to specify the initial state of the process we shall use an appropriate notation. For example $n_t^{\{0\}}$ and $k_t^{\{0\}}$ will indicate that we are referring to the process $\xi_t^{\{0\}}$. Similarly for the process ξ_t^{ν}, we write n_t^{ν} and k_t^{ν}.

For the ξ_t^ν process, $\xi_t^\nu(x)$ increases by 1 at rate $\lambda k_t^\nu(x)$ and decreases by 1 at rate $\xi_t^\nu(x)$. As ξ_t^ν is stationary, this implies that $\lambda \mathbb{E} k_t^\nu(x) = \mathbb{E}\xi_t^\nu(x)$ and since ξ_t^ν is spatially translation invariant we have

$$\lambda = \frac{\mathbb{E}\xi_t^\nu(x)}{\mathbb{E}k_t^\nu(x)} = \frac{\mathbb{E}\xi_t^\nu(0)}{\mathbb{E}k_t^\nu(0)} \quad . \tag{1.15}$$

Notice that these expectations are independent of t because of the stationarity of ξ_t^ν. For $t \geq 0$, let $A_t \subset \mathbb{Z}^d$ be finite sets of cardinality $|A_t| \to \infty$ as $t \to \infty$. It seems reasonable to expect that some form of the law of large numbers will ensure that as $t \to \infty$,

$$\frac{n_t^\nu(A_t)}{|A_t|} = \frac{\sum\limits_{x \in A_t} \xi_t^\nu(x)}{|A_t|} \sim \mathbb{E}\xi_t^\nu(0)$$

and

$$\frac{k_t^\nu(A_t)}{|A_t|} = \frac{\sum\limits_{x \in A_t} k_t^\nu(x)}{|A_t|} \sim \mathbb{E}k_t^\nu(0) \quad .$$

This would imply that

$$\frac{n_t^\nu(A_t)}{k_t^\nu(A_t)}$$

is a plausible estimator of λ on the basis of an observation of the process ξ_t^ν at a single time t. If, in addition to $|A_t| \to \infty$, we also require that $A_t \subset (1-\epsilon)tU$ for some $\epsilon > 0$, then the shape theorem suggests that conditional on $\xi_t^{\{0\}}$ surviving forever, the probabilistic behavior of $\xi_t^{\{0\}}$ and ξ_t^ν should be asymptotically the same on the set $A_t \subset \mathbb{Z}^d$. But this indicates that if we observe the process $\xi_t^{\{0\}}$ instead of ξ_t^ν, then

$$\frac{n_t^{\{0\}}(A_t)}{k_t^{\{0\}}(A_t)}$$

would be a plausible estimator of λ based on $\xi_t^{\{0\}}$, provided that $\xi_t^{\{0\}}$ survives. Unfortunately, the set U is unknown - and so is t in many applications - and hence we can not implement this estimation procedure directly. However, the shape theorem also suggests that if $\xi_t^{\{0\}}$ survives forever, the convex hull $\mathcal{C}(\xi_t^{\{0\}})$ of the set $\xi_t^{\{0\}}$ of infected sites behaves asymptotically like tU. Hence we may expect that if we define a mask

$$C_t = (1-\delta)\mathcal{C}(\xi_t^{\{0\}})$$

for some $\delta > 0$ and $\xi_t^{\{0\}}$ survives, then $|C_t \cap \mathbb{Z}^d| \to \infty$ and $C_t \subset (1-\epsilon)tU$ for some $\epsilon > 0$. Combining these ideas we arrive at

$$\widehat{\lambda}_t^{\{0\}} = \widehat{\lambda}_t^{\{0\}}(C_t) = \frac{n_t^{\{0\}}(C_t)}{k_t^{\{0\}}(C_t)} \tag{1.16}$$

as a plausible estimator of λ on the basis of an observation of $\xi_t^{\{0\}}$ at a single time t. In fact we shall use masks C_t which are obtained by shrinking the set $\mathcal{C}(\xi_t^{\{0\}})$ in a more general manner than through multiplication by $(1-\delta)$ (cf. Section 3).

The aim of this paper is to prove that $\widehat{\lambda}_t^{\{0\}}$ is a consistent and asymptotically normal estimator of λ on the event where $\xi_t^{\{0\}}$ survives forever. To do this we not only have the considerable problem of making the above heuristic argument precise, but in order to prove the asymptotic normality, we also have to show that for the $\xi_t^{\{0\}}$ process conditional on survival, distant sites evolve almost independently. The technical tools for dealing with these problems are provided in Fiocco & van Zwet (1998b).

We should stress at this point that shrinking $\mathcal{C}(\xi_t^{\{0\}})$ to obtain the mask C_t is absolutely essential to obtain an estimator that works well in practice. Without shrinking, the mask will contain the boundary area of the "blob" of infected points where equilibrium has not yet set in and the infected points are therefore less dense. This has the effect of lowering the estimator of λ and simulation shows that the resulting negative bias is considerable (cf. Section 6). ¿From a theoretical point of view we shall find that without shrinking - i.e. if $\delta = 0$ and hence $C_t = \mathcal{C}(\xi_t^{\{0\}})$ - we can still show consistency of the estimator $\widehat{\lambda}_t^{\{0\}}$, but not its asymptotic normality.

Having described our purpose, a few words concerning the motivation for this study may not be amiss. For more than two decades interacting particle systems have been a major object of study in probability theory. We believe that these models are important and can be applied to describe many situations of great practical interest. It would therefore seem timely to study these processes from a statistical point of view.

However, the probabilistic structure of interacting particle systems is highly complex and it therefore seemed wise to begin by restricting attention to the simplest non-trivial example, which is the contact process. It is non-trivial because it exhibits a phase change; it is relatively simple because each site can be in only two states and direct interactions are limited to immediate neighbors. Admittedly the contact process is perhaps too simple to serve as a model for many situations. However, we are inclined to think that any progress made for this problem will help to attack more complex processes in future.

2 Technical tools

In this section we provide the reader with a number of tools that will be used in this paper for establishing the properties of $\widehat{\lambda}_t^{\{0\}}$. The proofs may be found in Fiocco & van Zwet (1998b) which will henceforth be abbreviated F&vZ (1998b). Let $\mathcal{C}(\xi_t^{\{0\}})$ be the convex hull of the set of infected sites. Theorems 1.4 and 1.5 in F&vZ (1998b) provide the following bounds on this set.

Theorem 2.1 *For every* $\epsilon \in (0,1)$,

$$(1-\epsilon)tU \subset \mathcal{C}(\xi_t^{\{0\}}) \subset (1+\epsilon)tU \text{ eventually,} \tag{2.1}$$

a.s.on the set $\{\tau^{\{0\}} = \infty\}$. *Moreover, for every* $\epsilon \in (0,1)$ *and* $r > 0$, *there exists a positive number* $A_{r,\epsilon}$ *such that for every* $t > 0$

$$\mathbb{P}\Big((1-\epsilon)tU \subset C(\xi_t^{\{0\}}) \mid \tau^{\{0\}} = \infty\Big) \geq 1 - A_{r,\epsilon}t^{-r} \quad .$$

Before formulating the next result we need to introduce some notation. Let $H = \{0,1\}^{\mathbb{Z}^d}$ denote the state space for the contact process. For $f : H \to \mathbb{R}$ and $x \in \mathbb{Z}^d$, define

$$\begin{aligned} &\Delta_f(x) = \sup\Big\{|f(\eta) - f(\zeta)| : \eta, \zeta \in H \text{ and } \eta(y) = \zeta(y) \text{ for all } y \neq x\Big\} , \\ (2.2) \qquad & \\ &|||f||| = \sum_{x \in \mathbb{Z}^d} \Delta_f(x) . \end{aligned}$$

For $R_1, R_2 \subset \mathbb{Z}^d$, let $d(R_1, R_2)$ denote the L^1-distance of R_1 and R_2, thus

$$d(R_1, R_2) = \inf_{x \in R_1, y \in R_2} |x - y| = \inf_{x \in R_1, y \in R_2} \sum_{i=1}^{d} |x_i - y_i| \quad ,$$

Let

$$(2.3) \qquad D_R = \{f : H \to \mathbb{R}, |||f||| < \infty, f(\eta) \textit{ depends on } \eta \textit{ only through } \eta \cap R\} ,$$

i.e. D_R is the class of functions f with $|||f\,||| < \infty$ such that $f(\eta)$ depends on η only through $\eta(x)$ with $x \in R$.

Theorem 2.2 *There exist positive numbers* γ *and* C *such that for every* $R_1,\ R_2 \subset \mathbb{Z}^d$, $f \in D_{R_1}$ $g \in D_{R_2}$, *and* $t \geq 0$,

$$(2.4) \qquad \left|\operatorname{cov}\left(f(\xi_t^{\mathbb{Z}^d}), g(\xi_t^{\mathbb{Z}^d})\right)\right| \leq C|||f|||.|||g|||e^{-\gamma d(R_1,R_2)}.$$

In particular, there exist positive numbers γ *and* C *such that for all* $t \geq 0$, *and* x, $y \in \mathbb{Z}^d$,

$$(2.5) \qquad \left|\operatorname{cov}\left(\xi_t^{\mathbb{Z}^d}(x), \xi_t^{\mathbb{Z}^d}(y)\right)\right| \leq Ce^{-\gamma|x-y|},$$

and

$$(2.6) \qquad \left|\operatorname{cov}\left(k_t^{\mathbb{Z}^d}(x), k_t^{\mathbb{Z}^d}(y)\right)\right| \leq Ce^{-\gamma|x-y|} .$$

Proof. Theorem 1.7 in F&vZ (1998b) produces the first part of the theorem. Inequalities (2.5) and (2.6) follow because $\|f\| = \|g\| = 1$ or 8 respectively.

Obviously (2.5) and (2.6) imply that $\sigma^2(n_t^{\mathbb{Z}^d}(D))$ and $\sigma^2(k_t^{\mathbb{Z}^d}(D))$ are of order $|D|$ for large D. The following theorem extends this result to all moments of even order.

Theorem 2.3 *For any $k = 1, 2...$, there exists a number $C_k > 0$ such that for every $D \subset \mathbb{Z}^d$ and $t \geq 0$,*

$$\mu_{2k} = \mathbb{E}\Big(n_t^{\mathbb{Z}^d}(D) - \mathbb{E}n_t^{\mathbb{Z}^d}(D)\Big)^{2k} \leq C_k |D|^k \,. \tag{2.7}$$

and

$$\nu_{2k} = \mathbb{E}\Big(k_t^{\mathbb{Z}^d}(D) - \mathbb{E}k_t^{\mathbb{Z}^d}(D)\Big)^{2k} \leq C_k |D|^k \,. \tag{2.8}$$

Proof. The proof follows from Theorem 2.1 in F&vZ (1998b).

For $\lambda > \lambda_d$, let $\overline{\xi}_t^{\{0\}}$ denote the process $\xi_t^{\{0\}}$ conditioned on $\{\tau^{\{0\}} = \infty\}$. This is not a contact process but for large s and $t - s$ it can be defined for $t \geq s$ according to the graphical representation starting at time s on a set of large probability. This enables us to define a process $\overline{\xi}_t^{\{0\}}$ in such a way that it is coupled to a process $\xi_{t-s}^{\mathbb{Z}^d}$ for $t \geq s$ with large probability for large s and $t-s$. Theorem 1.6 in F&vZ (1998b) provides the following probability bound for equality of the processes on the set $(1-\epsilon)tU$, as well as for each individual site in $(1-\epsilon)tU$ separately.

Theorem 2.4 *For every $\epsilon \in (0,1)$ and $r > 0$ there exist numbers A_ϵ and $A_{r,\epsilon}$ depending on ϵ and (r, ϵ) respectively, such that for $s \wedge (t-s) \geq A_\epsilon$,*

$$\begin{aligned} \mathbb{P}\Big(\overline{\xi}_t^{\{0\}} \cap (1-\epsilon)tU &= \xi_{t-s}^{\mathbb{Z}^d} \cap (1-\epsilon)tU\Big) \\ &\geq 1 - A_{r,\epsilon}\Big(s^{-r} + \frac{s^d}{(t-s)^r}\Big) \,. \end{aligned} \tag{2.9}$$

Moreover, for every $x \in (1-\epsilon)tU$

$$\mathbb{P}\Big(\overline{\xi}_t^{\{0\}}(x) = \xi_{t-s}^{\mathbb{Z}^d}(x)\Big) \geq 1 - A_{r,\epsilon}\Big(s^{-r} + (t-s)^{-r}\Big) \,. \tag{2.10}$$

The final theorem in this section is a re-statement of Theorem 1.8 in F&vZ (1998b). It asserts that for the $\overline{\xi}_t^{\{0\}}$ process, distant sites evolve almost independently for large t.

Theorem 2.5 *For every $\epsilon \in (0,1)$ and $r > 0$ there exists a positive number $A_{r,\epsilon}$ such that for all $t \geq 0$ and all f and g satisfying*

$$f \in D_{R_1} \text{ with } R_1 \subset (1-\epsilon)tU \cap \mathbb{Z}^d \ ,$$

$$g \in D_{R_2} \text{ with } R_2 \subset \mathbb{Z}^d \ ,$$

$$\text{(2.11)} \qquad \left|\mathrm{cov}\left(f(\overline{\xi}_t^{\{0\}}), g(\overline{\xi}_t^{\{0\}})\right)\right| \leq A_{r,\epsilon} \, |\!|\!| f |\!|\!| \cdot |\!|\!| g |\!|\!| \, (d(R_1, R_2) \wedge t)^{-r} \quad .$$

3 Shrinking

As we have argued in the introduction we choose the mask C_t for computing the estimator $\widehat{\lambda}_t^{\{0\}}$ as a shrunken version of the convex hull $\mathcal{C}(\xi_t^{\{0\}})$, that is guaranteed to lie in $(1-\epsilon)tU$ with large probability. As an example we discussed the choice $C_t = (1-\delta)\mathcal{C}(\xi_t^{\{0\}})$ about which we shall have more to say later in this section (see Example (ii)). However, we also noted that it is possible to consider more general methods of shrinking and this is the topic of the present section.

For a set $A \subset \mathbb{R}^d$ the interior of A is denoted by $\overset{\circ}{A}$ and the discrete cardinality of A as $A_D = |A \cap \mathbb{Z}^d|$. Define a shrinking operation as follows.

Definition 3.1 *Suppose that to any convex set $V \subset \mathbb{R}^d$ there corresponds a convex set $V^- \subset \mathbb{R}^d$. Then the map $V \to V^-$ is called a shrinking if for every convex V and W with $0 \in \overset{\circ}{V}$,*

$$\text{(3.1)} \qquad V^- \subset V \ ,$$

$$\text{(3.2)} \qquad V \subset W \Longrightarrow V^- \subset W^- \ ,$$

$$\text{(3.3)} \qquad |(tV)^-|_D \to \infty \text{ as } t \to \infty \ ,$$

$$\text{(3.4)} \qquad \text{if } s,t \to \infty \text{ with } t/s \to 1, \text{ then } \frac{|(tV)^-|_D}{|(sV)^-|_D} \to 1 \ .$$

Property (3.3) guarantees that if V contains a ball centered at the origin and hence tV grows linearly in t in any direction, then the number of lattice points in $(tV)^-$ tends to infinity. By a standard argument one finds that (3.4) is equivalent to:
if $0 \in \overset{\circ}{V}$, then for every $\delta > 0$ there exist $\epsilon > 0$ and $t_0 > 0$ such that

$$\left| \frac{|[(1+\epsilon)tV]^-|_D}{|[(1-\epsilon)tV]^-|_D} - 1 \right| \leq \delta \quad \text{for all } t \geq t_0 \quad . \tag{3.5}$$

We shall base the estimator of λ on a shrunken version C_t of $\mathcal{C}(\xi_t^{\{0\}})$, i.e.

$$C_t = [\mathcal{C}\ (\xi_t^{\{0\}})]^- \tag{3.6}$$

and

$$\widehat{\lambda}_t^{\{0\}} = \widehat{\lambda}_t^{\{0\}}(C_t) = \frac{n_t^{\{0\}}(C_t)}{k_t^{\{0\}}(C_t)} \quad . \tag{3.7}$$

The set defined in (3.6) is called the random mask or window.

The reader will have noticed that we have defined a shrinking by listing its properties (3.1)- (3.4) when applied to a convex set V with $0 \in \overset{\circ}{V}$. For other convex sets V, we may define the convex set V^- in an arbitrary manner. The reason that we have extended the definition of a shrinking $V \to V^-$ to all convex V is that in e.g. (3.6), we cannot guarantee that $\mathcal{C}(\xi_t^{\{0\}})$ will have the origin as an interior point. Of course (2.1) ensures that eventually a.s. on $\{\tau^{\{0\}} = \infty\}$, 0 will be an interior point of $\mathcal{C}(\xi_t^{\{0\}})$ because $0 \in \overset{\circ}{U}$. Hence the fact that V^- is an arbitrary convex set if $0 \notin \overset{\circ}{V}$ will not influence the asymptotic behavior of our estimator $\widehat{\lambda}_t^{\{0\}}(C_t)$.

Together (3.6), (3.7) and Definition 3.1 will allow us to prove consistency of $\widehat{\lambda}_t^{\{0\}}$ on the set where $\xi_t^{\{0\}}$ survives forever. However, in order to prove strong consistency of $\widehat{\lambda}_t^{\{0\}}$, we need to strengthen assumption (3.3) and require that if $0 \in \overset{\circ}{V}$, then

$$for\ some \quad \delta > 0, \qquad \liminf_{t\to\infty} \frac{|(tV)^-|_D}{t^\delta} > 0 \quad . \tag{3.8}$$

To prove asymptotic normality of our estimator given $\{\tau^{\{0\}} = \infty\}$ we need to assume that, if $0 \in \overset{\circ}{V}$, then

$$V^- \subset (1-\delta)V \quad , \tag{3.9}$$

while at the same time strengthening (3.3) in a different direction and require that

$$(tV)^- \to \mathbb{R}^d \qquad as \ \ t \to \infty \quad . \tag{3.10}$$

We end this section by presenting various ways of shrinking that one may wish to apply to the convex hull of the set of infected sites $\mathcal{C}\ (\xi_t^{\{0\}})$ in order to obtain the mask C_t. Numerical results that show how the performance of the estimator $\widehat{\lambda}_t^{\{0\}}$ improves by restricting attention to sites in the mask C_t instead of considering the entire configuration, will be discussed in Section 6.

Examples of shrinking

(i) $V^- = V$.
This satisfies Definition 3.1 as well as (3.8) and (3.10) but not (3.9). In this case we do not shrink but simply choose $C_t = \mathcal{C}\ (\xi_t^{\{0\}})$ for computing $\widehat{\lambda}_t^{\{0\}}$.

(ii) $V^- = (1-\delta)V,\ 0 < \delta < 1$.
Obviously Definition 3.1 as well as (3.8)-(3.10) are satisfied. In determining the mask $C_t = (1-\delta)\mathcal{C}(\xi_t^{\{0\}})$ we have to face the problem that we observe the set $\xi_t^{\{0\}}$, but not necessarily the location of the origin. As C_t is determined by shrinking $\mathcal{C}(\xi_t^{\{0\}})$ towards the origin, we have to estimate the origin and shrink towards this estimated origin instead. An obvious estimate of the origin is the coordinate-wise average of all sites in $\mathcal{C}(\xi_t^{\{0\}})$, i.e. the center of gravity of this set of sites. In view of Theorem 2.1 and the fact that the set U is obviously symmetric with respect to the origin, it is easy to see that the estimate of the origin has error $o_P(t)$ on the set where $\xi_t^{\{0\}}$ survives forever. But this implies that shrinking $\mathcal{C}(\xi_t^{\{0\}})$ towards the estimated rather then the true origin will not affect the consistency of $\widehat{\lambda}_t^{\{0\}}$ in the conclusion of Theorem 4.1. The asymptotic normality of $\widehat{\lambda}_t^{\{0\}}$ in Theorem 5.2 will not be affected either by a slightly more complicated argument.

As we have already argued, the reason for shrinking the convex hull $\mathcal{C}(\xi_t^{\{0\}})$ to obtain the mask C_t, is that once the process $\xi_t^{\{0\}}$ has reached a certain site, it needs time to attain equilibrium. It seems reasonable to assume that the time needed for this is independent of the location of the site. Since $\mathcal{C}(\xi_t^{\{0\}})$ grows at a constant speed like tU, this leads us to consider choosing δ proportional to t^{-1}, i.e. inversely proportional to the linear dimension of the "blob". Notice that the fraction of lattice points deleted when shrinking $\mathcal{C}(\xi_t^{\{0\}})$ to obtain C_t, is roughly proportional to $(t^d - [(1-\delta)t]^d)/t^d \sim \delta d$ for small $\delta > 0$ and hence this fraction α could also be chosen inversely proportional to the linear dimension of the blob. Of course the argument only makes sense if the blob is of a certain minimum size.

(iii) $V^- = peeling(V)$.
This type of shrinking avoids the estimation of the origin of the picture. For an arbitrary convex set $V \subset \mathbb{R}^d$, the peeling procedure starts with the set $V_0 = \mathcal{C}(V \cap \mathbb{Z}^d)$, the convex hull of the lattice points of V. Notice that in the particular case we are considering, $V = \mathcal{C}(\xi_t^{\{0\}})$ and hence $V_0 = V$. The peeling of V is now obtained by removing all lattice points in the L^1-contour of V_0, constructing the convex hull of the remaining lattice points of V_0, and repeating this procedure k times until a fraction

α of the lattice points in V_0 has been removed. This amounts to stripping away the k outermost layers of the blob. Obviously peeling satisfies Definition 3.1 as well as (3.8)-(3.10). In view of the problems encountered in Example (ii), we prefer peeling over multiplication by $(1-\delta)$ as a shrinking operation. For more details about peeling the reader is referred to Fiocco (1997).

(vi) $V^- = B_{\{c,r\}}$.
The mask is computed by taking a Euclidean ball inside the set on infected sites with center c and radius r where the center is estimated by taking the coordinate-wise average of all sites in $\mathcal{C}(\xi_t^{\{0\}})$ and the radius r is computed by averaging the L^1-distances between the estimated center and the sites in $\mathcal{C}(\xi_t^{\{0\}})$.

It should be clear from these four examples that we have a great deal of freedom in choosing our mask as a shrunken version of $\mathcal{C}$ $(\xi_t^{\{0\}})$. In order to satisfy (3.1)–(3.4), we mainly have to watch out that we do not remove all but a bounded number of lattice points of $\mathcal{C}$ $(\xi_t^{\{0\}})$, and that for large sets the fraction α of lattice points deleted depends on the size of the set in a smooth manner. Conditions (3.8) and (3.10) are not likely to be violated for any sensible procedure either. Assumption (3.9) asserts that the shrinking is non-trivial.

4 The estimation problem: Consistency

In the proof of the consistency of $\widehat{\lambda}_t^{\{0\}}$ we shall not follow the same route that we used in Section 1 to arrive at the estimator $\widehat{\lambda}_t^{\{0\}}(C_t)$. Rather than introducing a new coupling to compare $\xi_t^{\{0\}}$ on $\{\tau^{\{0\}} = \infty\}$ with ξ_t^{ν}, we shall simply employ the standard graphical representation for comparison with $\xi_t^{\mathbb{Z}^d}$ instead. In Theorem 2.1 we showed that on $\{\tau^{\{0\}} = \infty\}$, $\mathcal{C}(\xi_t^{\{0\}})$ can be bracketed between two non-random convex sets. By applying the shape theorem (Theorem 1.2) we reduce the problem to one concerning the $\xi_t^{\mathbb{Z}^d}$ process on a non-random convex set and then show that the difference between the random and the non-random masks is negligible.

Let $A_t \subset \mathbb{Z}^d$ be a finite non-random set with $|A_t| \to \infty$ as $t \to \infty$. In analogy with (1.12) and (1.13) define

$$n_t^{\mathbb{Z}^d}(A_t) = \sum_{x \in A_t} \xi_t^{\mathbb{Z}^d}(x) \tag{4.1}$$

$$k_t^{\mathbb{Z}^d}(A_t) = \sum_{x \in A_t} k_t^{\mathbb{Z}^d}(x) \quad \textit{where} \quad k_t^{\mathbb{Z}^d}(x) = (1 - \xi_t^{\mathbb{Z}^d}(x)) \sum_{|x-y|=1} \xi_t^{\mathbb{Z}^d}(y) \ . \tag{4.2}$$

Lemma 4.1 *Suppose that for $t \geq 0$, the sets $A_t \subset \mathbb{Z}^d$ satisfy $A_t \subset A_{t'}$ if $t < t'$, $|A_t| < \infty$ and $|A_t| \to \infty$ for $t \to \infty$. Then as $t \to \infty$,*

$$\text{(4.3)} \qquad \frac{n_t^{\mathbb{Z}^d}(A_t)}{|A_t|} \xrightarrow{P} \mathbb{E}\xi^\nu(0) \quad ,$$

$$\text{(4.4)} \qquad \frac{k_t^{\mathbb{Z}^d}(A_t)}{|A_t|} \xrightarrow{P} \mathbb{E}k^\nu(0) \quad .$$

Moreover, if for some $\delta > 0$

$$\text{(4.5)} \qquad \liminf_{t\to\infty} \frac{|A_t|}{t^\delta} > 0 \ ,$$

then as $t \to \infty$,

$$\text{(4.6)} \qquad \frac{n_t^{\mathbb{Z}^d}(A_t)}{|A_t|} \longrightarrow \mathbb{E}\xi^\nu(0) \quad a.s. \ ,$$

$$\text{(4.7)} \qquad \frac{k_t^{\mathbb{Z}^d}(A_t)}{|A_t|} \longrightarrow \mathbb{E}k^\nu(0) \quad a.s. \ .$$

Proof. We shall only prove (4.3) and (4.6). The proof of (4.4) and (4.7) is almost exactly the same.

By Theorem 2.3 and the Markov inequality,

$$\text{(4.8)} \qquad \mathbb{P}\left(\left|\frac{n_t^{\mathbb{Z}^d}(A_t)}{|A_t|} - \frac{\mathbb{E}n_t^{\mathbb{Z}^d}(A_t)}{|A_t|}\right| \geq \epsilon\right) \leq C_{k,\epsilon}\ |A_t|^{-k}$$

for every $k = 1, 2, \ldots$ and appropriate $C_{k,\epsilon} > 0$. By (1.6),

$$\text{(4.9)} \qquad \frac{\mathbb{E}n_t^{\mathbb{Z}^d}(A_t)}{|A_t|} = \mathbb{E}\xi_t^{\mathbb{Z}^d}(0) \longrightarrow \mathbb{E}\xi^\nu(0)$$

as $t \to \infty$. Since $|A_t| \to \infty$ this proves (4.3).

For every $\epsilon > 0$ and $A \subset \mathbb{Z}^d$ we have

$$\text{(4.10)} \qquad \mathbb{P}\Big(\sup_{0\leq s\leq h}\left|n_{t+s}^{\mathbb{Z}^d}(A) - n_t^{\mathbb{Z}^d}(A)\right| \geq \epsilon|A|\Big) \leq \mathbb{P}\Big(Z \geq \epsilon|A|\Big)$$

where Z has a Poisson distribution with $\mathbb{E}Z = \mu = ch|A|$, where $c = 1 \vee 2d\lambda$. To see this, note that between times t and $t + h$ a change at any particular site in A occurs with rate at most c. As

$$\mathbb{E}e^Z = e^{(e-1)\mu} \leq e^{2\mu} \quad ,$$

we find that if $h \leq \epsilon/(4c)$,

$$\mathbb{P}(Z \geq \epsilon|A|) \leq e^{2\mu - \epsilon|A|} \leq e^{-\epsilon|A|/2} \quad . \tag{4.11}$$

Take $t_0 = 0$ and define $t_0 < t_1 < t_2 < \ldots$ recursively by

$$t_{m+1} = (t_m + \epsilon/(4c)) \wedge \inf\{t > t_m : A_{t-} \neq A_{t+}\}$$

where $A_{t-} = \lim_{s \uparrow t} A_s = \bigcup_{s<t} A_s$ and $A_{t+} = \lim_{s \downarrow t} A_s = \bigcap_{s>t} A_s$. Hence t_{m+1} is obtained by adding to t_m until one either arrives at $t_m + \epsilon/(4c)$ or encounters a change in A_t. Because A_t is non-decreasing, this implies that by passing from t_m to t_{m+1}, one either increases t by $\epsilon/(4c)$ or $|A_t|$ by at least 1. It follows that $t_m \to \infty$ as $m \to \infty$. To see this, note that either $t_m \to \infty$ or $|A_{t_m}| \to \infty$. Since $|A_t| < \infty$ for all t, we must have $t_m \to \infty$ in both cases. Obviously there exists $0 \leq k \leq m-1$ such that $t_m \geq k\epsilon/(4c)$ and $|A_{t_m+}| \geq |A_{t_m}| \geq |A_{t_m-}| \geq m-k-1$. By (4.5) this implies that

$$\liminf_m \frac{|A_{t_m-}|}{m^{\delta'}} > 0$$

for $\delta' = \delta \wedge 1$. It follows from (4.8) that for every $k = 1, 2\ldots$,

$$\mathbb{P}\left(\left|\frac{n_{t_m}^{\mathbf{Z}^d}(A_{t_m})}{|A_{t_m}|} - \mathbb{E}\xi_{t_m}^{\mathbf{Z}^d}(0)\right| \geq \epsilon\right) \leq C'_{k,\epsilon} m^{-\delta' k} \tag{4.12}$$

and the same is true with A_{t_m} replaced by A_{t_m-} or A_{t_m+}.

As $t_{m+1} - t_m \leq \epsilon/(4c)$ and $A_t = A_{t_m+}$ for $t_m < t < t_{m+1}$, (4.10) and (4.11) yield

$$\begin{aligned} \mathbb{P}\left(\sup_{t_m < t < t_{m+1}} \left|n_t^{\mathbf{Z}^d}(A_t) - n_{t_m}^{\mathbf{Z}^d}(A_{t_m+})\right| \geq \epsilon|A_{t_m+}|\right) &\leq e^{-\epsilon|A_{t_m+}|/2} \\ &\leq e^{-\epsilon C m^{\delta'}/2} \end{aligned} \tag{4.13}$$

for some $C > 0$ and $m > m_0$. By (4.12) with $k > 1/\delta'$, (4.13) and the Borel-Cantelli lemma we find

$$\frac{n_t^{\mathbf{Z}^d}(A_t)}{|A_t|} - \mathbb{E}\xi_t^{\mathbf{Z}^d}(0) \longrightarrow 0 \quad a.s.$$

and together with (4.9) this proves (4.6). □

Lemma 4.1 allows us to prove both the consistency and the strong consistency of $\widehat{\lambda}_t^{\{0\}}$ as $t \to \infty$.

Theorem 4.1 *Let $\widehat{\lambda}_t^{\{0\}}(C_t)$ be the estimator of λ for the process $\xi_t^{\{0\}}$ defined in (3.6)-(3.7) and Definition 3.1 . Then on the set where $\xi_t^{\{0\}}$ survives forever,*

$$\widehat{\lambda}_t^{\{0\}}(C_t) \xrightarrow{P} \lambda \quad as \quad t \to \infty \quad . \tag{4.14}$$

If, in addition (3.8) holds, then

$$\widehat{\lambda}_t^{\{0\}}(C_t) \longrightarrow \lambda \quad as \quad t \to \infty \tag{4.15}$$

a.s. on the set where $\xi_t^{\{0\}}$ survives forever.

Proof. We first prove (4.14). Both $n_t^{\{0\}}(A)$ and $k_t^{\{0\}}(A)$ are increasing in A and because $0 \in \overset{\circ}{U}$, Theorem 2.1, (3.2) and (3.6) ensure that

$$\frac{n_t^{\{0\}}([(1-\epsilon)tU]^-)}{k_t^{\{0\}}([(1+\epsilon)tU]^-)} \le \frac{n_t^{\{0\}}(C_t)}{k_t^{\{0\}}(C_t)} \le \frac{n_t^{\{0\}}([(1+\epsilon)tU]^-)}{k_t^{\{0\}}([(1-\epsilon)tU]^-)} \tag{4.16}$$

eventually a.s. on the set $\{\tau^{\{0\}} = \infty\}$.

In view of Theorem 1.2, (3.1) and the fact that $\xi_t^{\{0\}} \subset \xi_t^{\mathbf{Z}^d}$ in the graphical representation, we have eventually almost surely on the set where $\xi_t^{\{0\}}$ survives forever,

$$\frac{n_t^{\{0\}}(C_t)}{k_t^{\{0\}}(C_t)} \le \frac{n_t^{\{0\}}([(1+\epsilon)tU]^-)}{k_t^{\{0\}}([(1-\epsilon)tU]^-)} \le \frac{n_t^{\mathbf{Z}^d}([(1+\epsilon)tU]^-)}{k_t^{\mathbf{Z}^d}([(1-\epsilon)tU]^-)} \quad .$$

As $0 \in \overset{\circ}{U}$, (3.3) ensures that $|[(1+\epsilon)tU]^-|_D \to \infty$ and $|[(1-\epsilon)tU]^-|_D \to \infty$. Applying (4.3)-(4.4) we obtain

$$\frac{n_t^{\mathbf{Z}^d}([(1+\epsilon)tU]^-)}{|[(1+\epsilon)tU]^-|_D} \xrightarrow{P} \mathbb{E}\xi^\nu(0) \quad \text{as} \quad t \to \infty$$

and

$$\frac{k_t^{\mathbf{Z}^d}([(1-\epsilon)tU]^-)}{|[(1-\epsilon)tU]^-|_D} \xrightarrow{P} \mathbb{E}k^\nu(0) \quad \text{as} \quad t \to \infty \quad .$$

Since $\epsilon > 0$ can be chosen arbitrarily small and by (1.15) $\mathbb{E}\xi^\nu(0) \;/\; \mathbb{E}k_t^\nu(0) = \lambda$, (4.16) and (3.5) imply that for any $\delta > 0$ and $t \to \infty$,

$$\mathbb{P}(\widehat{\lambda}_t^{\{0\}} \ge \lambda(1+\delta)) \;,\; \xi_t^{\{0\}} \; survives forever) \to 0 \quad .$$

To deal with the left hand-side of (4.16), we note that by (3.2) and Theorem 1.2

$$\begin{aligned} k_t^{\{0\}}([(1+\epsilon)tU]^-) &= k_t^{\{0\}}([(1-\epsilon)tU]^-) + k_t^{\{0\}}([(1+\epsilon)tU]^-) - k_t^{\{0\}}([(1-\epsilon)tU]^-) \le \\ & \quad k_t^{\mathbf{Z}^d}([(1-\epsilon)tU]^-) + 2d\{|[(1+\epsilon)tU]^-|_D - |[(1-\epsilon)tU]^-|_D\}, \end{aligned}$$

eventually a.s. on $\{\tau^{\{0\}} = \infty\}$. Again applying (4.16), (4.3), (4.4) and (3.5) for sufficiently small $\epsilon > 0$, we find that for any $\delta > 0$ and $t \to \infty$,

$$\mathbb{P}(\widehat{\lambda}_t^{\{0\}} \leq \lambda(1-\delta)) \ , \ \xi_t^{\{0\}} \ survivesforever) \to 0.$$

This proves (4.14). The proof of (4.15) uses (4.6) and (4.7) instead of (4.3) and (4.4). This is allowed because (3.8) implies (4.5) for $A_t = [(1-\epsilon)tU]^- \cap \mathbb{Z}^d$ and $A_t = [(1+\epsilon)tU]^- \cap \mathbb{Z}^d$ since $0 \in \overset{\circ}{U}$. □

Remark 4.1 For the supercritical contact process, (4.14) may be written as

$$\mathbb{P}\{|\widehat{\lambda}_t^{\{0\}} - \lambda| \geq \epsilon \ \mid \ \tau^{\{0\}} = \infty) \to 0 \quad as \ t \to \infty \tag{4.17}$$

for every $\epsilon \geq 0$. From a statistical point of view this appears unsatisfactory since we shall never know whether the process will survive forever and hence whether $\widehat{\lambda}_t^{\{0\}}$ will probably be close to λ even for very large t. However, for the supercritical contact process (4.17) is obviously equivalent to

$$\mathbb{P}\{|\widehat{\lambda}_t^{\{0\}} - \lambda| \geq \epsilon \ \mid \ \xi_t^{\{0\}} \neq \emptyset\} \to 0 \tag{4.18}$$

for every $\epsilon > 0$, and this statement does have statistical relevance. Of course our result does not provide any information in the subcritical case ($\lambda \leq \lambda_d$).

5 The estimation problem: Asymptotic normality

This section is devoted to the proof of a conditional central limit theorem for the estimator $\widehat{\lambda}_t^{\{0\}} = \widehat{\lambda}_t^{\{0\}}(C_t)$ based on the random mask C_t. First we establish the joint asymptotic normality of

$$|A_t|^{-1/2}\Big(n_t^{\mathbb{Z}^d}(A_t) - |A_t|\mathbb{E}\xi^\nu(0), \ \ k_t^{\mathbb{Z}^d}(A_t) - |A_t|\mathbb{E}k^\nu(0)\Big)$$

for a non-random mask $A_t \subset \mathbb{Z}^d$ with $|A_t| < \infty$ for all $t \geq 0$, but $|A_t| \to \infty$ as $t \to \infty$. Next we show that this result carries over to the $\overline{\xi}_t^{\{0\}}$ process, i.e. the $\xi_t^{\{0\}}$ process conditioned on $\{\tau^{\{0\}} = \infty\}$. This proves the asymptotic normality of the estimator $\widehat{\lambda}_t^{\{0\}}(A_t)$ given $\{\tau^{\{0\}} = \infty\}$ for a non-random mask A_t. Then we show that the contribution to the standardized estimator which is due to the randomness of the mask $C_t = [\mathcal{C}(\xi_t^{\{0\}})]^-$ vanishes as $t \to \infty$. The asymptotic normality of

$$|C_t|^{-1/2}(\widehat{\lambda}_t^{\{0\}}(C_t) - \lambda)$$

given $\{\tau^{\{0\}} = \infty\}$ then follows.

A very general central limit theorem for a translation invariant random field was proved by Bolthausen (1982) under mixing conditions. Let $\zeta(x)$, $x \in \mathbb{Z}^d$, denote a real valued translation invariant random field, i.e. $\{\zeta(x) : x \in \mathbb{Z}^d\}$ is a collection of random variables and the joint law of the $\zeta(x)$ are invariant under integer valued shifts in $\mathbb{Z}^d$. It is assumed that $\mathbb{E}\zeta^2(x) < \infty$. For $x = (x_1, .., x_d)$, $y = (y_1, .., y_d) \in \mathbb{Z}^d$ define the L^∞-distance of x and y as

$$\rho(x, y) = \max_{1 \le i \le d} |x_i - y_i| \quad .$$

Let $A_n \in \mathbb{Z}^d$, $n = 1, 2...$, with $|A_n| < \infty$ for all n, $|A_n| \to \infty$ as $n \to \infty$ and

$$\frac{|\partial A_n|}{|A_n|} \to 0 \; as \; n \to \infty \; . \tag{5.1}$$

Here

$$\partial A_n = \{x \in A_n : \exists \;\; y \in \mathbb{Z}^d \setminus A_n \; with \; \rho(x, y) = 1\} \tag{5.2}$$

denotes the L^∞-contour of A_n in $\mathbb{Z}^d$. Consider

$$S_n = \sum_{x \in A_n} (\zeta(x) - \mathbb{E}\zeta(0)) \quad .$$

If $C \subset \mathbb{Z}^d$, let $\mathcal{B}_C$ be the sigma-algebra generated by $\{\zeta(x), x \in C\}$. For $C_1, C_2 \subset \mathbb{Z}^d$, let

$$\rho(C_1, C_2) = \inf\{\rho(x, y) : x \in C_1, \;\; y \in C_2\} \quad .$$

For $m \in \mathbb{N}, k, l \in \mathbb{N} \cup \{\infty\}$, define the mixing coefficients

$$\begin{aligned} \alpha_{k,l}(m) = \sup\{|\mathbb{P}(B_1 \cap B_2) - \mathbb{P}(B_1)\mathbb{P}(B_2)| : B_i \in \mathcal{B}_{C_i}, |C_1| \le k, \\ |C_2| \le l, \;\; \rho(C_1, C_2) \ge m\} \; . \end{aligned} \tag{5.3}$$

Let $N(0, \sigma^2)$ denote the univariate normal distribution with expectation μ and variance σ^2 and $N(\mu, \Sigma)$ the bivariate normal distribution with expectation vector μ and covariance matrix Σ. Part of Bolthausen's theorem reads as follows.

Lemma 5.1 *Suppose that, as $m \to \infty$,*

$$\sum_{m=1}^{\infty} m^{d-1} \alpha_{k,l}(m) < \infty \qquad for \quad k + l \le 4 \; , \tag{5.4}$$

$$\alpha_{1,\infty}(m) = o(m^{-d}) \; , \tag{5.5}$$

and that for some $\delta > 0$,

(5.6) $$\mathbb{E}|\zeta(x)|^{2+\delta} < \infty \quad \text{and} \quad \sum_{m=1}^{\infty} m^{d-1}\alpha_{1,1}(m)^{\delta/(2+\delta)} < \infty\ .$$

Then $\sum_{x\in\mathbb{Z}^d} |cov(\zeta(0),\zeta(x))| < \infty$. *If, in addition,* $\sigma^2 = \sum_{x\in\mathbb{Z}^d} cov(\zeta(0),\zeta(x)) > 0$ *and (5.1) holds, then* $|A_n|^{-1/2}S_n/\sigma$ *converges in distribution to* $N(0,1)$.

For our purpose we have to modify this result slightly. First of all, we allow a different stationary random field $\zeta_n(x)$ for each n so that S_n becomes

$$\tilde{S}_n = \sum_{x\in A_n} (\zeta_n(x) - \mathbb{E}\zeta_n(0))\ .$$

As a result, we also have to replace the assumptions of the lemma by versions which are uniform in n. This means that in the assumptions of the lemma we replace $\alpha_{k,l}(m)$ by the supremum over n of expression (5.3) for $\zeta_n(x)$. Similarly the integrability of $|\zeta_n(x)|^{2+\delta}$ in (5.3) is replaced by the uniform integrability of $|\zeta_n(x)|^{2+\delta}$. Then Bolthausen's proof goes through to show that $sup_n \sum_{x\in\mathbb{Z}^d} |cov(\zeta_n(0),\zeta_n(x))| < \infty$ and that $|A_n|^{-1/2}\tilde{S}_n/\sigma_n \xrightarrow{\mathcal{D}} N(0,1)$, provided that $liminf\, \sigma_n^2 > 0$, where $\sigma_n^2 = \sum_{x\in\mathbb{Z}^d} cov(\zeta_n(0),\zeta_n(x))$.

A second modification of Lemma 5.1 concerns assumption (5.5). It is clear from Bolthausen's proof that condition (5.5) may be replaced by

(5.7) $$\alpha_{1,l}(l^{1/(2d+1)}) = o(l^{-1/2}) \quad as\ l\to\infty\ .$$

With these modifications , Lemma 5.1 allows us to prove

Lemma 5.2 *For $t \geq 0$, let A_t be a finite subset of $\mathbb{Z}^d$ such that for each $\gamma > 0$,*

(5.8) $$|A_t| \to \infty \quad , \quad |\partial A_t|/|A_t| \to 0 \quad \text{and} \quad |A_t|e^{-\gamma t} \to 0 \quad as\ t\to\infty\ .$$

As $t\to\infty$ the random vector

(5.9) $$|A_t|^{-1/2}\Big(\sum_{x\in A_t}(\xi_t^{\mathbb{Z}^d}(x) - \mathbb{E}\xi^\nu(0)), \sum_{x\in A_t}(k_t^{\mathbb{Z}^d}(x) - \mathbb{E}k^\nu(0))\Big)$$

converges in distribution to $N(0,\Sigma)$ where

(5.10) $$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_2^2 \end{pmatrix}$$

and

$$\sigma_1^2 = \sum_{x\in \mathbf{Z}^d} cov(\xi^\nu(0), \xi^\nu(x)) \quad , \quad \sigma_2^2 = \sum_{x\in \mathbf{Z}^d} cov(k^\nu(0), k^\nu(x)) \quad ,$$

(5.11)

$$\sigma_{1,2} = \sum_{x\in \mathbf{Z}^d} cov(k^\nu(0), \xi^\nu(x)) \quad .$$

Proof. Let u and v be real numbers and define

$$\zeta_t(x) = u\xi_t^{\mathbf{Z}^d}(x) + vk_t^{\mathbf{Z}^d}(x) \ ,$$

Clearly $\{\zeta_t(x), x \in \mathbf{Z}^d\}$ is a real valued translation invariant random field for each t. Consider

$$\widetilde{S}_t = \sum_{x\in A_t} (\zeta_t(x) - \mathbb{E}\zeta_t(0)) \ .$$

The fact that $\widetilde{S}_t$ depends on a real valued index $t \to \infty$ instead of an integer $n \to \infty$ as in our version of Bolthausen's result, is of course immaterial in what follows. Note that $|\zeta_t(x) \le |u| + 4|v|$ so that all moments of $|\zeta_t(x)|$ are bounded independent of t.

Let us write $\alpha_{klt}(m)$ for the quantity defined in (5.3) computed for ζ_t. By Theorem 2.2 and because $\rho(x,y) \le d(x,y) = \sum_{i=1}^{d} |x_i - y_i|$, there exist positive C and γ such that

$$\alpha_{klt}(m) \le Ckle^{-\gamma m}$$

independent of t. This means that assumptions (5.4) , (5.6) and (5.7) are fulfilled uniformly in t. Note that (5.5) is not satisfied since we can not allow $l = \infty$, but as we have indicated, (5.7) serves just as well. Hence we have proved that

$$(5.12) \qquad |A_t|^{-1/2}\sigma_t^{-1} \sum_{x\in A_t} \Big(u(\xi_t^{\mathbf{Z}^d}(x) - \mathbb{E}\xi^{\mathbf{Z}^d}(0)) + v(k_t^{\mathbf{Z}^d}(x) - \mathbb{E}k_t^{\mathbf{Z}^d}(0)) \Big)$$

has a standard normal limit distribution provided that $\liminf \sigma_t^2 > 0$. Here

$$(5.13) \qquad \sigma_t^2 = \sum_{x\in \mathbf{Z}^d} cov\Big(u\xi_t^{\mathbf{Z}^d}(0) + vk_t^{\mathbf{Z}^d}(0), u\xi_t^{\mathbf{Z}^d}(x) + vk_t^{\mathbf{Z}^d}(x) \Big) \quad .$$

We also know that σ_t^2 is bounded for fixed u and v.

By (1.6) there exist positive C and γ such that

$$0 \le \mathbb{E}\xi_t^{\mathbf{Z}^d}(0) - \mathbb{E}\xi^\nu(0) \le Ce^{-\gamma t} \ ,$$

and hence also

$$|\mathbb{E}k_t^{\mathbf{Z}^d}(0) - \mathbb{E}k^\nu(0)| \le 4d(\mathbb{E}\xi_t^{\mathbf{Z}^d}(0) - \mathbb{E}\xi^\nu(0)) \le 4Cde^{-\gamma t} \quad .$$

As

$$|A_t|^{1/2}e^{-\gamma t} \to 0 \; as \; t \to \infty \quad ,$$

by (5.8), we see that in (5.12) we can replace $\mathbb{E}\xi_t^{\mathbb{Z}^d}(0)$ and $\mathbb{E}k_t^{\mathbb{Z}^d}(0)$ by $\mathbb{E}\xi^\nu(0)$ and $\mathbb{E}k^\nu(0)$ with impunity.

Again by Theorem 2.2 we find that for fixed u and v and all $x \in \mathbb{Z}^d$,

$$\left|cov\Big(u\xi_t^{\mathbb{Z}^d}(0) + vk_t^{\mathbb{Z}^d}(0), u\xi_t^{\mathbb{Z}^d}(x) + vk_t^{\mathbb{Z}^d}(x)\Big)\right| \le C' e^{-\gamma \sum_{i=1}^d |x_i|} \; .$$

for appropriate positive C' and γ. Because this bound is independent of t and as Theorem 1.1 and $\mathbb{P}(\tau^{\mathbb{Z}^d} = \infty) = 1$ imply that $\xi_t^{\mathbb{Z}^d} \xrightarrow{\mathcal{D}} \xi^\nu$ for $t \to \infty$, we also have

$$\left|cov\Big(u\xi^\nu(0) + vk^\nu(0), u\xi^\nu(x) + vk^\nu(x)\Big)\right| \le C' e^{-\gamma \sum_{i=1}^d |x_i|}$$

for all $x \in \mathbb{Z}^d$. But since $\xi_t^{\mathbb{Z}^d} \xrightarrow{\mathcal{D}} \xi^\nu$, this ensures that σ_t^2 in (5.13), converges for $t \to \infty$ to

$$\sigma^2 = \sum_{x \in \mathbb{Z}^d} cov\Big(u\xi^\nu(0) + vk^\nu(0), u\xi^\nu(x) + vk^\nu(x)\Big) \; . \tag{5.14}$$

Thus we have shown that for every fixed real u and v,

$$|A_t|^{-1/2} \sum_{x \in A_t} \Big(u(\xi_t^{\mathbb{Z}^d}(x) - \mathbb{E}\xi^\nu(0)) + v(k_t^{\mathbb{Z}^d}(x) - \mathbb{E}k^\nu(0))\Big)$$

converges in distribution to $N(0, \sigma^2)$ for $\sigma^2 > 0$. This remains true for $\sigma^2 = 0$ if we interpret $N(0,0)$ as the degenerate distribution at 0. In view of the Cramér-Wold device, this proves the lemma. □

Our next step is to show that the result of Lemma 5.2 continues to hold if the $\xi_t^{\mathbb{Z}^d}$ process is replaced by the conditional process $\bar{\xi}_t^{\{0\}} = (\xi_t^{\{0\}} | \tau^{\{0\}} = \infty)$.

Lemma 5.3 *Choose $\epsilon \in (0,1)$ and $A_t \subset \mathbb{Z}^d$ for $t \ge 0$ such that*

$$A_t \subset (1-\epsilon)tU \quad , |A_t| \to \infty \;\; and \;\; \frac{|\partial A_t|}{|A_t|} \to 0 \;\; as \;\; t \to \infty \; . \tag{5.15}$$

Then as $t \to \infty$, the conditional distribution of the vector

$$|A_t|^{-1/2}\Big[\sum_{x \in A_t} \Big(\xi_t^{\{0\}}(x) - \mathbb{E}\xi^\nu(0)\Big), \sum_{x \in A_t} \Big(k_t^{\{0\}}(x) - \mathbb{E}k^\nu(0)\Big)\Big] \tag{5.16}$$

given $\{\tau^{\{0\}} = \infty\}$, converges weakly to $N(0, \Sigma)$, with Σ given by (5.10)-(5.11).

Proof. For $0 < s < t$, define the random vectors

$$V_t = |A_t|^{-1/2}\Big[\sum_{x\in A_t}\Big(\xi_{t-s}^{\mathbb{Z}^d}(x) - \mathbb{E}\xi^\nu(0)\Big)\ ,\ \sum_{x\in A_t}\Big(k_{t-s}^{\mathbb{Z}^d}(x) - \mathbb{E}k^\nu(0)\Big)\Big]\ ,$$

$$W_t = |A_t|^{-1/2}\Big[\sum_{x\in A_t}\Big(\overline{\xi}_t^{\{0\}}(x) - \mathbb{E}\xi^\nu(0)\Big)\ ,\ \sum_{x\in A_t}\Big(\overline{k}_t^{\{0\}}(x) - \mathbb{E}k^\nu(0)\Big)\Big]\ .$$

As $A_t \subset (1-\epsilon)tU$, (2.9) of Theorem 2.4 ensures that for any $r > 0$ and $s \wedge (t-s) \geq A_\epsilon$, and for any Borel set $B \subset \mathbb{R}^2$,

$$\Big|\mathbb{P}(V_t \in B) - \mathbb{P}(W_t \in B)\Big| \leq A_{r,\epsilon}\Big(s^{-r} + \frac{s^d}{(t-s)^r}\Big)\ . \tag{5.17}$$

Fix $s > 0$ and let $t \to \infty$. As A_t satisfies assumption (5.15), $\tilde{A}_t = A_{s+t}$ satisfies assumption (5.8) and hence Lemma 5.2 implies that

$$V_t \xrightarrow{\mathcal{D}} N(0,\Sigma) \quad as\ t \to \infty\ , \tag{5.18}$$

with Σ as in (5.10)-(5.11). Obviously (5.17) yields for every $s \geq A_\epsilon$ and every Borel set $B \subset \mathbb{R}^2$,

$$\limsup_{t\to\infty} |\mathbb{P}(V_t \in B) - \mathbb{P}(W_t \in B)| \leq A_{r,\epsilon}s^{-r} \tag{5.19}$$

Since s can be taken arbitrarily large, we may combine (5.18) and (5.19) to find that

$$W_t \xrightarrow{\mathcal{D}} N(0,\Sigma) \quad as\ t \to \infty\ ,$$

which proves the lemma. □

For a non-random mask A_t satisfying (5.15), Lemma 5.3 yields the asymptotic normality of the estimator

$$\widehat{\lambda}_t^{\{0\}}(A_t) = \frac{n_t^{\{0\}}(A_t)}{k_t^{\{0\}}(A_t)}$$

conditioned on $\{\tau^{\{0\}} = \infty\}$. By (1.15) we have

$$|A_t|^{1/2}(\widehat{\lambda}_t^{\{0\}}(A_t) - \lambda) = |A_t|^{1/2}\Big(\frac{n_t^{\{0\}}(A_t)}{k_t^{\{0\}}(A_t)} - \frac{\mathbb{E}\xi^\nu(0)}{\mathbb{E}k(0)}\Big)$$

$$= \lambda|A_t|^{1/2}\Bigg[\frac{1 + \sum_{x\in A_t}\Big(\xi_t^{\{0\}}(x) - \mathbb{E}\xi^\nu(0)\Big)/\{|A_t|\mathbb{E}\xi^\nu(0)\}}{1 + \sum_{x\in A_t}\Big(k_t^{\{0\}}(x) - \mathbb{E}k^\nu(0)\Big)/\{|A_t|\mathbb{E}k^\nu(0)\}} - 1\Bigg]\ .$$

Hence Lemma 5.3 implies that

$$|A_t|^{1/2}(\widehat{\lambda}_t^{\{0\}}(A_t) - \lambda) \;=\; \lambda|A_t|^{-1/2}\left[\frac{\sum_{x\in A_t}\Big(\xi_t^{\{0\}}(x) - \mathbb{E}\xi^\nu(0)\Big)}{\mathbb{E}\xi^\nu(0)} - \frac{\sum_{x\in A_t}\Big(k_t^{\{0\}}(x) - \mathbb{E}k^\nu(0)\Big)}{\mathbb{E}k^\nu(0)}\right]$$
$$+\; \mathcal{O}_{\overline{\mathbb{P}}}(|A_t|^{-1/2})$$

where $\overline{\mathbb{P}}$ denotes the conditional probability given $\{\tau^{\{0\}} = \infty\}$. Another application of Lemma 5.3 establishes the following result.

Theorem 5.1 *Under the assumptions of Lemma 5.3,*

$$|A_t|^{1/2}(\widehat{\lambda}_t^{\{0\}}(A_t) - \lambda) \xrightarrow{\mathcal{D}} N(0, \sigma^2) \tag{5.20}$$

conditional on $\{\tau^{\{0\}} = \infty\}$. *Here*

$$\sigma^2 = \lambda^2\left[\frac{\sigma_1^2}{(\mathbb{E}\xi^\nu(0))^2} + \frac{\sigma_2^2}{(\mathbb{E}k^\nu(0))^2} - \frac{2\sigma_{1,2}}{\mathbb{E}\xi^\nu(0)\mathbb{E}k^\nu(0)}\right] \;, \tag{5.21}$$

where σ_1^2 , σ_2^2 *and* $\sigma_{1,2}$ *are given by (5.11).*

To complete the proof of the asymptotic normality of $\widehat{\lambda}_t^{\{0\}}$ based on the random mask $C_t = [\mathcal{C}(\xi_t^{\{0\}})]^-$ we have to show that the difference between the estimator computed on the random mask and the one based on a non-random window is negligible as $t \to \infty$. This means we still have to prove

$$|A_t|_D^{1/2}(\widehat{\lambda}_t^{\{0\}}(C_t) - \widehat{\lambda}_t^{\{0\}}(A_t)) \xrightarrow{\overline{P}} 0$$

where A_t is a non-random mask which is close to C_t and $\overline{P}(.) = P(.|\tau^0 = \infty)$. For $\epsilon > 0$ and $t > 0$ define

$$A_t = [(1-\epsilon)tU]^- \cap \mathbb{Z}^d \;, \qquad B_t = [(1+\epsilon)tU]^- \cap \mathbb{Z}^d \;, \tag{5.22}$$

i.e. A_t and B_t consist of the sites in shrunken versions of the sets $(1-\epsilon)tU$ and $(1+\epsilon)tU$ respectively, where the shrinking operation is defined in Definition 3.1.

Lemma 5.4 *For* $\epsilon \in (0,1)$ *define* A_t *and* B_t *as in (5.22) and let* $D_t = (B_t \setminus A_t) \cap C_t$, *with* $C_t = [\mathcal{C}(\xi_t^{\{0\}})]^-$ *as given by (3.6) and Definition 3.1. If the shrinking operation* $V \to V^-$ *satisfies (3.9) for some* $\delta \in (0,1)$, *then for every* $z > 0$

$$\lim_{\epsilon\to 0}\limsup_{t\to\infty} \mathbb{P}\Big(|A_t|^{-1/2}\Big|\sum_{x\in D_t}\Big(\xi_t^{\{0\}}(x) - \mathbb{E}\xi^\nu(0)\Big)\Big| \geq z \mid \tau^{\{0\}} = \infty\Big) = 0 \quad , \tag{5.23}$$

$$\lim_{\epsilon\to 0}\limsup_{t\to\infty} \mathbb{P}\Big(|A_t|^{-1/2}\Big|\sum_{x\in D_t}\Big(k_t^{\{0\}}(x) - \mathbb{E}k^\nu(0)\Big)\Big| \geq z \mid \tau^{\{0\}} = \infty\Big) = 0 \quad . \tag{5.24}$$

Proof. We shall only prove (5.23), the proof of (5.24) is almost the same. As before we write $\bar\xi_t^{\{0\}}$ for the conditional process $(\xi_t^{\{0\}}|\tau^{\{0\}}=\infty)$ and $\overline{\mathbb{P}}$ for the conditional probability $\mathbb{P}(\cdot|\,\tau^{\{0\}}=\infty)$.

Without loss of generality we assume that $\epsilon\le\delta/4$ so that $(1-\delta)(1+\epsilon)\le 1-3\delta/4$ and by (3.9),

$$B_t=[(1+\epsilon)tU]^-\cap\mathbb{Z}^d\subset(1-\delta)(1+\epsilon)tU\subset(1-\frac{3\delta}{4})tU\quad. \tag{5.25}$$

It follows from (2.10) of Theorem 2.4 with $s=t/2$ and $r=2d$ that for the coupling employed there,

$$\mathbb{P}(\bar\xi_t^{\{0\}}(x)=\xi_{t/2}^{\mathbb{Z}^d}(x))\ge 1-b'_\delta t^{-2d}$$

for all $t\ge b''_\delta$, all $x\in B_t$ and appropriate constants b'_δ and b''_δ. As $|\xi_t|\le 1$, this implies that

$$\left|\mathbb{E}\bar\xi_t^{\{0\}}(x)-\mathbb{E}\xi_{t/2}^{\mathbb{Z}^d}(x)\right|\le b'_\delta t^{-2d}$$

for $t\ge b''_\delta$ and $x\in B_t$. By (1.6),

$$\left|\mathbb{E}\xi_{t/2}^{\mathbb{Z}^d}(x)-\mathbb{E}\xi^\nu(0)\right|\le Ce^{-\gamma t/2}$$

and as $|D_t|\le|B_t|\le|tU|_D=\mathcal{O}(t^d)$ by (5.25), we find that

$$\lim_{t\to\infty}\sum_{x\in D_t}\left|\mathbb{E}\bar\xi_t^{\{0\}}(x)-\mathbb{E}\xi^\nu(0)\right|=0\quad.$$

As $|A_t|=|[(1-\epsilon)tU]^-|_D\to\infty$ for $t\to\infty$ by (3.3), we see that in order to prove (5.23), it is enough to show that

$$\lim_{\epsilon\to 0}\limsup_{t\to\infty}\mathbb{P}\Big(|A_t|^{-1/2}\Big|\sum_{x\in D_t}\Big(\bar\xi_t^{\{0\}}(x)-\mathbb{E}\bar\xi_t^{\{0\}}(x)\Big)\Big|\ge z\Big)=0\quad. \tag{5.26}$$

Define

$$C_t^*=\Big[\mathcal{C}\Big(\{\bar\xi_t^{\{0\}}\cup(1-\epsilon)tU\}\cap(1+\epsilon)tU\Big)\Big]^-\quad. \tag{5.27}$$

By (2.1), $(1-\epsilon)tU\subset\mathcal{C}(\bar\xi_t^{\{0\}})\subset(1+\epsilon)tU$ and hence

$$\begin{aligned}\mathcal{C}\Big(\{\bar\xi_t^{\{0\}}\cup(1-\epsilon)tU\}\cap(1+\epsilon)tU\Big)&=\mathcal{C}\Big(\bar\xi_t^{\{0\}}\cup(1-\epsilon)tU\Big)\\&=\mathcal{C}\Big(\mathcal{C}(\bar\xi_t^{\{0\}})\cup(1-\epsilon)tU\Big)=\mathcal{C}(\bar\xi_t^{\{0\}})\end{aligned}$$

eventually $\overline{\mathbb{P}}$-almost surely . It follows that

$$C_t^* = [\mathcal{C}(\overline{\xi}_t^{\{0\}})]^- = C_t \tag{5.28}$$

eventually a.s. ($\overline{\mathbb{P}}$). Obviously this implies that

$$\begin{aligned}\sum_{x\in D_t}\Big(\overline{\xi}_t^{\{0\}}(x) - \mathbb{E}\overline{\xi}_t^{\{0\}}(x)\Big) &= \sum_{x\in B_t\setminus A_t}\Big(\overline{\xi}_t^{\{0\}}(x) - \mathbb{E}\overline{\xi}_t^{\{0\}}(x)\Big)I_{C_t}(x)\\ &= \sum_{x\in B_t\setminus A_t}\Big(\overline{\xi}_t^{\{0\}}(x) - \mathbb{E}\overline{\xi}_t^{\{0\}}(x)\Big)I_{C_t^*}(x)\end{aligned}$$

eventually a.s. ($\overline{\mathbb{P}}$). Instead of (5.26) it is therefore sufficient to show that

$$\lim_{\epsilon\to 0}\limsup_{t\to\infty}\mathbb{P}\Big(|A_t|^{-1/2}\Big|\sum_{x\in B_t\setminus A_t}\Big(\overline{\xi}_t^{\{0\}}(x) - \mathbb{E}\overline{\xi}_t^{\{0\}}(x)\Big)I_{C_t^*}(x)\Big| \geq z\Big) = 0 \quad .$$

Clearly this will follow if we prove that

$$\lim_{\epsilon\to 0}\limsup_{t\to\infty}|A_t|^{-1}\mathbb{E}\Big[\sum_{x\in B_t\setminus A_t}\Big(\overline{\xi}_t^{\{0\}}(x) - \mathbb{E}\overline{\xi}_t^{\{0\}}(x)\Big)I_{C_t^*}(x)\Big]^2 = 0 \quad . \tag{5.29}$$

By (5.27) the random set C_t^* is determined by the random set $\{\overline{\xi}_t^{\{0\}} \cup (1-\epsilon)tU\} \cap (1+\epsilon)tU$ which is bracketed by the non-random convex sets $(1-\epsilon)tU$ and $(1+\epsilon)tU$. It follows that C_t^* is determined by the values of $\overline{\xi}_t^{\{0\}}(y)$ for sites $y \in (1+\epsilon)tU \setminus (1-\epsilon)tU$. Put differently, for every $x \in \mathbb{Z}^d$ the function $g_x : H \to \{0,1\}$ defined by

$$g_x(\overline{\xi}_t^{\{0\}}) = I_{C_t^*}(x) \tag{5.30}$$

satisfies

$$g_x \in D_R \quad \textit{with} \quad R = \{(1+\epsilon)tU \setminus (1-\epsilon)tU\} \cap \mathbb{Z}^d \quad , \tag{5.31}$$

and D_R defined by (2.3).

The expected value in (5.29) can be written as

$$\begin{aligned}&\mathbb{E}\Big[\sum_{x\in B_t\setminus A_t}\Big(\overline{\xi}_t^{\{0\}}(x) - \mathbb{E}\overline{\xi}_t^{\{0\}}(x)\Big)I_{C_t^*}(x)\Big]^2\\ &= \sum_{x,x'\in B_t\setminus A_t}\mathbb{E}\Big(\overline{\xi}_t^{\{0\}}(x) - \mathbb{E}\overline{\xi}_t^{\{0\}}(x)\Big)\Big(\overline{\xi}_t^{\{0\}}(x') - \mathbb{E}\overline{\xi}_t^{\{0\}}(x')\Big)I_{C_t^*}(x)I_{C_t^*}(x')\\ &= \sum_{x,x'\in B_t\setminus A_t}\mathbb{E}f_x(\overline{\xi}_t^{\{0\}})f_{x'}(\overline{\xi}_t^{\{0\}})g_x(\overline{\xi}_t^{\{0\}})g_{x'}(\overline{\xi}_t^{\{0\}})\end{aligned} \tag{5.32}$$

with $f_x(\overline{\xi}_t^{\{0\}}) = \overline{\xi}_t^{\{0\}}(x) - \mathbb{E}\overline{\xi}_t^{\{0\}}(x)$ and g_x defined by (5.30). Obviously

$$f_x.f_{x'} \in D_{\{x,x'\}} \quad , \quad g_x.g_{x'} \in D_R \tag{5.33}$$

in view of (5.31). If $x, x' \in B_t \setminus A_t$, then (5.25) ensures that $\{x, x'\} \subset (1 - 3\delta/4)tU$ and because $\epsilon \leq \delta/4$, (5.31) implies that $R \subset \{(1 - \delta/4)tU\}^c$. Hence, if $d(.,.)$ denotes L^1-distance, then

$$d(\{x, x'\}, R) \geq b_\delta'''t \quad \textit{for all} \quad x, x' \in B_t \setminus A_t \quad , \tag{5.34}$$

where b_δ''' is a positive number depending only on δ. Finally we use (2.2) to compute

$$|\!|\!| f_x . f_x' |\!|\!| = 2 \quad , \quad |\!|\!| g_x . g_x' |\!|\!| \leq |R| \leq a\epsilon t^d \leq a\delta t^d \tag{5.35}$$

for an appropriate constant $a > 0$. Combining (5.32)-(5.35) and invoking Theorem 2.5 with $r = 3d$ we obtain

$$\begin{aligned}
&|A_t|^{-1}\, \mathbb{E}\Big[\sum_{x \in B_t \setminus A_t} \Big(\overline{\xi}_t^{\{0\}}(x) - \mathbb{E}\overline{\xi}_t^{\{0\}}(x)\Big) I_{C_t^*}(x)\Big]^2 \\
&\leq |A_t|^{-1} \sum_{x,x' \in B_t \setminus A_t} \mathbb{E} f_x(\overline{\xi}_t^{\{0\}}) f_{x'}(\overline{\xi}_t^{\{0\}})\ \mathbb{E} g_x(\overline{\xi}_t^{\{0\}}) g_{x'}(\overline{\xi}_t^{\{0\}}) + M_t \\
&\leq |A_t|^{-1} \sum_{x,x' \in B_t \setminus A_t} \Big| cov\Big(\overline{\xi}_t^{\{0\}}(x), \overline{\xi}_t^{\{0\}}(x')\Big)\Big| + M_t \quad ,
\end{aligned}$$

where the remainder term M_t satisfies, for appropriate positive c_δ and c_δ',

$$\begin{aligned}
|M_t| &\leq |A_t|^{-1}\, |B_t \setminus A_t|^2\, c_\delta |\!|\!| f_x . f_x' |\!|\!| \cdot |\!|\!| g_x . g_x' |\!|\!| t^{-3d} \\
&\leq c_\delta' |A_t|^{-1} \to 0 \quad as \quad t \to \infty \quad ,
\end{aligned}$$

since $|B_t \setminus A_t| \leq |B_t| \leq |(1+\epsilon)tU|_D \leq |(1 + \delta/4)tU|_D = \mathcal{O}(t^d)$ and $|A_t| \to \infty$ by (3.3).

To prove (5.29) it therefore remains to be shown that

$$\lim_{\epsilon \to 0} \limsup_{t \to \infty} |A_t|^{-1} \sum_{x,x' \in B_t \setminus A_t} \Big| cov\Big(\overline{\xi}_t^{\{0\}}(x), \overline{\xi}_t^{\{0\}}(x')\Big)\Big| = 0 \quad . \tag{5.36}$$

Invoking Theorem 2.5 once more, this time with $r = d + 1$, we find that for $x, x' \in B_t \setminus A_t$, $x \neq x'$, and appropriate $c_\delta'' > 0$,

$$\Big| cov\Big(\overline{\xi}_t^{\{0\}}(x), \overline{\xi}_t^{\{0\}}(x')\Big)\Big| \leq c_\delta'' |x - x'|^{-(d+1)} \quad ,$$

since $x, x' \in B_t \setminus A_t$ implies $|x - x'| = \mathcal{O}(t)$. It follows that

$$\begin{aligned}
\sum_{x,x' \in B_t \setminus A_t} \Big| cov\Big(\overline{\xi}_t^{\{0\}}(x), \overline{\xi}_t^{\{0\}}(x')\Big)\Big| &\leq |B_t \setminus A_t| \Big(1 + c_\delta'' \sum_{x \in \mathbf{Z}^d, x \neq 0} |x|^{-(d+1)}\Big) \\
&\leq c_\delta'''\, |B_t \setminus A_t|
\end{aligned}$$

for some $c_\delta''' > 0$, as $\sum_{x \in \mathbf{Z}^d, x \neq 0} |x|^{-(d+1)}$ converges. Hence (5.36) holds if

$$\lim_{\epsilon \to 0} \limsup_{t \to \infty} \frac{|B_t \setminus A_t|}{|A_t|} = 0 \quad .$$

But since $A_t = [(1-\epsilon)tU]^- \cap \mathbb{Z}^d$ and $B_t = [(1+\epsilon)tU]^- \cap \mathbb{Z}^d$, this is a consequence of (3.5). This proves (5.36) and the lemma. $\square$

Before proving the central limit theorem for the estimator $\widehat{\lambda}_t^{\{0\}}(C_t)$ based on the random mask C_t, we have to verify that $|\partial A_t|/|A_t| \to 0$ for A_t defined in (5.22).

Lemma 5.5 *For $t \geq 0$, let W_t be bounded convex sets in $\mathbb{R}^d$ with $W_t \to \mathbb{R}^d$ as $t \to \infty$. Then*

$$\frac{|\partial(W_t \cap \mathbb{Z}^d)|}{|W_t \cap \mathbb{Z}^d|} \to 0 \quad as \quad t \to \infty \quad . \tag{5.37}$$

Proof. Let W be a bounded convex set in $\mathbb{R}^d$ with non-empty interior $\mathring{W}$. Let $V_d(W)$ and $A(W)$ denote the volume and the surface area of W, by which we mean the $d-$ and $(d-1)$-dimensional Lebesgue measure of W and its boundary respectively. As W is bounded and $\mathring{W} \neq \emptyset$, W contains a largest d-dimensional open hypersphere S with radius $r > 0$. We shall show that

$$\frac{A(W)}{V_d(W)} \leq \frac{d}{r} \quad . \tag{5.38}$$

Without loss of generality we shall assume that the center of S is located at the origin so that $S = \{x \in \mathbb{R}^d : \|x\|_2 < r\}$ where $\|x\|_2 = (\sum x_i^2)^{1/2}$ denotes the Euclidean norm.

The set W can be approximated by a convex polytope in such a way that both the volume and the surface area of the polytope are arbitrarily close to those of W (cf. Eggleston (1958), Theorem 33 and property (a) on page 88). We may therefore assume that W is a convex polytope. By Theorem 37 in Eggleston (1958) we have

$$V_d(W) = \frac{1}{d} \sum_{i=1}^{k} h_i V_{d-1}(F_i) \quad ,$$

where $F_1, ..., F_k$ are the faces of convex polytope W and h_i is the Euclidean distance of the origin to the $(d-1)-$dimensional hyperplane containing F_i. As $S \subset W$, we have $h_i \geq r$ for $i = 1, ..., k$. Hence

$$V_d(W) \geq \frac{r}{d} \sum_{i=1}^{k} V_{d-1}(F_i) = \frac{r}{d} A(W)$$

which establishes (5.38) for bounded convex W with $\mathring{W} \neq \emptyset$.

As $W_t \nearrow \mathbb{R}^d$, the convexity of W_t implies that W_t will contain a hypersphere with arbitrarily large radius r for sufficiently large t. Combining this with (5.38) we find that

$$\frac{A(W_t)}{V_d(W_t)} \to 0 \qquad as \quad t \to \infty \quad . \tag{5.39}$$

According to (5.2) there are disjoint unit hypercubes within a distance $2\sqrt{d}$ from the boundary of W centered at the sites in $\partial(W_t \cap \mathbb{Z}^d)$. Hence $|\partial(W_t \cap \mathbb{Z}^d)|/A(W_t)$ remains bounded as $t \to \infty$ and as $|W_t \cap \mathbb{Z}^d| \sim V_d(W_t)$, (5.39) yields (5.37).

We are now in a position to prove the main result of this paper.

Theorem 5.2 *Let $\widehat{\lambda}_t^{\{0\}}(C_t)$ be the estimator of λ for the process $\xi_t^{\{0\}}$ defined in (3.6)-(3.7) and Definition 3.1. If the shrinking operation $V \to V^-$ satisfies (3.9) for some $\delta \in (0,1)$ as well as (3.10), then as $t \to \infty$, the conditional distribution of*

$$|C_t|_D^{1/2}[\widehat{\lambda}_t^{\{0\}}(C_t) - \lambda] \tag{5.40}$$

given that $\{\tau^{\{0\}} = \infty\}$, converges weakly to $N(0,\sigma^2)$. Here σ^2 is given by (5.21) and (5.11).

Proof. In the proof we write $\overline{\xi}_t^{\{0\}}$ for the conditional process $(\xi_t^{\{0\}} | \tau^{\{0\}} = \infty)$. For $t \geq 0$, define

$$A_t = [(1-\epsilon)tU]^- \cap \mathbb{Z}^d \quad .$$

Since $0 \in \overset{\circ}{U}$, we have $A_t \subset (1-\epsilon)tU$ by (3.1) and $[(1-\epsilon)tU]^- \to \mathbb{R}^d$ by (3.10). Because $[(1-\epsilon)tU]^-$ is bounded and convex it follows that $|\partial A_t|/|A_t| \to 0$ as $t \to \infty$ by Lemma 5.5. Hence A_t satisfies condition (5.15) of Lemma 5.3 and we find that for every $\epsilon \in (0,1)$,

$$\begin{aligned} &\mathbb{P}\Big(|A_t|^{-1/2}\Big[\sum_{x\in A_t}\Big(\overline{\xi}_t^{\{0\}}(x) - \mathbb{E}\xi^\nu(0)\Big)\Big] \leq u,\ |A_t|^{-1/2}\Big[\sum_{x\in A_t}\Big(\overline{k}_t^{\{0\}}(x) - \mathbb{E}k^\nu(0)\Big)\Big] \leq v\Big) \\ &\longrightarrow \Phi(u,v) \quad as \quad t \to \infty \quad , \end{aligned} \tag{5.41}$$

where Φ denotes the two dimensional $N(0,\Sigma)$ distribution with Σ given by (5.10)-(5.11).

In view of (3.9) we may apply Lemma 5.4 to obtain

$$\begin{aligned} \Psi_1(\epsilon) &= \limsup_{t\to\infty} \mathbb{P}\Big(|A_t|^{-1/2}\Big|\sum_{x\in D_t}\Big(\overline{\xi}_t^{\{0\}}(x) - \mathbb{E}\xi^\nu(0)\Big)\Big| \geq z\Big) \to 0 \ , \\ \Psi_2(\epsilon) &= \limsup_{t\to\infty} \mathbb{P}\Big(|A_t|^{-1/2}\Big|\sum_{x\in D_t}\Big(\overline{k}_t^{\{0\}}(x) - \mathbb{E}k^\nu(0)\Big)\Big| \geq z\Big) \to 0 \end{aligned} \tag{5.42}$$

as $\epsilon \to 0$ for every $z > 0$. Here $D_t = (B_t \setminus A_t) \cap C_t$ with $B_t = [(1+\epsilon)tU]^- \cap \mathbb{Z}^d$. Notice that by (2.1) and (3.2) we have $A_t \subset C_t \cap \mathbb{Z}^d \subset B_t$ and hence

$$C_t \cap \mathbb{Z}^d = A_t \cup D_t \quad , \quad A_t \cap D_t = \emptyset \tag{5.43}$$

eventually a.s. on $\{\tau^{\{0\}} = \infty\}$.

Let us write

$$
(5.44) \qquad \begin{aligned} X_t &= \sum_{x \in C_t} \left(\bar{\xi}_t^{\{0\}}(x) - \mathbb{E}\xi^\nu(0) \right) , \\ Y_t &= \sum_{x \in C_t} \left(\bar{k}_t^{\{0\}}(x) - \mathbb{E}k^\nu(0) \right) . \end{aligned}
$$

By a standard argument (5.41)-(5.43) yields

$$
(5.45) \qquad \begin{aligned} &\Phi(u - z, v - z) - \Psi_1(\epsilon) - \Psi_2(\epsilon) \\ &\leq \liminf_{t \to \infty} \mathbb{P}(|A_t|^{-1/2} X_t \leq u, \ |A_t|^{-1/2} Y_t \leq v) \\ &\leq \limsup_{t \to \infty} \mathbb{P}(|A_t|^{-1/2} X_t \leq u, \ |A_t|^{-1/2} Y_t \leq v) \\ &\leq \Phi(u + z, v + z) + \Psi_1(\epsilon) + \Psi_2(\epsilon) \ . \end{aligned}
$$

Having sent $t \to \infty$ for fixed $\epsilon \in (0, 1)$ and $z > 0$, we now let $\epsilon \to 0$ for fixed $z > 0$. By (5.42) this removes $\Psi_1(\epsilon)$ and $\Psi_2(\epsilon)$ in (5.45), but it also allows us to replace $|A_t|$ by $|C_t|_D$ in (5.45). This follows because $A_t \subset C_t \cap \mathbb{Z}^d \subset B_t$ eventually a.s. on $\{\tau^{\{0\}} = \infty\}$ by (2.1) and (3.2), and $|B_t|/|A_t| \to 1$ as $t \to \infty$ and $\epsilon \to 0$ by (3.5). Finally we let z tend to zero to obtain

$$
\lim_{t \to \infty} \mathbb{P}(|C_t|_D^{-1/2} X_t \leq u, \ \ |C_t|_D^{-1/2} Y_t \leq v) = \Phi(u, v) \ .
$$

The theorem now follows in the same way as Theorem 5.1 follows from Lemma 5.3. □

6 The asymptotic variance of $\hat{\lambda}_t^{\{0\}}(C_t)$

If the variance σ^2 of the normal limit distribution in Theorem 5.2 were known, then this would allow us to assess the accuracy of the estimator or to set up asymptotic confidence intervals for λ of the form

$$
(6.46) \qquad \hat{\lambda}_t^{\{0\}}(C_t) - u_\alpha |C_t|_D^{-1/2} \sigma < \lambda < \hat{\lambda}_t^{\{0\}}(C_t) + u_\alpha |C_t|_D^{-1/2} \sigma \quad ,
$$

where u_α is the upper α-point of the standard normal distribution. This asymptotic confidence interval would be valid provided that $\xi_t^{\{0\}}$ survives forever but as we pointed out in Remark 4.1, it is enough that $\xi_t^{\{0\}} \neq \emptyset$, i.e. that the process has survived up to time t.

Since σ^2 is unknown we have to find an estimator of σ^2. One way to achieve this would be to estimate $\sigma^2 = \sigma^2(\lambda)$ as a function of λ by simulating $\xi_t^{\{0\}}$ a large number of times for each λ, each time computing the value of $\widehat{\lambda}_t^{\{0\}}(C_t)$ and using $|C_t|_D$ times the sample variance of these values as an estimate of $\sigma^2(\lambda)$. One could then use $\sigma^2(\widehat{\lambda}_t^{\{0\}}(C_t))$ as an estimate of σ^2. Of course in any particular instance it would be enough to carry out these simulations only for $\lambda = \widehat{\lambda}_t^{\{0\}}(C_t)$.

An alternative way to estimate σ^2 would be to use the observed process $\xi_t^{\{0\}}$ itself. First we subdivide the mask C_t into k subsets $C_{t,1}, ..., C_{t,k}$ of (approximately) equal size and compute the values $\widehat{\lambda}_t^{\{0\}}(C_{t,i})$ for $i = 1, ..., k$. We then use $k^{-1}|C_t|_D$ times the sample variance of these values as an estimate of σ^2.

An obvious advantage of the second method is that it is not as dependent on the model as the first. It is quite conceivable that the estimator $\widehat{\lambda}_t^{\{0\}}(C_t)$ is a useful statistic in a much broader class of models than the contact process. In this case the second method is more likely to produce a sensible result than the first.

7 Simulation results

In this section we shall present some simulation results for the supercritical contact process $\xi_t^{\{0\}}$ on the lattice $\mathbb{Z}^2$ starting at time $t = 0$ with a single infected site at the origin. We have performed a large number of runs of the process $\xi_t^{\{0\}}$ for different values of the parameter λ in the interval [0.42,4]. The value 0.42 is taken close to the critical value λ_2 in dimension 2. Simulation suggests that $\lambda_2 \sim 0.41$ (cf. Brower, Furman & Moshe (1978) and Grasseberger & de La Torre (1979)) .

The simulation procedure does not take time into account but computes $\xi_t^{\{0\}}$ at every time point $0 < t_1 < t_2 < ... < t_N$ when a change occurs. This means that given the configuration $\xi_{t_n}^{\{0\}}$ at time t_n of the n-th step, a 0 at site x is replaced by a 1 with probability $\lambda k_{t_n}^{\{0\}}(x) \;/\; (\lambda k_{t_n}^{\{0\}}(\mathbb{Z}^2) + n_{t_n}^{\{0\}}(\mathbb{Z}^2))$ and a 1 by a 0 with probability ${\xi_{t_n}}^{\{0\}}(x) \;/\; (\lambda k_{t_n}^{\{0\}}(\mathbb{Z}^2) + n_{t_n}^{\{0\}}(\mathbb{Z}^2))$ at time t_{n+1}. Here $n_{t_n}^{\{0\}}(\mathbb{Z}^2)$, $k_{t_n}^{\{0\}}(\mathbb{Z}^2)$ and $k_{t_n}^{\{0\}}(x)$ have the same meaning as in (1.12)–(1.14) and the remark following this, with the index t replaced by t_n. The simulation run is stopped after N steps at time t_N.

Our first goal in running these simulations is to show how the convergence of $\xi_t^{\{0\}}$ takes place. As we have seen the process should spread like a "blob in equilibrium". This means that after some time the process $\xi_t^{\{0\}}$ has settled down to the stationary limiting process in the middle of the blob of infected sites. Near the boundary of this set the process has not yet reached the equilibrium distribution and the infected sites are less dense than in the middle. In Figures 1 and 2 we show the process for $\lambda = 0.5$ and $N = 40,000$ and for $\lambda = 3$ and $N = 30,000$. Infected sites are indicated by colored 1 x 1 squares. An additional feature of these two figures is that for each infected site we have kept track of the number of steps since it was last infected and have indicated this by the tone of the color: the darker the color, the older the present infection at a site. If we view the contact process as a simplified model for the growth of a forest (a tree is present at an infected site and absent at a healthy site) then the tone of the color indicates the age of the tree.

Next we present some results for the estimate $\widehat{\lambda}_{t_n}^{\{0\}}(C_{t_n})$ of λ where we apply peeling as a shrinking operation, thus

$$C_{t_n} = peeling(\ \mathcal{C}(\xi_{t_n}^{\{0\}})) \quad .$$

Figure 3 shows how C_t is obtained from $\xi_t^{\{0\}}$.

Define the peeling fraction α as

$$\alpha = 1 - \frac{|C_{t_n}|_D}{|\mathcal{C}(\xi_{t_n}^{\{0\}})|_D} \quad .$$

Let $C_{\{t_n,\alpha\}}$ denote the mask obtained from the convex hull $\mathcal{C}(\xi_{t_n}^{\{0\}})$, with peeling fraction α. For fixed λ we have computed the values of $\widehat{\lambda}_{t_n}^{\{0\}}(C_{\{t_n,\alpha\}})$ for increasing n and α, averaged over 20 simulation runs with $N = 40,000$. The results for $\lambda = 0.42; 1; 1.5$ and 2 are shown in Tables 1–7.4.

As Table 1 shows, the estimates obtained for $\lambda = 0.42$ indicate rather large fluctuations for varying α and fixed N, especially for those processes with $N \leq 15,000$. Based on these experimental results it seems that a fraction α between 0.5 and 0.7 gives the best estimates of λ. Both these fluctuations and the rather large values of α are hardly surprising in view of the scattered character of the set $\xi_{t_N}^{\{0\}}$ for such small values of λ (cf. Figure 1).

Tables 2–7.4 show that the estimates at the beginning of the shrinking procedure are monotone increasing and then there is an interval where the values of $\widehat{\lambda}_{t_N}^{\{0\}}$ oscillate around the true parameter. Small values of α yield negative bias because near the boundary of the convex hull infected points are less dense than in the middle of the "blob". This produces large values of $k_{t_n}^{\{0\}}(C_{\{t_n,\alpha\}})$ relative to $n_{t_n}^{\{0\}}(C_{\{t_n,\alpha\}})$. Simulations suggest the interval $0.2 \leq \alpha \leq 0.4$ for the fraction of lattice points that should be removed, with 0.3 as reasonable compromise. Moreover, the optimal fraction α decreases as n increases This is in accordance with a remark made in Section 3 under (ii).

$\lambda = 0.42$									
n/ α	0.0	0.10	0.20	0.30	0.35	0.45	0.55	0.65	0.70
5000	0.414	0.430	—	0.450	0.443	0.439	0.449	0.438	0.431
10000	0.415	0.437	0.406	—	0.405	0.408	0.393	0.431	0.355
15000	0.427	0.443	0.432	—	0.373	0.387	0.432	0.415	0.372
20000	0.448	0.454	0.460	0.470	0.426	0.473	0.419	0.424	0.465
25000	0.414	0.426	0.408	0.460	0.432	0.444	0.377	0.430	0.427
30000	0.408	0.401	0.417	0.395	0.390	0.419	0.419	0.432	0.425
35000	0.446	0.450	0.444	0.455	0.463	0.454	0.489	0.462	0.464
40000	0.392	0.404	0.389	0.407	0.396	0.392	0.402	0.402	0.417

Table 1 *Values of* $\widehat{\lambda}_{t_n}^{\{0\}}(C_{\{t_n,\alpha\}})$ *for different numbers of steps n and peeling fraction* α.

$\lambda = 1$										
n/ α	0.0	0.05	0.10	0.15	0.20	0.30	0.40	0.50	0.60	0.70
5000	0.821	—	0.898	—	0.930	0.941	0.957	0.970	0.998	1.019
10000	0.882	—	0.947	0.967	0.980	1.002	1.007	1.030	1.023	1.000
15000	0.879	0.934	0.949	0.967	0.975	0.977	0.989	0.992	0.992	1.012
20000	0.894	0.940	0.957	0.968	0.985	0.985	0.991	0.989	0.984	0.980
25000	0.893	0.932	0.947	0.952	0.966	0.969	0.979	0.978	0.985	0.984
30000	0.904	0.944	0.956	0.967	0.980	0.990	0.999	1.013	1.024	1.034
35000	0.930	0.967	0.980	0.997	1.006	1.017	1.027	1.039	1.029	1.012
40000	0.924	0.959	0.972	0.982	0.995	1.013	1.029	1.042	1.037	1.041

Table 2 *Values of* $\widehat{\lambda}^{\{0\}}_{t_n}(C_{\{t_n,\alpha\}})$ *for different numbers of steps n and peeling fraction* α.

$\lambda = 1.5$										
n/ α	0.0	0.05	0.10	0.15	0.20	0.30	0.40	0.50	0.60	0.70
5000	1.159	—	1.303	—	1.360	1.434	1.497	1.521	1.537	1.592
10000	1.211	1.333	—	1.378	1.427	1.470	1.530	1.547	1.543	1.547
15000	1.252	1.348	1.387	1.457	1.478	1.522	1.518	1.520	1.530	1.384
20000	1.258	1.340	1.377	1.413	1.439	1.476	1.485	1.504	1.499	1.502
25000	1.253	1.323	1.352	1.382	1.408	1.462	1.466	1.480	1.474	1.464
30000	1.276	1.348	1.375	1.402	1.422	1.454	1.472	1.453	1.450	1.457
35000	1.299	1.365	1.393	1.420	1.467	1.483	1.496	1.498	1.506	1.508
40000	1.300	1.367	1.394	1.447	1.468	1.492	1.492	1.505	1.504	1.508

Table 3 *Values of* $\widehat{\lambda}^{\{0\}}_{t_n}(C_{\{t_n,\alpha\}})$ *for different numbers of steps n and peeling fraction* α.

$\lambda = 2$										
n/ α	0.0	0.05	0.10	0.15	0.20	0.30	0.40	0.50	0.60	0.70
5000	1.439	—	1.651	1.764	1.860	1.908	1.945	2.026	2.020	2.044
10000	1.516	1.680	1.763	1.812	1.883	1.921	1.966	1.977	1.969	1.964
15000	1.551	1.684	1.748	1.804	1.878	1.940	1.976	1.970	1.958	1.936
20000	1.586	1.711	1.774	1.816	1.865	1.929	1.973	1.982	2.017	1.989
25000	1.620	1.731	1.793	1.842	1.883	1.954	1.975	1.973	1.970	1.949
30000	1.684	1.790	1.841	1.894	1.980	2.032	2.041	2.069	2.054	2.040
35000	1.706	1.808	1.860	1.945	1.971	2.024	2.033	2.030	2.044	2.055
40000	1.680	1.777	1.876	1.921	1.946	1.998	2.012	1.989	1.968	1.961

Table 7.4 *Values of* $\widehat{\lambda}^{\{0\}}_{t_n}(C_{\{t_n,\alpha\}})$ *for different numbers of steps n and peeling fraction* α.

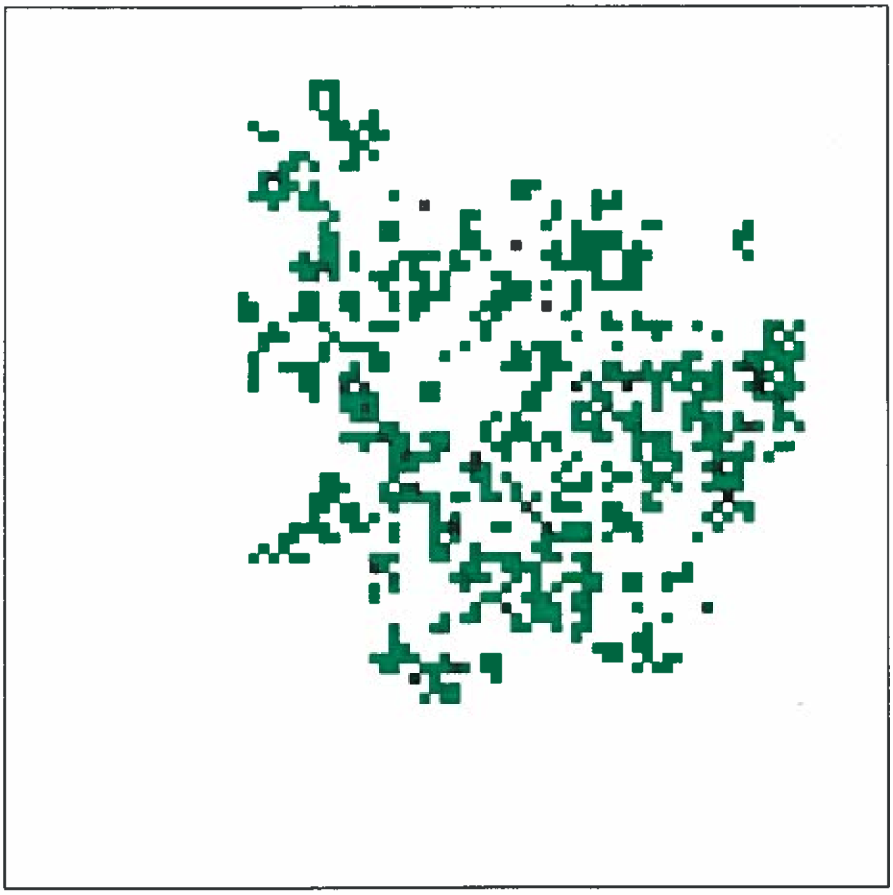

Figure 1: The process $\xi_{t_N}^{\{0\}}$ for $\lambda = 0.5$ and $N = 40,000$. Infected sites are represented by colored squares. The tone of the color indicates the age of the infection.

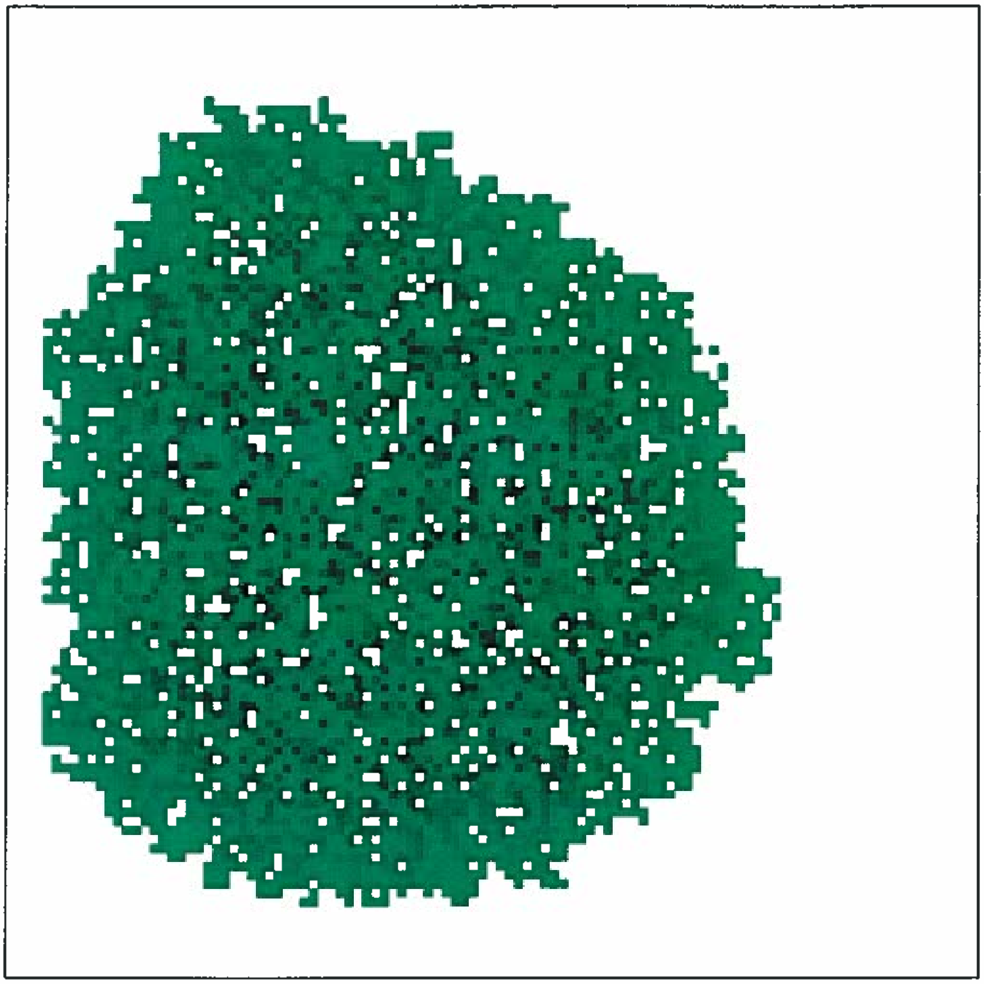

Figure 2: The process $\xi_{t_N}^{\{0\}}$ for $\lambda = 3$ and $N = 30,000$. Infected sites are represented by colored squares. The tone of the color indicates the age of the infection.

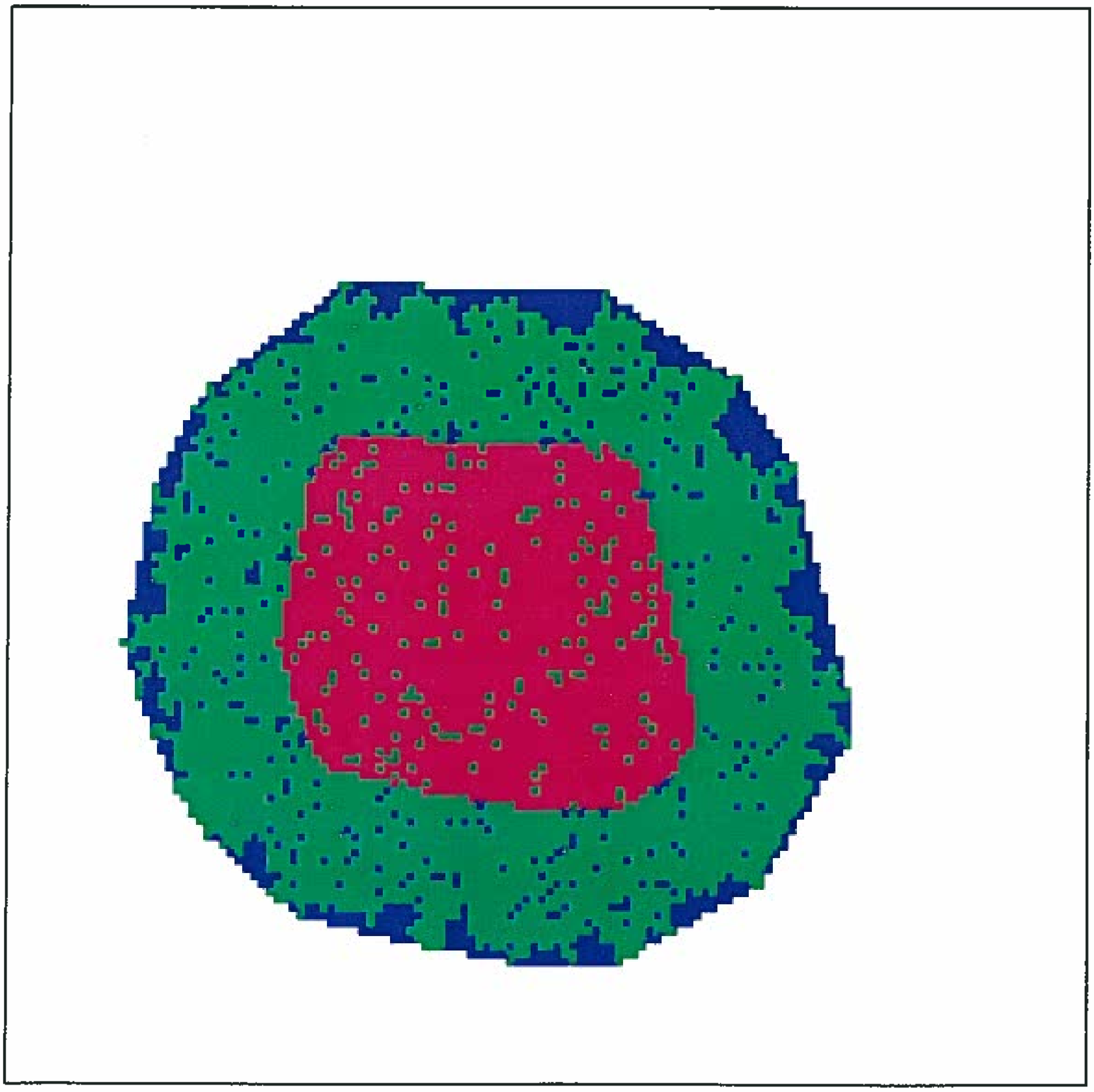

Figure 3: Mask obtained by peeling $\mathcal{C}(\xi_{t_N}^{\{0\}})$.

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