Report 1999-019

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ISSN 1389-2355

The Annals of Probability
2000, Vol. 28, No. 3, 1301–1335

FUNCTIONAL LIMIT LAWS FOR THE INCREMENTS OF KAPLAN-MEIER PRODUCT-LIMIT PROCESSES AND APPLICATIONS

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We prove functional limit laws for the increment functions of empirical processes based upon randomly right-censored data. The increment sizes we consider are classified into different classes covering the whole possible spectrum. We apply these results to obtain a description of the strong limiting behavior of a series of estimators of local functionals of lifetime distributions. In particular, we treat the case of kernel density and hazard rate estimators.

1. Statistical motivation and main results. In the right censorship model, the data set is given by $\{(Z_i, \delta_i)\colon\ 1 \le i \le n\}$, where $Z_i = \min(X_i, Y_i)$ and $\delta_i = \mathbb{1}_{\{X_i \le Y_i\}}$ for $i \ge 1$, with $\mathbb{1}_E$ denoting the indicator function of E. Here, $\{X_i\colon i \ge 1\}$ is a sequence of independent and identically distributed nonnegative lifetimes, and $\{Y_i\colon i \ge 1\}$ is an independent sequence of independent and identically distributed nonnegative censoring times, defined on the same probability space $(\Omega, \mathscr{A}, \mathbb{P})$. Set $X = X_1$, $Y = Y_1$, $Z = Z_1$, $\delta = \delta_1$, $F(x) = \mathbb{P}(X \le x)$, $G(x) = \mathbb{P}(Y \le x)$, $H(x) = \mathbb{P}(Z \le x) = 1 - (1 - F(x))(1 - G(x))$. We allow Y to be defective, that is, such that $\mathbb{P}(Y = \infty)$ is possibly positive, to cover the uncensored case corresponding to the particular case where $\mathbb{P}(Y = \infty) = 1$.

The problem of estimating F, together with local functionals of F such as the lifetime density $f(x) = (d/dx)F(x)$ or the hazard rate function $\lambda(x) = f(x)/(1 - F(x))$, assuming that they exist, has received much attention in the literature [see, e.g., Aalen (1976), Kalbfleisch and Prentice (1980), Gill (1980), Földes, Rejtő and Winter (1981), Burke, Csörgő and Horváth (1981, 1988), Csörgő and Horváth (1983), Anderson, Borgan, Gill and Keiding (1993)]. The nonparametric maximum likelihood estimator of F and G based upon the data are the product-limit (PL) estimators F_n and G_n, introduced in Kaplan and Meier (1958), and defined by

$$F_n(x) = 1 - \prod_{\substack{i:\ Z_{i,n} \le x \\ 1 \le i \le n}} \left\{1 - \frac{\delta_{i,n}}{n - i + 1}\right\}, \tag{1.1}$$

$$G_n(x) = 1 - \prod_{\substack{i:\ Z_{i,n} \le x \\ 1 \le i \le n}} \left\{1 - \frac{1 - \delta_{i,n}}{n - i + 1}\right\}, \tag{1.2}$$

Received May 1999.
[1]Supported in part by European Union HCM Grant ERB CHRX-CT 94-0693.
AMS 1991 *subject classifications.* Primary 62G05, 60F15, 60F17; secondary 62E20, 62P10.
Key words and phrases. Density and hazard rate estimation, functional law of the iterated logarithm, random censorship, strong limit theorems.

where $Z_{1,n} \le \cdots \le Z_{n,n}$ are the ordered $Z_1, \dots, Z_n$, and, for each $i = 1, \dots, n$, $\delta_{i,n}$ is the δ_j corresponding to $Z_{i,n} = Z_j$, $1 \le j \le n$ (we use the convention that $\prod_{\varnothing} = 1$). The Kaplan–Meier empirical process α_n and the Kaplan–Meier censoring process β_n are defined by

$$(1.3)\quad \alpha_n(x) = n^{1/2}(F_n(x) - F(x)) \quad \text{and} \quad \beta_n(x) = n^{1/2}(G_n(x) - G(x)),$$

for $n \ge 1$ and $x \in \mathbb{R}$. The aim of this paper is to describe the limiting behavior of the local oscillations of α_n (equivalently of β_n by the formal change of $\delta_{i,n}$ into $1 - \delta_{i,n}$, $1 \le i \le n$) through the study of the increment functions $\xi_n(h_n, t; I)$ and $\eta_n(h_n, t; I)$, defined by

$$(1.4)\quad \begin{aligned} \xi_n(h, t; s) &= \alpha_n(t + hs) - \alpha_n(t) \\ &= n^{1/2}\eta_n(h, t; s) - n^{1/2}(F(t + hs) - F(t)), \end{aligned}$$

$$(1.5)\quad \begin{aligned} \eta_n(h, t; s) &= F_n(t + hs) - F_n(t) \\ &= n^{-1/2}\xi_n(h, t; s) + (F(t + hs) - F(t)), \end{aligned}$$

for $h \ge 0$ and $s, t \in \mathbb{R}$. Here, $I(s) = s$ denotes the identity function and $\{h_n : n \ge 1\}$ is a sequence of positive constants satisfying conditions among the following, listed below. We will set $\log_2 u = \log_+(\log_+ u)$, $\log_+ u = \log(u \vee e)$, and denote by $u_n = \infty(v_n)$ (resp. $u_n \sim v_n$) the condition that $v_n/u_n \to 0$ (resp. $u_n/v_n \to 1$):

(H1) (i) $h_n \to 0$; (ii) $h_n \downarrow$; (iii) $nh_n \uparrow$;
(H2) $nh_n/\log_2 n \to \infty$;
(H3) (i) $nh_n/\log n \to \infty$; (ii) $(\log(1/h_n))/\log_2 n \to \infty$;
(H4) $(\log(1/h_n))/\log_2 n \to c \in [0, \infty)$;
(H5) $nh_n/\log n \to \gamma \in [0, \infty)$;
(H6) $(\log(1/h_n))/\log n \to d \in [1, \infty)$.

To motivate our forthcoming theorems, we start by an exposition of their implications in the framework of nonparametric estimation of $f = (d/dx)F$ by kernel estimators. Let K be a function (or kernel) fulfilling the assumptions:

(K1) K is of bounded variation on $\mathbb{R}$.
(K2) For some $0 < T < \infty$, $K(u) = 0$ for all $|u| \ge \frac{1}{2}T$.
(K3) $\int_{-\infty}^{\infty} K(u)\,du = 1$.

The kernel estimator of $f(x)$ [see, e.g., Watson and Leadbetter (1964a, b), Tanner and Wong (1983)] is given by

$$(1.6)\quad f_n(x) = \int_{-\infty}^{\infty} h_n^{-1} K((t - x)/h_n)\,dF_n(t).$$

Set, for all $x \in \mathbb{R}$,

$$(1.7)\quad \widehat{\mathbb{E}} f_n(x) = \int_{-\infty}^{\infty} h_n^{-1} K((t - x)/h_n)\,dF(t).$$

In the uncensored case, $\widehat{\mathbb{E}}f_n(x) = \mathbb{E}f_n(x)$, where $\mathbb{E}$ denotes the usual expectation. Otherwise, in general, $\widehat{\mathbb{E}}f_n(x)$ and $\mathbb{E}f_n(x)$ may differ. Note further that, under (K1) and (K2),

$$f_n(x) - \widehat{\mathbb{E}}f_n(x) = -h_n^{-1}n^{-1/2}\int_{-T}^{T}\xi_n(h_n, x; u)\,dK(u), \tag{1.8}$$

which follows from (1.3) and (1.4) and (1.6) and (1.7), after integrating by parts.

The following additional notation and assumptions will be needed. Let $L(x) = \mathbb{P}(V \le x)$ be the distribution function of a random variable V. We denote by $L^{-1}(u) = \inf\{x\colon L(x) \ge u\}$ for $0 < u < 1$ the quantile function of L, and by $T_L = \sup\{x\colon L(x) < 1\}$ the upper endpoint of the distribution of V. Throughout the sequel, we shall assume that the upper endpoints T_F and T_G of the distributions of X and Y are such that $\Theta = \min(T_F, T_G) > 0$, and let a, a', b, b' be specified constants such that $0 < a' < a < b < b' < \Theta$. Unless otherwise specified, we assume that F and G fulfill the conditions (F1) and (F2):

(F1) $F(0) = G(0) = 0$;
(F2) (i) F and G are continuous on $[a', b']$;
(ii) $f = (d/dx)F$ is defined, continuous and strictly positive on $[a', b']$.

Throughout, Ψ will denote a specified continuous and (strictly) positive function on $[a', b']$. We assume that Ψ_n is an estimator of Ψ fulfilling assumptions among (C1) and (C2):

(C1) $\sup_{a\le x\le b}|\Psi_n(x)/\Psi(x) - 1| \to 0$ in probability as $n \to \infty$;
(C2) $\sup_{a\le x\le b}|\Psi_n(x)/\Psi(x) - 1| \to 0$ almost surely as $n \to \infty$.

As follows from the results of Deheuvels and Einmahl (1996), under (H1), (H2), (K1)–(K3), (F1), (F2) and (C2), for any specified $x_0 \in [a, b]$, we have

$$\begin{aligned}&\limsup_{n\to\infty} \pm\left\{\frac{nh_n}{2\log_2 n}\right\}^{1/2}(f_n(x_0) - \widehat{\mathbb{E}}f_n(x_0))\left\{\Psi_n(x_0) \times \frac{1 - G(x_0)}{f(x_0)}\right\}^{1/2}\\ &\qquad = \{\Psi(x_0)\}^{1/2}\left\{\int_{-\infty}^{\infty}K^2(u)\,du\right\}^{1/2} \quad \text{a.s.}\end{aligned} \tag{1.9}$$

In this paper, we will prove the following basic limit law concerning f_n. Below, we use the convention $c/(c+1) = 1$ when (H3) holds, that is, when $c = \infty$.

THEOREM 1.1. *Under* (H1)(i), (H3) *or* (H4), (K1)–(K3), (F1)–(F2) *and* (C1), *we have*

$$\begin{aligned}
&\lim_{n\to\infty}\left\{\frac{nh_n}{2\{\log(1/h_n)+\log_2 n\}}\right\}^{1/2}\\
(1.10)\qquad &\times \sup_{a\le x\le b} \pm(f_n(x)-\widehat{\mathbb{E}}f_n(x))\left\{\Psi_n(x)\times\frac{1-G(x)}{f(x)}\right\}^{1/2}\\
&=\left(\frac{c}{c+1}\right)^{1/2}\left\{\sup_{a\le x\le b}\Psi(x)\right\}^{1/2}\left\{\int_{-\infty}^{\infty}K^2(u)\,du\right\}^{1/2}\quad \textit{in probability.}
\end{aligned}$$

If, in addition, (H1)(ii)–(iii) *and* (C2) *hold, then*

$$\begin{aligned}
&\limsup_{n\to\infty}\left\{\frac{nh_n}{2\{\log(1/h_n)+\log_2 n\}}\right\}^{1/2}\\
(1.11)\qquad &\times \sup_{a\le x\le b} \pm(f_n(x)-\widehat{\mathbb{E}}f_n(x))\left\{\Psi_n(x)\times\frac{1-G(x)}{f(x)}\right\}^{1/2}\\
&=\left\{\sup_{a\le x\le b}\Psi(x)\right\}^{1/2}\left\{\int_{-\infty}^{\infty}K^2(u)\,du\right\}^{1/2}\quad \textit{a.s.}
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{n\to\infty}\left\{\frac{nh_n}{2\{\log(1/h_n)+\log_2 n\}}\right\}^{1/2}\\
(1.12)\qquad &\times \sup_{a\le x\le b} \pm(f_n(x)-\widehat{\mathbb{E}}f_n(x))\left\{\Psi_n(x)\times\frac{1-G(x)}{f(x)}\right\}^{1/2}\\
&=\left(\frac{c}{c+1}\right)^{1/2}\left\{\sup_{a\le x\le b}\Psi(x)\right\}^{1/2}\left\{\int_{-\infty}^{\infty}K^2(u)\,du\right\}^{1/2}\quad \textit{a.s.}
\end{aligned}$$

REMARK 1.1. (i) The assumptions in Theorem 1.1 allow in particular the following possible choices of interest, denoted by $\Psi^{(j)}$, $j = 1$–5, for Ψ, where ψ is an auxiliary continuous and positive function on $[a', b']$:

$$\begin{aligned}
&\Psi^{(1)}(x)=1,\quad \Psi^{(2)}(x)=\frac{1}{1-G(x)},\\
(1.13)\qquad &\Psi^{(3)}(x)=f(x),\quad \Psi^{(4)}(x)=\frac{f(x)}{1-G(x)},\\
&\Psi^{(5)}(x)=\frac{f(x)\psi(x)}{(1-F(x))^2(1-G(x))}=\frac{\lambda(x)\psi(x)}{1-H(x)}.
\end{aligned}$$

(ii) For each of the above choices of Ψ, an estimator Ψ_n of Ψ fulfilling (C1), (C2) is obtained by replacing in the definition (1.13) of Ψ any one among the functions $f(x)$, $F(x)$ or $G(x)$ by $f_n(x)$, $F_n(x)$ or $G_n(x)$, respectively. The fact that (C1), (C2) hold for either of these functions is readily verified. First, it is straightforward from (1.10) and (1.11), (1.12), taken with $\Psi_n = \Psi = 1$, that, under the assumptions of Theorem 1.1, $\sup_{a\le x\le b} |f_n(x)/f(x)-1| \to 0$. Second, the fact that

$$\sup_{a\le x\le b} |(1-F_n(x))/(1-F(x))-1| \to 0 \quad \text{and}$$
$$\sup_{a\le x\le b} |(1-G_n(x))/(1-G(x))-1| \to 0 \quad \text{a.s.},$$

is a simple consequence of the strong uniform consistency of the PL estimators F_n and G_n of F and G [see, e.g., Gu and Lai (1990), Chen and Lo (1997) and the references therein].

(iii) In particular, the replacement of F by F_n in $\Psi^{(5)}$ corresponds to estimators of the hazard rate function $\lambda(x) = f(x)/(1-F(x))$, which will be considered in more detail in Section 3.

The following results in the literature are related to (1.11) and (1.12). Under more stringent assumptions than that given above, Diehl and Stute (1988) showed that, under (H3),

$$\lim_{n\to\infty}\left\{\frac{nh_n}{2\log(1/h_n)}\right\}^{1/2}\sup_{a\le x\le b}|f_n(x)-\widehat{\mathbb{E}}f_n(x)|\left\{\frac{1-G(x)}{f(x)}\right\}^{1/2} = \left\{\int_{-\infty}^{\infty} K^2(u)\,du\right\}^{1/2} \quad \text{a.s.}, \tag{1.14}$$

which follows from (1.11) and (1.12), taken with $\Psi = 1$ and $c = \infty$. Their results were extended by Xiang (1994), who established (1.14) under the additional assumptions on K that $K(u) = K(-u)\ \forall\ u \in \mathbb{R}$, and for $h_n = n^{-\gamma}$ with $0 < \gamma < 1$. Related statements are to be found in Stute and Wang (1993), Lo, Mack and Wang (1989), Schäfer (1986), Liu and Van Ryzin (1985), Padgett and McNichols (1984) and Mielniczuk (1986). It is to be noted that in the uncensored case, corresponding to when $G(x) = 0$ for all $x \in \mathbb{R}$ (see the discussion in Section 3 below), the conclusion of Theorem 1.1 is obtained by combining Theorem 4.1 of Deheuvels and Mason (1992) (for $c = \infty$) with Theorem 3.3 of Deheuvels (1992) (for $0 \le c < \infty$) [see also Deheuvels (1974), Hall (1981), Stute (1982b) and Xu (1993)]. For further descriptions of limiting properties of Kaplan–Meier empirical processes with applications, we refer to Chen and Lo (1997), Csörgő (1996), Földes and Rejtő (1981), Gijbels and Wang (1993), Lo and Singh (1986), Major and Rejtő (1988), Müller and Wang (1994), Patil (1993), Stute (1995, 1996), Yandell (1983) and the references therein.

Theorem 1.1 and related applications (see Section 3 in the sequel) will be shown to be direct consequences of general functional limit laws for the increments $\xi_n(h_n, t; I)$ of the Kaplan–Meier empirical process α_n [recall (1.3) and (1.4)], which constitute the main results of this paper. To present the first of these limit laws in the forthcoming Theorem 1.2, we need to introduce some notation and vocabulary. We closely follow Deheuvels (1992) where additional details can be found, concerning the topological aspects of the function spaces we consider.

Denote by $(B[0,1],\mathscr{U})$ [resp. $(AC[0,1],\mathscr{U})$] the set $B[0,1]$ (resp. $AC[0,1]$) of all bounded (resp. absolutely continuous) functions l on $[0,1]$, endowed with the uniform topology $\mathscr{U}$ defined by the sup-norm $\|l\| = \sup_{0\le t\le 1}|l(t)|$. For each $l \in AC[0,1]$, denote by $\dot{l} = (d/ds)l$ the Lebesgue derivative of l. For each $l \in B[0,1]$ set

$$|l|_H = \begin{cases} \left\{\int_0^1 \dot{l}^2(s)\,ds\right\}^{1/2}, & \text{if } l \in AC[0,1] \text{ and } l(0)=0,\\ \infty, & \text{otherwise.}\end{cases} \tag{1.15}$$

For each $\eta \ge 0$, set

$$\mathscr{S}_\eta = \{l \in AC[0,1]: |l|_H^2 \le \eta\}. \tag{1.16}$$

Observe that $\mathscr{S} = \mathscr{S}_1$ is the Strassen set [see, e.g., Strassen (1964)] and that $\mathscr{S}_\eta = \eta^{1/2}\mathscr{S}_1$, where, here and elsewhere, we set $\lambda\mathscr{G} = \{\lambda l: l \in \mathscr{G}\}$. The following inequality is a direct application of the Schwarz inequality [see, e.g., (2.36), page 2021 in Deheuvels (1997)]. For any $l \in \mathscr{S}_\eta$,

$$\|l\| \le |l|_H \le \eta^{1/2}. \tag{1.17}$$

Define a sequence of random subsets of $B[0,1]$ by setting, for each $n \ge 1$,

$$\begin{aligned}\mathscr{K}_n^{\pm}(\Psi_n) = \Big\{ &\pm\{2h_n(\log_+(1/h_n)+\log_2 n)\}^{-1/2}\\ &\times \xi_n(h_n,x;I)\left\{\Psi_n(x)\times\frac{1-G(x)}{f(x)}\right\}^{1/2} : a \le x \le b\Big\} \subseteq B[0,1].\end{aligned} \tag{1.18}$$

In what follows, we shall describe the limiting behavior of $\mathscr{K}_n^{\pm}(\Psi_n)$ as $n \to \infty$, making use of the following vocabulary and definitions.

Let $(\mathscr{E},\mathscr{T})$ denote a set $\mathscr{E}$, endowed with the topology $\mathscr{T}$ induced by a metric $d(l,g)$, with $l, g \in \mathscr{E}$. For each $\varepsilon > 0$ and $A \subseteq \mathscr{E}$, $A \ne \varnothing$, set $A^\varepsilon = \{g \in \mathscr{E}: \exists\, l \in A,\ d(l,g) < \varepsilon\}$. Introduce the Hausdorff set-metric pertaining to $\mathscr{T}$ by setting, for each $A, B \subseteq \mathscr{E}$,

$$\mathscr{D}_{\mathscr{T}}(A,B) = \begin{cases}\inf\{\varepsilon>0: A \subseteq B^\varepsilon \text{ and } B \subseteq A^\varepsilon\}, & \text{if such as } \varepsilon > 0 \text{ exists;}\\ \infty, & \text{otherwise.}\end{cases} \tag{1.19}$$

Consider now a sequence $\{\mathscr{B}_n \subseteq \mathscr{E}: n \ge 1\}$ of nonvoid subsets of $\mathscr{E}$ for which there exits a compact subset $\mathscr{K}$ of $\mathscr{E}$, such that the following property holds. For each $\varepsilon > 0$, there exists an $n_\varepsilon < \infty$ such that $\mathscr{B}_n \subseteq \mathscr{K}^\varepsilon$ for all $n \ge n_\varepsilon$. Under this assumption, we will say that $\mathscr{B}_n$ has *limit set* equal to $\mathscr{B} \subseteq \mathscr{K}$, if $\mathscr{B}$ consists of all limits as $j \to \infty$ of convergent sequences $l_{n_j} \in \mathscr{B}_{n_j}$ with $1 \le n_1 < n_2 < \cdots$ and $n_j \to \infty$. Likewise, we will say that $\mathscr{B}_n$ minimally covers $\mathscr{B}' \subseteq \mathscr{K}$ if $\mathscr{B}'$ consists of all limits as $n \to \infty$ of convergent sequences $l_n \in \mathscr{B}_n$. $\mathscr{B}'$ is possibly void, whereas such is not the case for $\mathscr{B}$. Both $\mathscr{B}$ and $\mathscr{B}'$ are closed (and hence compact when nonvoid) subsets of $(\mathscr{E},\mathscr{T})$, with $\varnothing \subseteq \mathscr{B}' \subseteq \mathscr{B} \subseteq \mathscr{K}$. When $\mathscr{B} = \mathscr{B}'$, we will say that $\mathscr{B}_n$ completely covers $\mathscr{B}$.

REMARK 1.2. (i) It is noteworthy [see, e.g., Deheuvels (1992)] that a sequence $\mathscr{B}_n \subseteq \mathscr{E}$ has limit set $\mathscr{B} \neq \varnothing$ and minimally covers $\mathscr{B}' \subseteq \mathscr{B}$, if and only if the following properties (a)–(c) hold.

(a) $\mathscr{B}$ is a compact subset of $(\mathscr{E}, \mathscr{T})$;

(b) For each $\varepsilon > 0$, we have for all n sufficiently large,

$$\mathscr{B}' \subseteq \mathscr{B}_n^\varepsilon \quad \text{and} \quad \mathscr{B}_n \subseteq \mathscr{B}^\varepsilon. \tag{1.20}$$

(c) For each $\varepsilon > 0$, $l' \notin \mathscr{B}'$ and $l \in \mathscr{B}$ we have

$$l' \notin \mathscr{B}_n^\varepsilon \text{ i.o (in } n) \quad \text{and} \quad l \in \mathscr{B}_n^\varepsilon \text{ i.o (in } n). \tag{1.21}$$

In particular, (a) implies that $\mathscr{B}_n$ completely covers $\mathscr{B}$ if and only if

$$\mathscr{D}_{\mathscr{T}}(\mathscr{B}_n, \mathscr{B}) \to 0. \tag{1.22}$$

(ii) The assumption (a) that $\mathscr{B}$ is compact is essential for the equivalence in Remark 1.2(i) to be fulfilled. When combined with (b), the condition (a) implies that each sequence $\{l_n\colon n \geq 1\}$, with $l_n \in \mathscr{B}_n$ for each $n \geq 1$, is relatively compact in $(\mathscr{E}, \mathscr{T})$ with limit set included in $\mathscr{B}$. The latter property is not necessarily satisfied when (a) does not hold.

Our main result may now be stated in the following Theorem 1.2, which will be shown later on to imply Theorem 1.1. In the statement of this theorem, $\mathscr{D}_{\mathscr{U}}$ stands for the Hausdorff set distance (1.19) pertaining to the sup-norm on $B[0,1]$, and $\mathscr{K}_n^{\pm}(\Psi_n)$ is as in (1.18).

THEOREM 1.2. *Assume that* H1(i), (H3) *or* (H4), (F1), (F2) *and* (C1) *hold. Set*

$$M = \sup_{a \leq x \leq b} \Psi(x). \tag{1.23}$$

Then

$$\lim_{n\to\infty} \mathscr{D}_{\mathscr{U}}\big(\mathscr{K}_n^{\pm}(\Psi_n), \mathscr{S}_{Mc/(c+1)}\big) = 0 \quad \textit{in probability.} \tag{1.24}$$

If, in addition to these assumptions, (H1)(ii), (iii) *and* (C2) *hold, then, with probability* 1, *in* $(B[0,1], \mathscr{U})$, *the sequence* $\{\mathscr{K}_n^{\pm}(\Psi_n)\colon n \geq 1\}$ *has limit set equal to* $\mathscr{S}_M$, *and minimally covers* $\mathscr{S}_{Mc/(c+1)}$. *In particular, under* (H3), *we have* $c = \infty$, $c/(c+1) = 1$,

$$\lim_{n\to\infty} \mathscr{D}_{\mathscr{U}}(\mathscr{K}_n^{\pm}(\Psi_n), \mathscr{S}_M) = 0 \quad \textit{a.s.,} \tag{1.25}$$

and $\{\mathscr{K}_n^{\pm}(\Psi_n)\colon n \geq 1\}$ *completely covers* $\mathscr{S}_M$ *with probability* 1.

REMARK 1.3. (i) In the uncensored case where $G(x) = 0$ for all x, and for $F = I$ (i.e., when X has the uniform $[0,1]$ distribution), Theorem 1.2 reduces to a combination of Theorem 3.1 in Deheuvels and Mason (1992) and Theorem 1.3 in Deheuvels (1992) [see also (1.11) in the latter]. The extension of the latter results to the case of an arbitrary F can be obtained by relatively simple arguments in this simplified setting.

(ii) In the statement of Theorem 1.2, $\mathscr{K}_n^{\pm}(\Psi_n)$, $\mathscr{S}_{Mc/(c+1)}$ and $\mathscr{S}_M$ are subsets of $B[0,1]$. We note that the conclusion of this theorem is unchanged if we work in the setting of $B[C,D]$ instead of $B[0,1]$, where $-\infty < C < D < \infty$ are arbitrary constants, after the appropriate notational changes. The choice of $C = 0$ and $D = 1$ will be used here and later on for convenience.

(iii) Starting with Finkelstein (1971), there have been a great many papers giving functional limit laws for uncensored empirical processes, taken either globally or locally. In addition to the previously mentioned references, we may add that of Mason (1988), Deheuvels and Mason (1990, 1991, 1994, 1995), Einmahl (1992, 1997) and Einmahl and Mason (1997, 1998).

Among other results, Deheuvels and Einmahl (1996) have shown that, under the assumptions (H1), (H2) and (F1), (F2) for each $x_0 \in [a,b]$, if $M_0 = f(x_0)/(1 - G(x_0))$, the sequence

$$\mathscr{F}_n = \Big\{\{2h_n \log_2 n\}^{-1/2}\xi_n(h_n, x_0; I)\Big\} \subseteq B[0,1], \tag{1.26}$$

is almost surely relatively compact and has limit set in $(B[0,1], \mathscr{U})$ equal to $\mathscr{S}_{M_0}$. A comparison of (1.26) with the conclusion of Theorem 1.2 gives emphasis on the fact that the present work completes the study of the local Kaplan–Meier empirical process in the neighborhood of a fixed point by that of the same process on the specified interval $[a,b]$.

In the remainder of this paper, we will describe the limiting behavior of the random sets of increment functions $\{\xi_n(h_n, t; I)\colon a \le t \le b\}$ for sequences h_n which are not covered by the assumptions of Theorems 1.1 and 1.2. It is convenient to distinguish the following main ranges of interest depending on the rate of convergence of h_n to 0. We will speak namely of:

1. Large increments when $(\log(1/h_n))/\log_2 n \to c \in [0,\infty)$.
2. Standard increments when $nh_n/\log n \to \infty$ and $(\log(1/h_n))/\log_2 n \to \infty$.
3. Intermediate increments when $nh_n/\log n \to \gamma \in (0,\infty)$.
4. Small increments when $nh_n/\log n \to 0$.

Following Deheuvels (1996), we distinguish two subclasses of small increments. We speak of:

4a. Fairly small increments when $(\log(1/h_n))/\log n \to 1$ (and $nh_n/\log n \to 0$).
4b. Extremely small increments when $(\log(1/h_n))/\log n \to 1 + 1/\kappa$ for some $\kappa \in [0,\infty)$.

The convention $1 + 1/\kappa = \infty$ is used when $\kappa = 0$. Large and standard increments are treated in Theorems 1.1 and 1.2, and the limit laws in the other cases are stated in Section 2. Section 3 collects a series of applications of the theorems of Sections 1, 2. In particular, in Section 3, we describe the limiting behavior of a classical nonparametric estimator of the hazard rate function $f/(1-F)$. The proofs of our main results are postponed until Section 4.

2. Nonstandard local functional laws for censored processes.

2.1. *Introduction: intermediate increments.* The assumption (H3)(i) in Theorems 1.1 and 1.2 limits the validity of these theorems to sequences h_n fulfilling $h_n = \infty(n^{-1}\log n)$. It is the purpose of this section to complete the study of local increments of α_n by a description of the limiting behavior corresponding to sequences such that $h_n = O(n^{-1}\log n)$. We call such results nonstandard by following the vocabulary of Deheuvels and Mason (1990, 1991, 1995). First, we will consider the borderline case, where the condition (H5) holds for some constant $\gamma \in (0,\infty)$. We assume namely that, as $n \to \infty$,

$$nh_n/\log n \to \gamma \in (0,\infty). \tag{2.1}$$

The sequences $\{h_n: n \geq 1\}$ fulfilling (2.1) will be called intermediate sequences. We will discuss later small sequences corresponding to when $\gamma = 0$ in (H5) (see, e.g., Section 2.2). In either of these cases depending upon the value of $\gamma < \infty$ in (H5), it is more convenient to work on

$$\eta_n(h_n, t; I) = F_n(t + h_n I) - F_n(t), \tag{2.2}$$

rather than with $\xi_n(h_n, t; I)$. This fact is captured in Remark 2.1.

REMARK 2.1. Recalling the definitions (1.4) and (1.5), we may check that, under (2.1) and (F2)(ii), we have as $n \to \infty$, uniformly over $t \in [a, b]$,

$$\begin{aligned}\frac{n}{\log n}\eta_n(h_n, t; I) &= (1 + o(1))(2\gamma)^{1/2}(2h_n\{\log(1/h_n) + \log_2 n\})^{-1/2}\\ &\quad \times \xi_n(h_n, t; I) + (1 + o(1))\gamma f(t) I.\end{aligned} \tag{2.3}$$

As follows from (2.3), a functional limit law dealing, under (2.1), with the random functions

$$\frac{n}{\log n}\eta_n(h_n, t; I) \quad \text{for } t \in [a, b],$$

is equivalent, after a simple change of scale and centering, to a functional limit law dealing with

$$(2h_n\{\log(1/h_n) + \log_2 n\})^{-1/2}\xi_n(h_n, t; I) \quad \text{for } t \in [a, b].$$

Throughout the remainder of this subsection, we assume (2.1). In view of Remark 2.1, our aim is to describe the limiting behavior of the set of random functions defined in (2.4) below, in terms of $\eta_n(h_n, t; I)$. Similarly to Section 1, we let $\{\Xi(x): a' \leq x \leq b'\}$ denote a continuous and (strictly) positive function, and $\Xi_n(x)$, for $a \leq x \leq b$, an estimator of $\Xi(x)$ such that

(X.1) $\sup_{a\leq x\leq b}|\Xi(x)/\Xi(x) - 1| \to 0$ almost surely as $n \to \infty$.

Given Ξ_n as above, set

$$\mathscr{L}_n(\Xi_n) = \left\{\frac{n}{\log n}\eta_n(h_n, x; I)\Xi_n(x): a \leq x \leq b\right\}. \tag{2.4}$$

Our main result, stated in Theorem 2.1 in the sequel, establishes a strong limit law for $\mathscr{L}_n(\Xi_n)$. We start by some preliminary results and notation which are needed in the present framework. Denote by $I_{\mathrm{RC}}[0,1]$ the set of all right-continuous distribution functions $l(x) = \mathscr{M}([0,x])$ for $x \in \mathbb{R}$, of nonnegative bounded Radon measures $\mathscr{M}$ with support in $[0,1]$, and set $I_{\mathrm{AC}}[0,1] = I_{\mathrm{RC}}[0,1] \cap AC[0,1] \cap \{l: l(0) = 0\}$. We will endow $I_{\mathrm{RC}}[0,1]$ with either the uniform topology $\mathscr{U}$, or with the weak topology $\mathscr{W}$, conveniently defined via the Lévy metric, for $l, g \in I_{\mathrm{RC}}[0,1]$,

$$d_L(l,g) = \inf\{\varepsilon > 0: l(x-\varepsilon) - \varepsilon < g(x) < l(x+\varepsilon) + \varepsilon \ \forall\, x \in \mathbb{R}\}. \tag{2.5}$$

Consider a function $\Phi: \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ fulfilling the following condition, for some $\mu \in \mathbb{R}$:

(Cμ) (i) $\Phi(\mu) = 0$.
(ii) Φ is convex and $\Phi(\alpha) \geq 0$, all α.
(iii) $\Phi(\alpha)/\alpha \to \infty$ as $\alpha \to \infty$.
(iv) $\Phi(\alpha) = \infty$ for $\alpha < 0$.

Introduce the functional on $B[0,1]$ defined by

$$J_\Phi(l) = \begin{cases} \displaystyle\int_0^1 \Phi(\dot{l}(u))\,du, & \text{if } l(0) = 0 \text{ and } l \in AC[0,1] \\ & \quad \text{with } \dot{l} = (d/du)l; \\ \infty, & \text{otherwise.} \end{cases} \tag{2.6}$$

It is noteworthy that, under (Cμ), $J_\Phi(l) < \infty \Rightarrow l \in I_{\mathrm{AC}}[0,1]$. Consider the subsets of $I_{\mathrm{AC}}[0,1]$ defined, for $\rho \geq 0$, by

$$L_\Phi = \{l \in B[0,1]: J_\Phi(l) < \infty\} \quad \text{and} \quad B_\Phi(\rho) = \{l \in B[0,1]: J_\Phi(l) \leq \rho\}. \tag{2.7}$$

LEMMA 2.1. *Under* (Cμ), *the mapping $l \in B[0,1] \mapsto J_\Phi(l)$ is lower semi-continuous with respect to the uniform topology $\mathscr{U}$.*

PROOF. Let $\{l_n: n \geq 1\} \subseteq B[0,1]$ and $l \in B[0,1]$ be such that $\|l_n - l\| \to 0$. We need only show that

$$J_\Phi(l) \leq \liminf_{n\to\infty} J_\Phi(l_n). \tag{2.8}$$

There is nothing to prove if $\liminf_{n\to\infty} J_\Phi(l_n) = \infty$, so that we may assume, without loss of generality, that $\liminf_{n\to\infty} J_\Phi(l_n) < \infty$. If such is the case, then $l_{n_k} \in I_{\mathrm{AC}}[0,1]$ along a subsequence $\{n_k: k \geq 1\}$, which implies in turn that $l \in I_{RC}[0,1]$. The fact that $\|l_n - l\| \to 0$ then obviously implies that $d_L(l_{n_k}, l) \to 0$, so that we may infer (2.8) from Lemma 3.3 of Lynch and Sethuraman (1987). □

LEMMA 2.2. *Under* (Cμ), *for each $\rho > 0$, $B_\Phi(\rho)$ is a convex and compact subset of* $(B[0,1], \mathscr{U})$.

PROOF. We first observe that $B_\Phi(\rho) \neq \varnothing$ since the linear function $l(t) = \mu t$, $t \in [0,1]$ is such that $J_\Phi(l) = 0$, and therefore belongs to $B_\Phi(\rho)$. To establish the convexity of $B_\Phi(\rho)$, consider $l_1, l_2 \in B_\Phi(\rho)$ and $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 = 1$. The convexity of Φ obviously implies that

$$\int_0^1 \Phi(\lambda_1 \dot{l}_1(u) + \lambda_2 \dot{l}_2(u))\,du \leq \lambda_1 \int_0^1 \Phi(\dot{l}_1(u))\,du + \lambda_2 \int_0^1 \Phi(\dot{l}_2(u))\,du \leq \rho,$$

so that $\lambda_1 l_1 + \lambda_2 l_2 \in B_\Phi(\rho)$. To show that $B_\Phi(\rho)$ is relatively compact in $(B[0,1], \mathscr{U})$, we make use of the Arzelà–Ascoli theorem. First, we show that $B_\Phi(\rho)$ is uniformly equicontinuous. The convexity inequality for integrals shows that, for any $0 \leq c < d \leq 1$ and $l \in B_\Phi$,

$$(2.9) \qquad (d-c)\Phi\left(\frac{l(d)-l(c)}{d-c}\right) \leq \int_c^d \Phi(\dot{l}(u))\,du \leq J_\Phi(l) \leq \rho.$$

Since (Cμ) implies the existence, for any $\varepsilon > 0$, of an $\alpha_\varepsilon > 0$ such that $\Phi(\alpha) \geq (\rho/\varepsilon)|\alpha|$ for all $|\alpha| \geq \alpha_\varepsilon$, it follows from (2.9) that

$$(2.10) \qquad \begin{aligned} \left|\frac{l(d)-l(c)}{d-c}\right| \geq \alpha_\varepsilon \;\Rightarrow\; \frac{\rho}{d-c} \geq \Phi\left(\frac{l(d)-l(c)}{d-c}\right) \geq \frac{\rho}{\varepsilon}\left|\frac{l(d)-l(c)}{d-c}\right| \\ \Rightarrow\; \left|l(d)-l(c)\right| \leq \varepsilon. \end{aligned}$$

On the other hand, if we choose $0 \leq c < d \leq 1$ such that $|d-c| \leq \varepsilon/\alpha_\varepsilon$, we see that

$$(2.11) \qquad \left|\frac{l(d)-l(c)}{d-c}\right| \leq \alpha_\varepsilon \;\Rightarrow\; \left|l(d)-l(c)\right| \leq |d-c|\alpha_\varepsilon \leq \varepsilon.$$

By combining (2.10) and (2.11), we obtain that $|d-c| \leq \varepsilon/\alpha_\varepsilon \Rightarrow |l(d)-l(c)| \leq \varepsilon$, which establishes the equicontinuity of $B_\Phi(\rho)$. Since $l(0) = 0$ for all $l \in B_\Phi(\rho)$ the uniform boundedness of $B_\Phi(\rho)$ is trivial, whence the relative compactness of $B_\Phi(\rho)$. We conclude by an application of Lemma 2.1, which entails that $B_\Phi(\rho)$ is a closed, nonvoid and relatively compact (and hence compact) subset of $(B[0,1], \mathscr{U})$. □

For each $v > 0$ and $x \in \mathbb{R}$, set

$$(2.12) \qquad \mathbf{h}_v(x) = v\mathbf{h}(x/v) \quad \text{where } \mathbf{h}(x) = \begin{cases} x\log x - x + 1, & \text{for } x > 0, \\ 1, & \text{for } x = 0, \\ \infty, & \text{for } x < 0. \end{cases}$$

For each $c > 0$ set

$$(2.13) \qquad \delta_c^- = \sup\{x < 1 \colon \mathbf{h}(x) \geq 1/c\} \quad \text{and} \quad \delta_c^+ = \inf\{x > 1 \colon \mathbf{h}(x) \geq 1/c\}.$$

We observe that $\mathbf{h}_v$ fulfills (Cv) for all $v > 0$. In view of (2.6), set, for all $v > 0$,

$$(2.14) \qquad J_v(l) = J_{\mathbf{h}_v}(l) = \begin{cases} \int_0^1 v\mathbf{h}(\dot{l}(u)/v)\,dv, & \text{if } l \in I_{\mathrm{AC}}[0,1], \\ \infty, & \text{otherwise.} \end{cases}$$

For each $v > 0$ and $\rho > 0$, let $\Delta_v(\rho)$ and Δ_v denote the sets of functions defined by

$$(2.15) \qquad \Delta_v(\rho) = B_{\mathbf{h}_v}(\rho) = \{l \in B[0,1] \colon J_v(l) \leq \rho\} \quad \text{and} \quad \Delta_v = \Delta_v(1).$$

LEMMA 2.3. *We have, for each $w > 0$ and $\rho > 0$,*

$$\inf_{l\in\Delta_w(\rho)} l(1) = w\delta^-_{w/\rho} \quad \textit{and} \quad \sup_{l\in\Delta_w(\rho)} l(1) = w\delta^+_{w/\rho}. \tag{2.16}$$

The proof follows readily from Example 6, page 62 in Deheuvels and Mason (1991).

LEMMA 2.4. *Fix $0 < A \le B < \infty$. Let $\kappa(v)$ and $\beta(v)$ be positive and continuous functions of $v \in [A, B]$. Then, the set $\Delta(\kappa, \beta)$ defined by*

$$\Delta(\kappa,\beta) = \bigcup_{A\le v\le B} \kappa(v)\Delta_{\beta(v)} \tag{2.17}$$

is a compact subset of $(I_{\text{AC}}[0, 1], \mathscr{U})$.

PROOF. We first establish that $\Delta(\kappa, \beta)$ is relatively compact in $(I_{\text{AC}}[0, 1], \mathscr{U})$. Toward this aim, we observe that, by the assumptions of the lemma,

$$0 < \kappa' := \inf_{A\le v\le B} \kappa(v) \le \kappa'' := \sup_{A\le v\le B} \kappa(v) < \infty,$$

$$0 < \beta' := \inf_{A\le v\le B} \beta(v) \le \beta'' := \sup_{A\le v\le B} \beta(v) < \infty.$$

Since, by (2.14), for each $l \in \Delta_{\beta(v)}$ and $\lambda > 0$,

$$J_{\lambda\beta(v)}(\lambda l) = \lambda J_{\beta(v)}(l) \le \lambda,$$

we have $\lambda\Delta_{\beta(v)} \subseteq \Delta_{\lambda\beta(v)}(\lambda)$, and hence, for each $A \le v \le B$,

$$\kappa(v)\Delta_{\beta(v)} \subseteq \bigcup_{\kappa'\le\lambda\le\kappa''} \Delta_{\lambda\beta(v)}(\lambda) \subseteq \bigcup_{\kappa'\beta'\le w\le\kappa''\beta''} \Delta_w(\kappa''). \tag{2.18}$$

Now, making use of (2.12), we observe that, for any $0 < u \le w$ and $l \in I_{\text{AC}}[0, 1]$,

$$J_u(l) = J_w(l) + u - w + l(1)\log(w/u) \le J_w(l) + l(1)\log(w/u), \tag{2.19}$$

By combining (2.19), taken with $u = \kappa'\beta'$, with (2.16), we obtain readily that, uniformly over all $l \in \Delta_w(\kappa'')$, with $\kappa'\beta' \le w \le \kappa''\beta''$,

$$J_{\kappa'\beta'}(l) \le R := \kappa'' + \left\{ \sup_{\kappa'\beta'\le w\le\kappa''\beta''} w\delta^+_{w/\kappa''} \right\} \log(\kappa''\beta''/\kappa'\beta').$$

By combining this last inequality with (2.18) and (2.17), we see that

$$\Delta(\kappa,\beta) \subseteq \Delta_{\kappa'\beta'}(R).$$

In view of (2.15) and Lemma 2.2, $\Delta_{\kappa'\beta'}(R)$ is a compact subset of $(I_{\text{AC}}[0, 1], \mathscr{U})$, whence the relative compactness of $\Delta(\kappa, \beta)$.

We next assume that $g \in \Delta_{\kappa'\beta'}(R)$ is such that, for some sequence $\{g_n\} \subseteq I_{\text{AC}}[0, 1]$ and some sequence $\{v_n\} \subseteq \mathbb{R}$, we have $\|g_n - g\| \to 0$ as $n \to \infty$ with $g_n \in \kappa(v_n)\Delta_{\beta(v_n)}$ and $A \le v_n \le B$ for each $n \ge 1$. By eventually replacing $\{g_n\}$ by an appropriate subsequence, we may and do assume that, for some $v \in [A, B]$, $v_n \to v$ as $n \to \infty$. Set $g_n = \kappa(v_n)l_n$ and $g = \kappa(v)l$, so that $l_n \in \Delta_{\beta(v_n)} \Leftrightarrow J_{\beta(v_n)}(l_n) \leq 1$. By application of (2.19), we obtain readily from the continuity of β that $|J_{\beta(v)}(l_n) - J_{\beta(v_n)}(l_n)| = o(1)$, and hence, that $\liminf_{n\to\infty} J_{\beta(v)}(l_n) \leq 1$. Now, the continuity of κ entails that $\|l_n - l\| \to 0$. By an application of Lemma 2.1 taken with $\Phi = \mathbf{h}_v$ [see, e.g., (2.8)], it follows that $J_{\beta(v)}(l) \leq 1$. We have therefore $g = \kappa(v)l \in \kappa(v)\Delta_{\beta(v)} \in \Delta(\kappa, \beta)$. This proves that $\Delta(\kappa, \beta)$ is closed in $(I_{\mathrm{AC}}[0,1], \mathscr{U})$. Since this set is also nonvoid and relatively compact, it is therefore compact. □

We are now ready to state the main result of this section. Recall the definition (2.4).

THEOREM 2.1. *Under* (H5) *with* $\gamma \in (0, \infty)$, (F1), (F2) *and* (X1), *we have*

$$\lim_{n\to\infty} \mathscr{D}_{\mathscr{U}}(\mathscr{L}_n(\Xi_n), \mathscr{L}(\Xi)) = 0 \quad a.s., \tag{2.20}$$

where

$$\mathscr{L}(\Xi) = \bigcup_{a\leq x\leq b} \left\{\frac{\Xi(X)}{1-G(x)}\right\}\Delta_{\gamma f(x)(1-G(x))}. \tag{2.21}$$

REMARK 2.2. As follows from Lemma 2.4, the limit set $\mathscr{L}(\Xi)$ in (2.21) is a compact subset of $(I_{\mathrm{AC}}[0,1], \mathscr{U})$. On the other hand, this set is not necessarily convex. We note that the compactness of $\mathscr{L}(\Xi)$ is not straightforward, this set being defined, via (2.21), as a union of an uncountable collection of compact subsets of $(I_{\mathrm{AC}}[0,1], \mathscr{U})$. In view of Remark 1.2(ii), this property will turn out to be essential for proving that $\mathscr{L}_n(\Xi_n)$ completely covers $\mathscr{L}(\Xi)$.

The proof of Theorem 2.1 is postponed until Section 4. Applications are given in Section 3.

2.2. *Small increments.* We now turn to the case of small increments, that is, when $nh_n/\log n \to 0$. Consider the following random sets. Let Ξ and Ξ_n be as in Section 2.1, and set

$$\mathscr{L}^{[s]}(\Xi_n) = \left\{\left(\frac{n}{\log n}\log\left(\frac{\log n}{nh_n}\right)\right)\eta_n(h_n, x; I)\Xi_n(x) \colon a \leq x \leq b\right\}, \tag{2.22}$$

$$\mathscr{L}^{[es]}(\Xi_n) = \{n\eta_n(h_n, x; I)\Xi_n(x) \colon a \leq x \leq b\}. \tag{2.23}$$

Introduce the following compact subsets of $I_{\mathrm{RC}}[0,1]$. Set, for each integer $k \geq 0$,

$$\mathbb{L}^{[s]} = \{\phi \in I_{\mathrm{RC}}[0,1] \colon \phi(1) \leq 1\}, \tag{2.24}$$

$$\mathbb{L}^{[es]}(k) = \{\phi \in I_{\mathrm{RC}}[0,1] \colon \phi(x) \in \{0, \dots, k\}\}. \tag{2.25}$$

For the statement of the next theorems, we will make use of the following notation. Recall the definition (2.5) of the Lévy distance d_L. For any $A \subseteq I_{\mathrm{RC}}[0,1]$ and $\varepsilon > 0$, we set

$$A^{[\varepsilon]} = \{l \in I_{\mathrm{RC}}[0,1] \colon \exists\, g \in A \colon d_L(l, g) < \varepsilon\}. \tag{2.26}$$

For any $\kappa \geq 0$, we denote by $\lceil \kappa \rceil \geq \kappa > \lceil \kappa \rceil - 1$ the upper integer part of κ, and use the convention that $1/0 = \infty$.

THEOREM 2.2. *Let* (F1), (F2) *and* (X1) *be satisfied. Assume that* (H1)(i), (ii), (H5) *with* $\gamma = 0$ *and* (H6) *with* $d = 1$ *hold, that is, that*

$$h_n \downarrow 0, nh_n/\log n \to 0 \quad \text{and} \quad (\log(1/h_n))/\log n \to 1. \tag{2.27}$$

Let

$$T = \sup_{a \leq x \leq b} \left\{ \frac{\Xi(x)}{1 - G(x)} \right\}. \tag{2.28}$$

Then, we have

$$\lim_{n \to \infty} d_L(\mathscr{L}^{[s]}(\Xi_n), T\mathbb{L}^{[s]}) = 0 \quad a.s. \tag{2.29}$$

THEOREM 2.3. *Let* (F1), (F2) *and* (X1) *be satisfied. Assume that* (H1)(i), (ii), (H6) *with* $d = 1 + 1/\kappa \in (1, \infty]$ *hold, that is, that*

$$h_n \downarrow 0 \quad \text{and} \quad (\log(1/h_n))/\log n \to 1 + 1/\kappa. \tag{2.30}$$

Define, for each integer $k \geq 0$, *the compact subset of* $I_{\mathrm{RC}}[0, 1]$,

$$\mathscr{R}(k) = \bigcup_{a \leq x \leq b} \left\{ \frac{\Xi(x)}{1 - G(x)} \right\} \mathbb{L}^{[es]}(k). \tag{2.31}$$

(i) *Then, whenever* κ *is noninteger, we have*

$$\lim_{n \to \infty} d_L(\mathscr{L}^{[es]}(\Xi_n), \mathscr{R}(\lceil \kappa \rceil)) = 0 \quad a.s. \tag{2.32}$$

(ii) *When* $\kappa = 0$, *we have*

$$\lim_{n \to \infty} d_L(\mathscr{L}^{[es]}(\Xi_n), \mathscr{R}(1)) = 0 \quad a.s. \tag{2.33}$$

(iii) *When* $\kappa = k \geq 1$ *is integer, for each* $\varepsilon > 0$, *there exists almost surely an* $n(\varepsilon) < \infty$ *such that, for all* $n \geq n(\varepsilon)$,

$$\mathscr{L}^{[es]}(\Xi_n) \subseteq \mathscr{R}(k+1)^{[\varepsilon]} \quad \text{and} \quad \mathscr{R}(k) \subseteq \mathscr{L}^{[es]}(\Xi_n)^{[\varepsilon]}. \tag{2.34}$$

The proofs of Theorems 2.2 and 2.3 are postponed until Section 4.

3. Applications.

3.1. *Introduction.* We will make use of the following analytical proposition to derive a series of applications of our theorems. With the notation of Section 1, let $\mathscr{B}_n$ be a sequence of nonvoid subsets of $(\mathscr{E}, \mathscr{T})$.

PROPOSITION 3.1. *Assume that $\mathscr{B}_n$ has limit set $\mathscr{B}$ and minimally covers $\mathscr{B}' \neq \varnothing$. Let Γ: $\mathscr{E} \to \mathbb{R}$ be a $\mathscr{T}$-continuous mapping. Then, we have*

$$\liminf_{n\to\infty}\left\{\sup_{l\in\mathscr{B}_n}\Gamma(l)\right\} = \sup_{l\in\mathscr{B}'}\Gamma(l) \quad \textit{and} \quad \limsup_{n\to\infty}\left\{\sup_{l\in\mathscr{B}_n}\Gamma(l)\right\} = \sup_{l\in\mathscr{B}}\Gamma(l). \tag{3.1}$$

PROOF. Let $d(\cdot,\cdot)$ be the distance defining $\mathscr{T}$. Set

$$L_1 = \limsup_{n\to\infty}\left\{\sup_{l\in\mathscr{B}_n}\Gamma(l)\right\} \quad \text{and} \quad L_2 = \sup_{l\in\mathscr{B}}\Gamma(l).$$

There exists a sequence of indices $n_j \to \infty$ and $l_{n_j} \in \mathscr{B}_{n_j}$ such that $\Gamma(l_{n_j}) \to L_1$. By eventually replacing n_j by an appropriate subsequence, we may assume the existence of $l \in \mathscr{B}$ such that $d(l_{n_j}, l) \to 0$. The continuity of Γ implies therefore that $\Gamma(l_{n_j}) \to \Gamma(l) = L_1 \le L_2$. On the other hand, since $\mathscr{B}$ is compact, there exists a $g \in \mathscr{B}$ such that $\Gamma(g) = L_2$. Since, by definition of L_2, g is the limit of some sequence $g_{m_j} \in \mathscr{B}_{m_j}$ with $m_j \to \infty$, the definition of L_1 entails that $\Gamma(g) = L_2 \le L_1$. The inequality $L_1 \le L_2$ implies therefore that $L_1 = L_2$. Set now

$$L_3 = \liminf_{n\to\infty}\left\{\sup_{l\in\mathscr{B}_n}\Gamma(l)\right\} \quad \text{and} \quad L_4 = \sup_{l\in\mathscr{B}'}\Gamma(l).$$

Since $\mathscr{B}' \neq \varnothing$ is compact, there exists a $g' \in \mathscr{B}'$ such that $\Gamma(g') = L_4$, and a sequence $g'_n \in \mathscr{B}_n$ with $d(g', g'_n) \to 0$. This implying that $\Gamma(g'_n) \to \Gamma(g') = L_4$, it follows that $L_3 \ge L_4$. Suppose now that $L_3 > L_4$, and select an $\varepsilon > 0$ so small that $\Gamma(l) < L_3 - \frac{1}{2}(L_3 - L_4)$ for all $l \in (\mathscr{B}')^\varepsilon$. Since we have $\mathscr{B}_n \subseteq (\mathscr{B}')^\varepsilon$ i.o. in n, we also have $\sup_{l\in\mathscr{B}_n}\Gamma(l) \le L_3 - \frac{1}{2}(L_3 - L_4)$ i.o. in n, which is in contradiction with the definition of L_3. We have therefore $L_3 = L_4$. $\square$

3.2. *Oscillations of the local Kaplan–Meier empirical process.* We start by investigating the oscillation modulus of α_n. In the uncensored case, that is, when $G(x) = 0$ for all $x \in \mathbb{R}$, many papers have been devoted to this problem, among which we may cite those of Stute (1982a, b), Mason, Shorack and Wellner (1983), Deheuvels and Mason (1992) and Deheuvels (1992, 1996). We refer to Deheuvels (1997), Shorack and Wellner (1986) and Csörgő and Horváth (1994) for further references and details on the subject. In the censored case, a partial description is to be found in Schäfer (1986).

Recalling (1.3), (1.4) we first establish the following corollary of Theorem 1.2 concerning the oscillations of α_n. Set

$$\Omega_n^{\pm}(h) = \sup_{\substack{a\le x\le b\\ 0\le s\le h}} \pm(\alpha_n(x+s) - \alpha_n(x)) = \sup_{\substack{a\le x\le b\\ 0\le s\le 1}} \pm\xi_n(h,x;s) \tag{3.2}$$

and

$$\Omega_n(h) = \sup_{\substack{a\le x\le b\\ 0\le s\le h}} |\alpha_n(x+s) - \alpha_n(x)| = \sup_{\substack{a\le x\le b\\ 0\le s\le 1}} |\xi_n(h,x;s)|. \tag{3.3}$$

Set, for convenience,

$$b_n = \left(2h_n\{\log(1/h_n) + \log_2 n\}\right)^{1/2}. \tag{3.4}$$

COROLLARY 3.1. *Assume that* (H1), (H3) *or* (H4), *and* (F1), (F2) *hold. Then, we have*

$$\limsup_{n\to\infty} b_n^{-1}\Omega_n^{\pm}(h_n) = \limsup_{n\to\infty} b_n^{-1}\Omega_n(h_n) = \sup_{a\le x\le b}\left\{\frac{f(x)}{1-G(x)}\right\}^{1/2} \quad a.s. \tag{3.5}$$

and

$$\begin{aligned}\liminf_{n\to\infty} b_n^{-1}\Omega_n^{\pm}(h_n) &= \liminf_{n\to\infty} b_n^{-1}\Omega_n(h_n)\\ &= \left(\frac{c}{c+1}\right)^{1/2}\sup_{a\le x\le b}\left\{\frac{f(x)}{1-G(x)}\right\}^{1/2} \quad a.s.\end{aligned} \tag{3.6}$$

PROOF. The mapping,

$$l \in B[0,1] \mapsto \Gamma(f) = \sup_{0\le t\le 1} l(t),$$

is obviously $\mathscr{U}$-continuous. Recalling the notation (1.3) and (1.18), we see that, if $\Psi = f/(1-G)$,

$$\limsup_{n\to\infty} b_n^{-1}\Omega_n^{+}(h_n) = \sup_{l\in\mathscr{K}_n^{+}(\Psi)} \Gamma(f).$$

Setting $M = \sup_{a\le x\le b}\Psi(x)$, we infer from Theorem 1.2 and Proposition 3.1 that, almost surely,

$$\limsup_{n\to\infty} b_n^{-1}\Omega_n^{+}(h_n) = \sup_{l\in\mathscr{S}_M}\Gamma(l) = M^{1/2},$$

$$\liminf_{n\to\infty} b_n^{-1}\Omega_n^{+}(h_n) = \sup_{l\in\mathscr{S}_{Mc/(c+1)}}\Gamma(l) = M^{1/2}\left(\frac{c}{c+1}\right)^{1/2},$$

which yields (3.5) and (3.6) for Ω_n^{+}. The proofs of (3.5) and (3.6) in the other cases are similar and omitted.

Similar results to that given above can be derived for the intermediate and small increments corresponding to the sequences $\{h_n: n \ge 1\}$ considered in Section 2. We will restrict ourselves here to the intermediate sequences of Theorem 2.1.

COROLLARY 3.2. *Assume that* (H5), *with* $\gamma > 0$ *and* (F1), (F2) *hold. Then, we have*

$$(3.7)\quad \lim_{n\to\infty}\frac{n^{1/2}\Omega_n(h_n)}{\log n}=\lim_{n\to\infty}\frac{n^{1/2}\Omega_n^+(h_n)}{\log n}=\gamma\sup_{a\le x\le b}f(x)\big(\delta^+_{\gamma f(x)(1-G(x))}-1\big)\quad a.s.$$

and

$$(3.8)\quad \lim_{n\to\infty}\frac{n^{1/2}\Omega_n^-(h_n)}{\log n}=\gamma\sup_{a\le x\le b}f(x)\big(1-\delta^-_{\gamma f(x)(1-G(x))}\big)\quad a.s.$$

PROOF. In view of Remark 2.1, we make use of the version of Theorem 2.1 holding for a centered form of $\eta_n(h_n, t; I)$. By combining (2.2) and (2.3) with (2.20) and (2.21), we see that, under the assumptions of Theorem 2.1, the almost sure limit set in $(B[0,1], \mathscr{U})$ of

$$(3.9)\quad (\mathscr{L}_n(\Xi_n))^{(c)} := \left\{\Xi_n(x)\left(\frac{n^{1/2}}{\log n}\right)\xi_n(h_n, x; I) \colon a\le x\le b\right\}$$

is given by

$$(3.10)\quad (\mathscr{L}_n(\Xi))^{(c)} := \bigcup_{a\le x\le b}\left\{\Xi(x)\left(\frac{l}{1-G(x)}-\gamma f(x)I\right)\colon l\in\Delta_{\gamma f(x)(1-G(x))}\right\}.$$

We choose $\Xi_n \equiv \Xi \equiv 1$ in (3.9) and (3.10) and consider, as in the just-given proof of Corollary 3.2, the functional $l \mapsto \Gamma(l) = \sup_{0\le t\le 1} l(t)$. By combining (3.2) with (3.9) and (3.10), we see that

$$(3.11)\quad \lim_{n\to\infty}\frac{n^{1/2}\Omega_n^+(h_n)}{\log n}=\lim_{n\to\infty}\left\{\sup_{l\in(\mathscr{L}_n(\Xi))^{(c)}}\Gamma(l)\right\}=\sup_{l\in(\mathscr{L}_n(\Xi))^{(c)}}\Gamma(l)\quad \text{a.s.}$$

In view of (3.11), equality (3.7) for Ω_n^+ (and likewise for Ω_n^-) follows readily from the observation that $u(\delta_u^+ - 1)$ is increasing in $u > 0$, in combination with (2.18) in Deheuvels and Mason (1991), which yields that

$$(3.12)\quad \sup\{\pm l(t)\colon l\in\Delta_v\} = \pm vt\delta^{\pm}_{vt}.$$

Equality (3.7) for Ω_n follows similarly by taking $\Gamma(l) = \sup_{0\le t\le 1}|l(t)|$ and making use of the inequality $\delta_u^+ - 1 > 1 - \delta_u^-$ for $u > 0$. Then (3.8) follows along the same lines via (3.12), with $\Gamma(l) = \sup_{0\le t\le 1}(-l(t))$, and making use of the observation that $u(1-\delta_u^-)$ is increasing in $u > 0$. □

3.3. *Nonparametric estimation of the hazard rate function.* Let (F1), (F2) be satisfied, and denote the hazard rate (function) pertaining to F by

$$(3.13)\quad \lambda(x) = \frac{f(x)}{1-F(x)}\quad \text{for } a'\le x\le b'.$$

We will consider the estimator λ_n of λ defined by

$$(3.14)\quad \lambda(x) = \frac{f(x)}{1-F_n(x)},$$

where $f_n(x)$ is as in (1.6) and F_n as in (1.1). Recalling the definition (1.7) of $\widehat{\mathbb{E}}f_n(x)$, we have the following theorem, which largely extends the results of Zhang (1996). Let Ψ and Ψ_n be as in Section 1.

THEOREM 3.1. *Under* (H1)(i), (H3) *or* (H4), (K1)–(K3), (F1), (F2) *and* (C1), *we have*

$$\begin{aligned}\lim_{n\to\infty}&\left\{\frac{nh_n}{2\{\log(1/h_n)+\log_2 n\}}\right\}^{1/2}\sup_{a\le x\le b}\pm\left(\lambda_n(x)-\frac{\widehat{\mathbb{E}}f_n(x)}{1-F(x)}\right)\\ &\times\left\{\Psi_n(x)\times\frac{1-H(x)}{\lambda(x)}\right\}^{1/2}\\ &=\left(\frac{c}{c+1}\right)^{1/2}\left\{\sup_{a\le x\le b}\Psi(x)\right\}^{1/2}\left\{\int_{-\infty}^{\infty}K^2(u)\,du\right\}^{1/2}\quad\text{in probability.}\end{aligned}\tag{3.15}$$

If, in addition, (H1)(ii), (iii) *and* (C2) *hold, then*

$$\begin{aligned}\limsup_{n\to\infty}&\left\{\frac{nh_n}{2\{\log(1,h_n)+\log_2 n\}}\right\}^{1/2}\sup_{a\le x\le b}\pm\left(\lambda_n(x)-\frac{\widehat{\mathbb{E}}f_n(x)}{1-F(x)}\right)\\ &\times\left\{\Psi_n(x)\times\frac{1-H(x)}{\lambda(x)}\right\}^{1/2}\\ &=\left\{\sup_{a\le x\le b}\Psi(x)\right\}^{1/2}\left\{\int_{-\infty}^{\infty}K^2(u)\,du\right\}^{1/2}\quad a.s.\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}\liminf_{n\to\infty}&\left\{\frac{nh_n}{2\{\log(1,h_n)+\log_2 n\}}\right\}^{1/2}\sup_{a\le x\le b}\pm\left(\lambda_n(x)-\frac{\widehat{\mathbb{E}}f_n(x)}{1-F(x)}\right)\\ &\times\left\{\Psi_n(x)\times\frac{1-H(x)}{\lambda(x)}\right\}^{1/2}\\ &=\left(\frac{c}{c+1}\right)^{1/2}\left\{\sup_{a\le x\le b}\Psi(x)\right\}^{1/2}\left\{\int_{-\infty}^{\infty}K^2(u)\,du\right\}^{1/2}\quad a.s.\end{aligned}\tag{3.17}$$

PROOF. An application of the law of the iterated logarithm for α_n [see, e.g., Földes and Rejtő (1981)], in combination with Theorem 1.1, Remark 1.1 and the formal replacement of Ψ_n by

$$\left\{\frac{1-F_n(x)}{1-F(x)}\right\}^2\Psi_n(x),$$

reduces the proof of (3.15)–(3.17) to showing that, under the assumptions of the theorem,

$$\lim_{n\to\infty} n^{-1/2}(\log_2 n)^{1/2}\left\{\frac{nh_n}{\log(1/h_n)+\log_2 n}\right\}^{1/2}$$
$$= \lim_{n\to\infty}\left\{\frac{h_n \log_2 n}{\log(1/h_n)+\log_2 n}\right\}^{1/2} = 0,$$

which is obvious from (H1)(i). □

4. Proofs.

4.1. *Preliminary results and notation.* We will work here under slightly more general assumptions than in the previous sections. As in Section 1, we let X, Y be nonnegative independent random variables, with Y being allowed to be defective [i.e., with $\mathbb{P}(Y=\infty)$ possibly positive], we set $Z = \min(X, Y)$ and $\delta = \mathbb{1}_{\{X\le Y\}}$. Unless otherwise specified, we will allow the distribution functions $F(x) = \mathbb{P}(X \le x)$ and $G(x) = \mathbb{P}(Y \le x)$ to be discontinuous, so that the following conventions will be needed. For any function L, we will set, whenever the corresponding limits exist

$$L(x-) = \lim_{t\uparrow x} L(t) \quad \text{and} \quad L(x+) = \lim_{t\downarrow x} L(t). \tag{4.1}$$

Whenever L is of bounded variation on $[c, d]$, we will use the following convention for the Lebesgue–Stieltjes integral: for any $c \le y \le z \le d$, we set

$$L(z+) - L(y+) = \int_y^z dL(u) = \int_{y+}^{z+} dL(u). \tag{4.2}$$

Recalling that the above defined F, G are right continuous, that is, such that $F(x) = F(x+)$ and $G(x) = G(x+)$, we will set for convenience $F_-(x) = F(x-)$ and $G_-(x) = G(x-)$. Set

$$T_F = \sup\{x\colon F(x) < 1\},\ T_G = \sup\{x\colon G(x) < 1\} \quad \text{and assume}$$
$$\Theta = \min(T_F, T_G) > 0.$$

Throughout the sequel, we will assume the conditions (F1), (F2), which are stated below for convenience:

(F1) $F(0) = G(0) = 0$.
(F2) (i) F and G are continuous on $[a', b']$.
(ii) $f = (d/dx)F$ is defined, continuous and strictly positive on $[a', b']$.

The distribution function of $Z = \min(X, Y)$, denoted by $H(x) = \mathbb{P}(Z \le x) = H(x+)$, may be decomposed into

$$H(x) = 1 - (1 - F(x))(1 - G(x)) = H^{(1)}(x) + H^{(0)}(x), \tag{4.3}$$

where

$$(4.4)\qquad H^{(1)}(x) = \mathbb{P}(Z \le x \text{ and } \delta = 1) = \int_0^x (1 - G_-(t))\, dF(t) = H^{(1)}(x+),$$

$$(4.5)\qquad H^{(0)}(x) = \mathbb{P}(Z \le x \text{ and } \delta = 0) = \int_0^x (1 - F_-(t))\, dG(t) = H^{(0)}(x+).$$

Set $H_\pm^{(0)}(x) = H^{(0)}(x\pm)$ and $H_\pm^{(1)}(x) = H^{(1)}(x\pm)$. We set further

$$(4.6)\qquad p = \mathbb{P}(\delta = 1) = \int_0^\infty (1 - G_-(t))\, dF(t) = H_-^{(1)}(\infty) = 1 - H_-^{(0)}(\infty).$$

Our assumptions $\Theta > 0$ and (F2) exclude $p = 0$, but allow $p = 1$ when $P(Y = \infty) = 1$, that is, when $G(x) = 0$ for all $x \in \mathbb{R}$, which corresponds to uncensored data. In the latter case, the results of this section will turn out to be direct consequences of similar theorems for the uniform empirical process due to Deheuvels and Mason (1992) and Deheuvels (1992). Therefore, we will assume from now on without loss of generality in our proofs that $0 < p < 1$. Keeping in mind that $H^{(1)}(x)$ [resp. $H^{(0)}(x)$] increases from 0 to p (resp. $1 - p$) as x increases from 0 to ∞, denote the quantile functions of $H^{(1)}$ and $H^{(0)}$ by

$$(4.7)\qquad Q^{(1)}(s) = \inf\{x\colon H^{(1)}(x) \ge s\} \quad \text{for } 0 < s < p,$$

$$(4.8)\qquad Q^{(0)}(s) = \inf\{x\colon H^{(0)}(x) \ge s\} \quad \text{for } 0 < s < 1 - p.$$

Let $\{X_n\colon n \ge 1\}$ and $\{Y_n\colon n \ge 1\}$ be two independent sequences of independent and identically distributed random variables with $X = X_1$ and $Y = Y_1$. Set $Z_n = \min(X_n, Y_n)$, $Z = Z_1$, and $\delta_n = \mathbb{1}_{\{X_n \le Y_n\}}$, $\delta = \delta_1$, for $n \ge 1$. For each $n \ge 1$, define the empirical counterparts of H, $H^{(1)}$ and $H^{(0)}$ by

$$(4.9)\qquad H_n(x) = n^{-1} \sum_{i=1}^n \mathbb{1}_{\{Z_i \le x\}} = H^{(1)}(x) + H^{(0)}(x) = 1 - n^{-1} N_{n-}(x),$$

where

$$(4.10)\qquad \begin{aligned} N_n(x) &= \sum_{i=1}^n \mathbb{1}_{\{Z_i \ge x\}} = n(1 - H_{n-}(x)), \\ N_{n-}(x) &= N_n(x-) = \sum_{i=1}^n \mathbb{1}_{\{Z_i > x\}}, \end{aligned}$$

$$(4.11)\qquad \begin{aligned} H_n^{(1)}(x) &= n^{-1} \sum_{i=1}^n \delta_i \mathbb{1}_{\{Z_i \le x\}} \quad \text{and} \\ H_n^{(0)}(x) &= n^{-1} \sum_{i=1}^n (1 - \delta_i) \mathbb{1}_{\{Z_i \le x\}}. \end{aligned}$$

Set $H_{n\pm}(x) = H_n(x\pm)$, $H_{n\pm}^{(1)}(x) = H_n^{(1)}(x\pm)$ and $H_{n\pm}^{(0)}(x) = H_n^{(0)}(x\pm)$. Introduce the empirical cumulated hazard rate function defined by

$$(4.12)\qquad \Lambda_n(x) = \int_0^x \frac{1}{1 - H_{n-}(u)}\, dH_n^{(1)}(u) = \Lambda_n(x+) \quad \text{for } x \ge 0.$$

We note that the true cumulated hazard rate function may be defined, for $x \geq 0$, by

$$\begin{aligned}\Lambda(x) &= \int_0^x \frac{1}{1-F_-(u)}\,dF(u) = \int_0^x \frac{1-G_-(u)}{1-H_-(u)}\,dF(u)\\ &= \int_0^x \frac{1}{1-H_-(u)}\,dH^{(1)}(u).\end{aligned} \tag{4.13}$$

The Kaplan–Meier PL estimators F_n and G_n of F and G based upon $\{(Z_i, \delta_i)\colon 1 \leq i \leq n\}$ are such that [see, e.g., Shorack and Wellner (1986), page 295]

$$\begin{aligned}F_n(x) &= 1 - \prod_{i\colon Z_{i,n}\leq x,\, 1\leq i\leq n}\left(1-\frac{\delta_{i,n}}{n-i+1}\right) = \int_0^x (1-F_{n-}(u))\,d\Lambda_n(u)\\ &= \int_0^x \frac{1-F_{n-}(u)}{1-H_{n-}(u)}\,dH_n^{(1)}(u) = \int_0^x \frac{1}{1-G_{n-}(u)}\,dH_n^{(1)}(u)\end{aligned} \tag{4.14}$$

and likewise

$$\begin{aligned}G_n(x) &= 1 - \prod_{i\colon Z_{i,n}\leq x,\, 1\leq i\leq n}\left(1-\frac{1-\delta_{i,n}}{n-i+1}\right)\\ &= \int_0^x \frac{1}{1-F_n(u)}\,dH_n^{(0)}(u),\end{aligned} \tag{4.15}$$

where we set $F_{n\pm}(x) = F_n(x\pm)$ and $G_{n\pm}(x) = F_n(x\pm)$. Now, introduce the empirical processes $\alpha_n = \alpha_{n+}$ and $\beta_n = \beta_{n+}$ where, for each $n \geq 1$ and $x \in \mathbb{R}$,

$$\begin{aligned}\alpha_{n\pm}(x) &= n^{1/2}(F_{n\pm}(x) - F_{\pm}(x)) \quad \text{and}\\ \beta_{n\pm}(x) &= n^{1/2}(G_{n\pm}(x) - G_{\pm}(x)).\end{aligned} \tag{4.16}$$

Define likewise

$$\mathscr{H}_n^{(j)}(x) = n^{1/2}\big(H_n^{(j)}(x) - H^{(j)}(x)\big) \quad \text{for } j = 0, 1. \tag{4.17}$$

We may write

$$\begin{aligned}\alpha_n(x) &= n^{1/2}(F_n(x) - F(x)) = n^{1/2}\left\{\int_0^x dF_n(u) - \int_0^x dF(u)\right\}\\ &= n^{1/2}\left\{\int_0^x \frac{1}{1-G_{n-}(u)}\,dH_n^{(1)}(u) - \int_0^x \frac{1}{1-G_{n-}(u)}\,dH^{(1)}(u)\right.\\ &\qquad \left. + \int_0^x \frac{1-G_-(u)}{1-G_{n-}(u)}\,dF(u) - \int_0^x \frac{1-G_{n-}(u)}{1-G_{n-}(u)}\,dF(u)\right\}\\ &= \int_0^x \frac{1}{1-G_{n-}(u)}\,d\mathscr{H}_n^{(1)}(u) + \int_0^x \frac{\beta_{n-}(u)}{1-G_{n-}(u)}\,dF(u) =: \alpha_n'(x) + \alpha_n''(x).\end{aligned} \tag{4.18}$$

The following lemma establishes that, in the range of increments which we consider, the oscillations of α_n'' can be neglected.

LEMMA 4.1. *Fix any $0 < R < \Theta$. Assume that there exists a version $f(x) = (d/dx)F(x)$ of the Lebesgue derivative of F uniformly bounded on $[0, R]$. Then, there exists a constant $C_1(R) < \infty$ such that, almost surely for all n sufficiently large and uniformly over all $0 \le s \le t \le R$,*

$$(4.19)\qquad |\alpha_n''(t) - \alpha_n''(s)| = \left|\int_s^t \frac{\beta_{n-}(u)}{1 - G_{n-}(u)}\, dF(u)\right| \le C_1(R)(\log_2 n)^{1/2}|t-s|.$$

PROOF. We recall from the law of the iterated logarithm of Földes and Rejtő (1981) [see also Csörgő and Horváth (1983) and Gu and Lai (1990), (1.15)] that, for any specified $0 \le R < \Theta$,

$$(4.20)\qquad C_2(R) = \limsup_{n\to\infty} (\log_2 n)^{-1/2} \sup_{0\le u\le R} |\beta_n(u)| < \infty \quad \text{a.s.}$$

Set $C_3(R) = \sup_{0\le u\le R} |f(u)|$. Making use of (4.20), we obtain trivially that, almost surely for all large n,

$$\begin{aligned}(\log_2 n)^{-1/2}&\left|\int_s^t \frac{\beta_{n-}(u)}{1 - G_{n-}(u)}\, dF(u)\right| \\ &\le \frac{1}{1 - G_{n-}(R)} \times (\log_2 n)^{-1/2} \sup_{0\le u\le R} |\beta_{n-}(u)| \times \{F(t) - F(s)\} \\ &\le \frac{1}{1-G(R)} \times 2C_2(R)C_3(R) \times |t-s| =: C_1(R)|t-s|,\end{aligned}$$

which is (4.19). □

We will make use of the following fact, stated in (2.13), (2.14), (2.15) in Deheuvels and Einmahl (1996), to evaluate the increments of α_n' in (4.18).

FACT 4.1. On a suitably enlarged probability space $(\Omega, \mathscr{A}, \mathbb{P})$, it is possible to define $\{X_n: n \ge \mathbb{P}\}$ and $\{Y_n: n \ge 1\}$ jointly with a sequence $\{U_n: n \ge 1\}$ of independent random variables with a uniform distribution on $(0, 1)$, such that the following properties hold. For each $n \ge 1$, set

$$(4.21)\qquad U_n(s) = n^{-1}\sum_{i=1}^{n} \mathbb{1}_{\{U_i \le s\}} \quad \text{and} \quad a_n(s) = n^{1/2}(U_n(s) - s) \quad \text{for } s \in \mathbb{R}.$$

We have, almost surely,

$$(4.22)\qquad H_n^{(1)}(x) = U_n(H^{(1)}(x)) \quad \text{for } 0 < H^{(1)}(x) < p$$

and

$$(4.23)\qquad H_n^{(0)}(x) = U_n(H^{(0)}(x) + p) - U_n(p) \quad \text{for } 0 < H^{(0)}(x) < 1 - p.$$

In the remainder of this section, we will work on the probability space of Fact 4.1. Recalling (4.17), we set, in view of (4.21), (4.22), for each $h > 0$,

$$\begin{aligned}\omega_n^{(1)}(h) &= \sup_{\substack{a\le s,\, t\le b'\\ |t-s|\le h}} \left|\mathscr{H}_n^{(1)}(t) - \mathscr{H}_n^{(1)}(s)\right| \\ &= \sup_{\substack{a\le s,\, t\le b'\\ |t-s|\le h}} \left|a_n(H^{(1)}(t)) - a_n(H^{(1)}(s))\right|.\end{aligned} \tag{4.24}$$

In view of (4.18), consider now

$$\begin{aligned}A_{n,1}(s,t) &= \alpha_n'(t) - \alpha_n'(s) - \frac{1}{1-G_-(s)}\int_s^t d\mathscr{H}_n^{(1)}(u) \\ &= \int_s^t \left(\frac{1}{1-G_{n-}(u)} - \frac{1}{1-G_-(s)}\right) d\{\mathscr{H}_n^{(1)}(u) - \mathscr{H}_n^{(1)}(s)\} \\ &= \left(\frac{1}{1-G_{n-}(t)} - \frac{1}{1-G_-(s)}\right)\{\mathscr{H}_n^{(1)}(t) - \mathscr{H}_n^{(1)}(s)\} \\ &\quad - \int_s^t \{\mathscr{H}_n^{(1)}(u) - \mathscr{H}_n^{(1)}(s)\}\, d\left\{\frac{1}{1-G_{n-}(u)}\right\}.\end{aligned} \tag{4.25}$$

LEMMA 4.2. *Assume that* (F2)(i) *holds. Then, there exists a function $C_4(h) \to 0$ as $h \to 0$, together with a constant C_5 such that, almost surely for all n sufficiently large,*

$$\sup_{\substack{a\le s,\, t\le b'\\ |t-s|\le h}} |A_{n,1}(s,t)| \le \omega_n^{(1)}(h) \times \left\{C_5 n^{-1/2}(\log_2 n)^{1/2} + C_4(h)\right\}. \tag{4.26}$$

PROOF. Making use of the assumption (F2)(i) of continuity of G, we see that

$$C_4(h) := 2 \sup_{\substack{a\le s,\, t\le b'\\ |t-s|\le h}} \left|\frac{1}{1-G_-(t)} - \frac{1}{1-G_-(t)}\right| \to 0 \quad \text{as } h \to 0. \tag{4.27}$$

By combining (4.20) and (4.24), we obtain readily that there exists a constant C_5 such that, almost surely for all n sufficiently large,

$$\sup_{a\le t\le b'} \left|\frac{1}{1-G_{n-}(t)} - \frac{1}{1-G_-(t)}\right| \le (1/3)C_5 n^{-1/2}(\log_2 n)^{1/2}. \tag{4.28}$$

Thus, by combining (4.20) with (4.27) and (4.28), we see that, almost surely for all large n,

$$\begin{aligned}
&\sup_{\substack{a\le s,\, t\le b' \\ |t-s|\le h}} \left| \left(\frac{1}{1-G_{n-}(t)} - \frac{1}{1-G_{-}(s)} \right) \{ \mathscr{K}_n^{(1)}(t) - \mathscr{K}_n^{(1)}(s) \} \right| \\
&\quad \le \omega_n^{(1)}(h) \times \left\{ \sup_{a\le t\le b'} \left| \frac{1}{1-G_{n-}(t)} - \frac{1}{1-G_{-}(t)} \right| \right. \\
&\qquad \left. + \sup_{\substack{a\le s,\, t\le b' \\ |t-s|\le h}} \left| \frac{1}{1-G_{-}(t)} - \frac{1}{1-G_{-}(t)} \right| \right\} \\
&\quad \le \omega_n^{(1)}(h) \times \left\{ (1/3)C_5 n^{-1/2} (\log_2 n)^{1/2} + (1/2) C_4(h) \right\}.
\end{aligned} \tag{4.29}$$

Next, we observe that

$$\begin{aligned}
&\sup_{\substack{a\le s,\, t\le b' \\ |t-s|\le h}} \left| \int_s^t \{ \mathscr{K}_n^{(1)}(u) - \mathscr{K}_n^{(1)}(s) \} \, d\left\{ \frac{1}{1-G_{n-}(t)} \right\} \right| \\
&\quad \le \omega_n^{(1)}(h) \times \sup_{\substack{a\le s,\, t\le b' \\ |t-s|\le h}} \left| \frac{1}{1-G_{n-}(t)} - \frac{1}{1-G_{n-}(s)} \right| \\
&\quad \le \omega_n^{(1)}(h) \times \left\{ (2/3)C_5 n^{-1/2} (\log_2 n)^{1/2} + (1/2) C_4(h) \right\}.
\end{aligned} \tag{4.30}$$

We conclude (4.26) by combining (4.29) with (4.30). □

Set

$$\begin{aligned}
\xi_n^{(1)}(h, t; s) &= \frac{1}{1-G_{-}(t)} \left\{ \mathscr{K}_n^{(1)}(t+hs) - \mathscr{K}_n^{(1)}(t) \right\} \\
&= \frac{1}{1-G_{-}(t)} \left\{ \alpha_n(H^{(1)}(t+hs)) - \alpha_n(H^{(1)}(t)) \right\}.
\end{aligned} \tag{4.31}$$

The following lemma combines Lemmas 4.1 and 4.2. In the statement of this result, the constant $C_1(R)$ is defined as in Lemma 4.1 for $0 < R < \Theta$, whereas C_5 and $C_4(h)$ are as in Lemma 4.2.

LEMMA 4.3. *Assume that* (F2)(i) *holds. Then, there exist constants $C_1 = C_1(b)$ and C_5, and there exists a function $C_4(h) \to 0$ as $h \to 0$ such that the following property holds. There exists almost surely an $n_0 < \infty$ such that, for all $n \ge n_0$ and $h > 0$,*

$$\begin{aligned}
&\sup_{a\le t\le b} \| \xi_n(h, t; I) - \xi_n^{(1)}(h, t; I) \| \\
&\quad \le \omega_n^{(1)}(h) \left\{ C_5 n^{-1/2} (\log_2 n)^{1/2} + C_4(h) \right\} + C_1 h (\log_2 n)^{1/2}.
\end{aligned} \tag{4.32}$$

For the proof, combine (4.19) and (4.26) with (4.25) and (4.31).

In the following sections, we will make use of Lemma 4.3 to show that (4.32) allows a formal replacement of $\xi_n(h,t;s)$ by $\xi_n^{(1)}(h,t;s)$ in the proofs of our results.

4.2. *Proofs of Theorems* 1.2 *and* 1.1. This subsection is devoted to the proofs of Theorems 1.2 and 1.1, so that we assume throughout, unless otherwise specified, that (F1), (F2), (H1) and either (H3) or (H4) hold. We will work on the probability space of Fact 4.1, and make an instrumental use of the following useful facts which combine results from Stute (1982a), Mason, Shorack and Wellner (1983), Deheuvels and Mason (1992) and Deheuvels (1992). Recalling the notation (1.16) and (4.21), we consider the random sets of increment functions defined, for $0 \le c_1 \le c_2 \le 1$, $n \ge 1$ and $\lambda > 0$, by

$$\mathscr{E}_n(c_1, c_2; \lambda) = \left\{ b_n^{-1} \zeta_n(\lambda h_n, t; I) \colon c_1 \le t \le c_2 \right\}, \tag{4.33}$$

where $b_n = (2h_n\{\log(1/h_n) + \log_2 n\})^{1/2}$ is as in (3.4), and where we set, for $s, t \in \mathbb{R}$ and $h > 0$,

$$\zeta_n(h, t; s) = a_n(t + hs) - a_n(t). \tag{4.34}$$

For each $h > 0$, set

$$\omega_n(h) = \sup_{\substack{a \le s,\, t \le 1 \\ |s-t| \le h}} |a_n(t) - a_n(s)|. \tag{4.35}$$

FACT 4.2. Assume that (H1) and (H3) or (H4) hold. Then, for any $\lambda > 0$,

$$\limsup_{n\to\infty} b_n^{-1} \omega_n(\lambda h_n) = \lambda^{1/2} \quad \text{a.s.} \tag{4.36}$$

FACT 4.3. Let (H1)(i) and (H3) or (H4) be satisfied. Then, for any $0 \le c_1 < c_2 \le 1$ and $\lambda > 0$, we have

$$\lim_{n\to\infty} \mathscr{D}_{\mathscr{Q}}(\mathscr{E}_n(c_1, c_2; \lambda), \mathscr{S}_{\lambda c/(c+1)}) = 0 \quad \text{in probability.} \tag{4.37}$$

If, in addition, (H1)(ii) holds, then, for any $\varepsilon > 0$, almost surely for all n sufficiently large,

$$\mathscr{S}_{\lambda c/(c+1)} \subseteq \mathscr{E}_n(c_1, c_2; \lambda)^{\varepsilon} \quad \text{and} \quad \mathscr{E}_n(c_1, c_2; \lambda) \subseteq \mathscr{S}_{\lambda}^{\varepsilon}. \tag{4.38}$$

Moreover, for each $l \in \mathscr{S}_\lambda$, we have, infinitely often with probability 1,

$$l \in \mathscr{E}_n(c_1, c_2; \lambda)^{\varepsilon}. \tag{4.39}$$

LEMMA 4.4. *Assume that* (H1) *and* (H3) *or* (H4) *hold. Then, under* (F1), (F2),

$$\lim_{n\to\infty} b_n^{-1} \sup_{a \le t \le b} \| \xi_n(h_n, t; I) - \xi_n^{(1)}(h_n, t; I) \| = 0 \quad \text{a.s.} \tag{4.40}$$

PROOF. Setting $D = \max_{a\le t\le b'} f(t)(1-G(t))$, we have uniformly over $a \le s, t \le b'$,

$$|H^{(1)}(t) - H^{(1)}(s)| \le D|t-s|.$$

This inequality, in combination with (4.24) and (4.35), implies that, for all large n,

$$\omega_n^{(1)}(h_n) \le \omega_n(Dh_n)$$

whence, by (4.36),

$$\limsup_{n\to\infty} b_n^{-1}\omega_n^{(1)}(h_n) \le D^{1/2} \quad \text{a.s.}$$

By combining this last inequality with (4.32), and the observation that, under our assumptions,

$$C_5 n^{-1/2}(\log_2 n)^{1/2} + C_4(h_n) \to 0,$$

and, via (3.4),

$$b_n^{-1}C_1 h_n(\log_2 n)^{1/2} = O(h_n^{1/2}) \to 0,$$

we conclude readily (4.40). □

Let $N \ge 1$ be an arbitrary, but fixed, integer which will be specified later on. For $1 \le i \le N$, set $t_{i,N} = a + (i-1)N^{-1}(b-a)$. For each $1 \le i \le N$, set $\lambda_{i,N} = f(t_{i,N})(1-G(t_{i,N}))$, and, for $t \in [t_{i,N}, t_{i+1,N}]$,

$$(4.41)\qquad \begin{aligned}\xi_{n,N}^{(1)}(h_n, t; s) &= \frac{1}{1-G(t)}\Big\{\alpha_n\Big(H^{(1)}(t) + sh_n f(t_{i,N})(1-G(t_{i,N}))\Big) \\ &\qquad - \alpha_n\Big(H^{(1)}(t)\Big)\Big\} \\ &= \frac{1}{1-G(t)}\zeta_n(\lambda_{i,N}h_n, H^{(1)}(t); I).\end{aligned}$$

LEMMA 4.5. *Assume that* (H1) *and* (H3) *or* (H4) *hold. Then, under* (F1), (F2) *for any $\varepsilon > 0$, there exists an $N_0 = N_0(\varepsilon) < \infty$, such that, for all $N \ge N_0$,*

$$(4.42)\qquad \limsup_{n\to\infty} b_n^{-1} \sup_{a\le t\le b} \|\xi_{n,N}^{(1)}(h_n, t; I) - \xi_n^{(1)}(h_n, t; I)\| \le \varepsilon \quad a.s.$$

PROOF. Set

$$e_N = \max_{1\le i\le N}\left(\sup_{t_{i,N}\le t\le t_{i+1,N}+h_n} \Big|f(t)(1-G(t)) - f(t_{i,N})(1-G(t_{i,N}))\Big|\right),$$

and observe from the mean value theorem that, for all $1 \le i \le N$, $t \in [t_{i,N}, t_{t+1,N}]$, and $s \in [0,1]$, for all large n,

$$|H^{(1)}(t + h_n s) - \{H^{(1)}(t) + sh_n f(t_{i,N})(1-G(t_{i,N}))\}| \le e_N h_n.$$

This, in turn, implies that

$$\limsup_{n\to\infty} b_n^{-1} \sup_{a\le t\le b} \left\| \xi_n^{(1)}(h_n, t; I) - \xi_{n,N}^{(1)}(h_n, t; I) \right\|$$
$$\le \sup_{a\le t\le b} \left\{ \frac{1}{1-G(t)} \right\} \limsup_{n\to\infty} b_n^{-1} \omega_n(e_N h_n) = \left\{ \frac{1}{1-G(b)} \right\} e_N^{1/2} \quad \text{a.s.}$$

Since a choice of N sufficiently large ensures e_N to be as small as desired, we conclude (4.42). □

Let R denote a continuous and (strictly) positive function on $[a', b']$. Set

$$R_{i,N} = \inf_{t_{i,N}\le t\le t_{i+1,N}} \frac{R(t)}{1-G(t)} \quad \text{and} \quad S_{i,N} = \sup_{t_{i,N}\le t\le t_{i+1,N}} \frac{R(t)}{1-G(t)}.$$

So that, if we let $\mathscr{M}_N^{\pm}$ and $\mathscr{M}$ be defined by

$$\mathscr{M}_N^- = \max_{1\le i\le N} R_{i,N} \lambda_{i,N}^{1/2}, \qquad \mathscr{M}_N^+ = \max_{1\le i\le N} S_{i,N} \lambda_{i,N}^{1/2}, \tag{4.43}$$

$$\mathscr{M} = \sup_{a\le t\le b} \left\{ \frac{R(t)}{1-G(t)} \right\} \left\{ f(t)(1-G(t)) \right\}^{1/2} = \sup_{a\le t\le b} R(t) \left\{ \frac{f(t)}{1-G(t)} \right\}^{1/2}, \tag{4.44}$$

It is straightforward that, as $N \to \infty$,

$$|\mathscr{M}_N^{\pm} - \mathscr{M}| \to 0. \tag{4.45}$$

Set

$$\mathscr{K}_{n,N} = \left\{ R(t) b_n^{-1} \xi_{n,N}^{(1)}(h_n, t; I) \colon a \le t \le b \right\}. \tag{4.46}$$

LEMMA 4.6. *Assume that* (H1) *and* (H3) *or* (H4) *hold. Then, under* (F1), (F2) *for any* $\varepsilon > 0$, *there exists an* $N_1 = N_1(\varepsilon) < \infty$, *such that for all* $N \ge N_1$, *almost surely for all n sufficiently large,*

$$\mathscr{M}\mathscr{S}_{c/(c+1)} \subseteq \mathscr{K}_{n;N}^{\varepsilon} \quad \textit{and} \quad \mathscr{K}_{n;N} \subseteq (\mathscr{M}\mathscr{S})^{\varepsilon}. \tag{4.47}$$

Moreover, for any $l \in \mathscr{S}$, *we have, infinitely often with probability* 1,

$$\mathscr{M} l \in \mathscr{K}_{n;N}^{\varepsilon}. \tag{4.48}$$

PROOF. Recall (4.34) and (4.41). Fix an arbitrary $\varepsilon > 0$. For any fixed $1 \le i \le N$, set $c_1 = H^{(1)}(t_{i,N})$, $c_2 = H^{(1)}(t_{i+1,N})$, $\rho = \max_{1\le i\le N} S_{i,N}$,

$$\mathscr{E}_{n,i,N} = \{ b_n^{-1} \zeta_n(\lambda_{i,N} h_n, u; I) \colon c_1 \le u \le c_2 \}$$

and

$$\mathscr{K}_{n,i,N} = \left\{ R(t) b_n^{-1} \xi_{n,N}^{(1)}(h_n, t; I) \colon t_{i,N} \le t \le t_{i+1,N} \right\} \tag{4.49}$$

$$= \left\{ \frac{R(t)}{1-G(t)} b_n^{-1} \zeta_n(\lambda_{i,N} h_n, H^{(1)}(t); I) \colon t_{i,N} \le t \le t_{i+1,N} \right\}. \tag{4.50}$$

Our assumptions [in particular (F2)(ii)] imply that $0 \le c_1 < c_2 \le p \le 1$ [recall (4.6)]. Thus by (4.38) there exists almost surely an $n_0 = n_0(\varepsilon, i, N)$ such that, for all $n \ge n_0$,

$$\lambda_{i,N}^{1/2}\mathscr{S}_{c/(c+1)} \subseteq \mathscr{E}_{n;i,N}^{\varepsilon/(2\rho)} \quad \text{and} \quad \mathscr{E}_{n;i,N} \subseteq \left(\lambda_{i,N}^{1/2}\mathscr{S}\right)^{\varepsilon/(2\rho)},$$

and hence, that

$$\mathscr{R}_{i,N}\lambda_{i,N}^{1/2}\mathscr{S}_{c/(c+1)} \subseteq \mathscr{K}_{n;i,N}^{\varepsilon/2} \quad \text{and} \quad \mathscr{K}_{n;i,N} \subseteq \left(\mathscr{S}_{i,N}\lambda_{i,N}^{1/2}\mathscr{S}\right)^{\varepsilon/2}.$$

Here, we have made use of the fact that, for any $r > 0$, $rA^\varepsilon = (rA)^{r\varepsilon}$. Since $\mathscr{K}_{n;N} = \bigcup_{i=1}^N \mathscr{K}_{n;i,N}$, it follows that, for all $n \ge n_0$,

$$\text{(4.51)} \qquad \mathscr{M}_N^- \mathscr{S}_{c/(c+1)} \subseteq \mathscr{K}_{n;N}^{\varepsilon/2} \quad \text{and} \quad \mathscr{K}_{n;N} \subseteq (\mathscr{M}_N^+\mathscr{S})^{\varepsilon/2}.$$

Since $l \in \mathscr{S}_\lambda \Rightarrow \|l\| \le \lambda^{1/2}$, it is straightforward that

$$\mathscr{D}_{\mathscr{Q}}(\mathscr{M}_N^\pm \mathscr{S}_\lambda, \mathscr{M}\mathscr{S}_\lambda) \le |\mathscr{M}_N^\pm - \mathscr{M}|\lambda^{1/2}.$$

By combining this inequality, taken with either $\lambda = 1$ or $\lambda = c/(c+1)$, with (4.46) and (4.51), we obtain readily (4.47).

For the proof of (4.48), we select an arbitrary $l \in \mathscr{S}$, then make use of (4.43), to show that we have, infinitely often (in n) with probability 1,

$$\lambda_{i,N}^{1/2} l \in \mathscr{E}_{n;i,N}^{\varepsilon/(2\rho)}$$

and hence

$$\mathscr{M}_N^- l = R_{i,N}\lambda_{i,N}^{1/2} l \in \mathscr{K}_{n;i,N}^{\varepsilon/2} \subseteq \mathscr{K}_{n;N}^{\varepsilon/2}.$$

We conclude (4.48) by choosing N so large that $|\mathscr{M}_N^- - \mathscr{M}| < \varepsilon/2$. □

Proof of Theorem 1.2. We have now all the ingredients in hand to prove Theorem 1.2. First, we let Ψ in (C1), (C2) and R be related via

$$\text{(4.52)} \quad R(t) = \left\{\Psi(t) \times \frac{1 - G(t)}{f(t)}\right\}^{1/2} \quad \Leftrightarrow \quad \Psi(t) = R^2(t) \times \left\{\frac{f(t)}{1 - G(t)}\right\}.$$

This shows, via (4.44) and (1.23), that

$$M = \sup_{a \le t \le b} \Psi(t) = \mathscr{M}^2.$$

We will omit the proof of the "in probability part" in (1.24), since it is similar to but easier than the proof of the "almost sure part," which we present now.

We combine (1.18), (4.40), (4.42), (4.44), (4.46) and (4.47) to show that, for any $\varepsilon > 0$, a choice of N sufficiently large ensures that, almost surely for all large n,

$$\mathscr{K}_n^+(\Psi) \subseteq (\mathscr{M}\mathscr{S})^\varepsilon = \mathscr{S}_M^\varepsilon \quad \text{and} \quad \mathscr{M}\mathscr{S}_{c/(c+1)} = \mathscr{S}_{Mc/(c+1)} \subseteq \mathscr{K}_n^+(\Psi)^\varepsilon.$$

Moreover, by (4.48), it holds that, for any $l \in \mathscr{S}$ and $\varepsilon > 0$, we have almost surely,

$$\mathscr{M}l = M^{1/2}l \in \mathscr{K}_n^+(\Psi)^\varepsilon \quad \text{infinitely often}$$

in the "+" case with $\mathscr{K}_n^+(\Psi_n)$ replaced by $\mathscr{K}_n^+(\Psi)$. From (C1) or (C2) it is immediate that the theorem holds for $\mathscr{K}_n^+(\Psi_n)$ itself. The proof for the "−" case follows along the same lines and will be omitted.

PROOF OF THEOREM 1.1. First note that it follows immediately from (1.8) that

$$\left\{\frac{nh_n}{2\{\log(1, h_n) + \log_2 n\}}\right\}^{1/2} \pm (f_n(x) - \widehat{\mathbb{E}}f_n(x))\left\{\Psi_n(x) \times \frac{1-G(x)}{f(x)}\right\}^{1/2}$$
$$= \int_{-T}^{T} \mp\{2h_n(\log(1/h_n) + \log_2 n)\}^{-1/2}\xi_n(h_n, x; u)$$
$$\times \left\{\Psi_n(x) \times \frac{1-G(x)}{f(x)}\right\}^{1/2} dK(u).$$

Define Γ: $l \in B[-T, T] \cap M[-T, T] \mapsto \mathbb{R}$ (here, $M[-T, T]$ stands for the set of measurable functions on $[-T, T]$) by

$$\Gamma(l) = -\int_{-T}^{T} l(u)\, dK(u)$$

and let $\mathscr{B}_n = \mathscr{K}_n^\pm(\Psi_n)$. Then the versions of Theorem 1.2 and Proposition 3.1 obtained with the formal replacements of $B[0, 1]$ by $B[-T, T]$, yield the statements in (1.11) and (1.12), once we show that for $\eta \geq 0$,

$$\sup_{l\in S_\eta}\left\{-\int_{-T}^{T} l(u)\, dK(u)\right\} = \left\{\eta\int_{-T}^{T} K^2(u)\, du\right\}^{1/2}.$$

This, however, is well known [see, e.g., Section 4.2 in Deheuvels and Mason (1992)]. Likewise (1.24) implies (1.10); we omit details.

4.3. *Proof of Theorem* 2.1. We consider here a sequence of numbers $\{h_n: n \geq 1\}$ fulfilling condition (H5) with $\gamma > 0$, that is, such that, as $n \to \infty$,

$$nh_n/\log n \to \gamma \in (0, \infty). \tag{4.53}$$

We will again work on the probability space of Fact 4.1 and we will use the following fact from Mason, Shorack and Wellner (1983) and Deheuvels and Mason (1992). Recalling (4.21), define for $0 \leq c_1 < c_2 \leq 1$, $n \geq 1$ and $\lambda > 0$,

$$\tilde{\mathscr{E}}_n(c_1, c_2; \lambda) = \left\{\frac{n}{\log n}(U_n(t + h_n s) - U_n(t)) \colon c_1 \leq t \leq c_2\right\}$$

and

$$\tilde{\omega}_n(h) = \sup_{\substack{0\leq s<t\leq 1\\ t-s\leq h}} (U_n(t) - U_n(s)).$$

FACT 4.4. If (4.53) holds, then for any $0 \le c_1 < c_2 \le 1$ and $\lambda > 0$, we have

$$\lim_{n\to\infty} \mathscr{D}_{\mathscr{Q}}(\widetilde{\mathscr{E}_n}(c_1, c_2; \lambda), \Delta_{\lambda\gamma}) = 0 \quad \text{a.s.} \tag{4.54}$$

In particular,

$$\lim_{n\to\infty} \frac{n}{\log n} \tilde{\omega}_n(\lambda h_n) = \lambda\gamma\delta^+_{\lambda\gamma}, \tag{4.55}$$

with δ_c^+ as in (2.13).

The proof of Theorem 2.1 essentially follows the lines of the proof of Theorem 1.2 and will therefore not be given in full detail. The main difference with Theorem 1.2 is that in $\mathscr{L}_n(\Xi_n)$, F_n is not centered by F. Write

$$\begin{aligned} \eta_n^{(1)}(h, t; s) &= \frac{1}{1 - G_-(t)}\left\{H_n^{(1)}(t + hs) - H_n^{(1)}(t)\right\} \\ &= \frac{1}{1 - G_-(t)}\left\{U_n(H^{(1)}(t + hs)) - U_n(H^{(1)}(t))\right\}. \end{aligned} \tag{4.56}$$

Now under (F1), (F2) it can be shown along the same lines as Lemmas 4.1–4.4 that

$$\lim_{n\to\infty} \frac{n}{\log n} \sup_{a\le t\le b} \|\eta_n(h_n, t; I) - \eta_n^{(1)}(h_n, t; I)\| = 0 \quad \text{a.s.} \tag{4.57}$$

So it suffices to study $\eta_n^{(1)}$ instead of η_n for the present theorem.

PROOF OF THEOREM 2.1. We first show that for any $\varepsilon > 0$, there exists almost surely a finite $N(\varepsilon)$ such that for all $n \ge N(\varepsilon)$

$$\mathscr{L}_n(\Xi) \subseteq \mathscr{L}(\Xi)^{\varepsilon}. \tag{4.58}$$

As in Lemma 4.5 let $N \ge 1$ be an arbitrary but fixed integer. Set $t_{i,N} = a + (i-1)N^{-1}(b-a)$, $\lambda_{i,N} = f(t_{i,N})(1 - G(t_{i,N}))$, for $1 \le i \le n$, and write for $t \in [t_{i,N}, t_{i+1,N}]$, and $s \in \mathbb{R}$,

$$\eta_{n,N}^{(1)}(h_n, t; s) = \frac{1}{1 - G(t_{i,N})}\left\{U_n(H^{(1)}(t) + sh_n\lambda_{i,N}) - U_n(H^{(1)}(t))\right\}. \tag{4.59}$$

Write again $D = \max_{a\le t\le b'} f(t)(1 - G(t))$. Note that under (F1), (F2) for any $\tau > 0$, there exists an $N_0 = N_0(\varepsilon) < \infty$, such that, with $e_n \to 0$, for all $N \ge N_0$,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{n}{\log n} \sup_{a\le t\le b} \|\eta_{n,N}^{(1)}(h_n, t; I) - \eta_n^{(1)}(h_n, t; I)\| \\ &\quad\le \frac{1}{1 - G(b)} \limsup_{n\to\infty} \frac{n}{\log n} \tilde{\omega}_n(e_N h_n) \\ &\qquad + \max_{1\le i\le N} \frac{G(t_{i+1,N}) - G(t_{i,N})}{(1 - G(b))^2} \limsup_{n\to\infty} \frac{n}{\log n} \tilde{\omega}_n(Dh_n) \\ &\quad\le \frac{1}{1 - G(b)} \gamma e_N \delta^+_{\gamma e_N} + \max_{1\le i\le N} \frac{G(t_{i+1,N}) - G(t_{i,N})}{(1 - G(b))^2} D\gamma\delta^+_{D\gamma} \le \tau \quad \text{a.s.,} \end{aligned} \tag{4.60}$$

since $\lim_{c\downarrow 0} c\delta_c^+ = 0$, $\lim_{N\to\infty} e_N = 0$, and since G is uniformly continuous on $[a, b]$.

For any fixed $1 \le i \le N$ write

$$\tilde{\mathscr{E}}_{n,i,N} = \left\{\frac{n}{\log n}\{U_n(t + \lambda_{i,N} h_n s) - U_n(t)\}: c_1 \le t \le c_2\right\},$$

with as before $c_1 = H^{(1)}(t_{i,N})$, $c_2 = H^{(1)}(t_{i+1,N})$, and

$$\begin{aligned}\mathscr{L}_{n;i,N} &= \left\{\Xi(t_{i,N})\frac{n}{\log n}\eta_{n,N}^{(1)}(h_n, t; I): t_{i,N} \le t \le t_{i+1,N}\right\}\\ &= \left\{\frac{\Xi(t_{i,N})}{1 - G(t_{i,N})}\frac{n}{\log n}\{U_n(H^{(1)}(t) + Ih_n\lambda_{i,N})\right.\\ &\qquad\qquad \left. - U_n(H^{(1)}(t))\}: t_{i,N} \le t \le t_{i+1,N}\right\}.\end{aligned}$$

Consider

$$\mathscr{L}_{n;N} = \left\{\Xi(t_{i,N})\frac{n}{\log n}\eta_{n,N}^{(1)}(h_n, t; I): a \le t \le b\right\},$$

with $t_{i,N}$ such that $t_{i,N} \le t \le t_{i+1,N}$. Now by Fact 4.4 we have almost surely for all large n,

$$\tilde{\mathscr{E}}_{n;i,N} \subseteq \Delta_{\gamma\lambda_{i,N}}^{\varepsilon/2\bar{\rho}},$$

with $\bar{\rho} = \max_{1\le i\le N} \widetilde{S}_{i,N}$, $\widetilde{S}_{i,N} = \Xi(t_{i,N})/(1 - G(t_{i,N}))$, and hence

$$\mathscr{L}_{n;i,N} \subseteq \widetilde{S}_{i,N}\Delta_{\gamma\lambda_{i,N}}^{\varepsilon/2\bar{\rho}}.$$

Since $\mathscr{L}_{n;N} \subseteq \bigcup_{i=1}^N \mathscr{L}_{n;i,N}$, this implies

$$\mathscr{L}_{n;N} \subseteq \bigcup_{i=1}^{N} \mathscr{S}_{i,N}\Delta_{\gamma\lambda_{i,N}}^{\varepsilon/2\bar{\rho}} \subseteq \mathscr{L}(\Xi)^{\varepsilon/2}. \tag{4.61}$$

Combining (4.57), (4.60) and (4.61) yields (4.58).

Next we will show that for every $\varepsilon > 0$, there exists almost surely a finite $N(\varepsilon)$ such that for all $n \ge N(\varepsilon)$,

$$\mathscr{L}(\Xi) \subseteq \mathscr{L}_n(\Xi)^{\varepsilon}. \tag{4.62}$$

Since $\mathscr{L}(\Xi)$ is compact, it suffices to show that we have for an arbitrary $l \in \mathscr{L}(\Xi)$ that $l \in \mathscr{L}_n(\Xi)^{\varepsilon}$. Let $t_0 \in [a, b]$ be such that $l \in (\Xi(t_0)/(1 - G(t_0)))\Delta_{\gamma f(t_0)(1-G(t_0))}$. With $N \ge 1$ as before let $I_{0,N}$ be an interval of length $1/N$ having t_0 as one of the endpoints. Now combining (4.57) and a slight modification of (4.60) we obtain for any $\tau > 0$ for N large enough,

$$\lim_{n\to\infty}\frac{n}{\log n}\sup_{t\in I_{0,N}}\|\eta_n(h_n, t; I) - \eta_{n,N}^{(1)}(h_n, t; I)\| \le \tau \quad \text{a.s.,} \tag{4.63}$$

with the formal replacements of $\lambda_{i,N}$ by $\lambda_0 = f(t_0)(1 - G(t_0))$ and $t_{i,N}$ by t_0 in (4.59). Now (4.62) easily follows from Fact 4.4.

Finally from (X1) it is immediate that in (4.58) and (4.62), $\mathscr{L}_n(\Xi)$ can be replaced by $\mathscr{L}_n(\Xi_n)$. $\square$

4.4. *Proofs of Theorems* 2.2 *and* 2.3. The proofs of Theorems 2.2 and 2.3 will make an instrumental use of Theorems 1 and 3, respectively, of Deheuvels (1996). We only present a short proof of Theorem 2.2. The proof of Theorem 2.3 follows along the same lines and will therefore be omitted.

PROOF OF THEOREM 2.2. Similarly to the previous proofs we can show that it suffices to prove Theorem 2.2 with $\eta_n(h_n, t; I)$ replaced by $\eta_n^{(1)}(h_n, t; I)$, as defined in (4.56). We now have from Theorem 1 in Deheuvels (1996), with D as before,

$$\limsup_{n\to\infty} \sup_{a\le t\le b} \frac{n}{\log n} \log \frac{\log n}{nh_n} \Xi_n(t) \frac{1}{1-G(t)} \Big\{ U_n(H^{(1)}(t+h_n)) - U_n(H^{(1)}(t)) \Big\}$$
$$\le \limsup_{n\to\infty} \left\{ \sup_{a\le t\le b} \frac{\Xi_n(t)}{1-G(t)} \right\} \sup_{a\le t\le b} \frac{n}{\log n} \log \frac{\log n}{nh_n}$$
$$\times \Big\{ U_n(H^{(1)}(t) + Dh_n) - U_n(H^{(1)}(t)) \Big\}$$
$$\le \limsup_{n\to\infty} \left\{ \sup_{a\le t\le b} \frac{\Xi_n(t)}{1-G(t)} \right\} \sup_{0\le t\le 1-Dh_n} \frac{n}{\log n} \log \frac{\log n}{nh_n}$$
$$\times \Big\{ U_n(u + Dh_n) - U_n(u) \Big\} \le T \quad \text{a.s}$$

This proves that for any $\varepsilon > 0$, there exists almost surely an $n(\varepsilon)$ such that for all $n \ge n(\varepsilon)$,

$$\mathscr{L}^{[s]}(\Xi_n) \subseteq (T\mathbb{L}^{[s]})^{[\varepsilon]}.$$

So it remains to show that for any $\varepsilon > 0$, there exists a.s. an $n(\varepsilon)$ such that for all $n \ge n(\varepsilon)$,

$$(T\mathbb{L}^{[s]}) \subseteq \mathscr{L}^{[s]}(\Xi_n)^{[\varepsilon]}. \tag{4.64}$$

It is obvious again from Theorem 1 in Deheuvels (1996), (X1) and the continuity of Ξ and G that we have for any t_0 with $\Xi(t_0)/(1 - G(t_0)) = T$ that

$$\lim_{\lambda\downarrow 0} \limsup_{n\to\infty} \sup_{t\in[a,b]\cap[t_0-\lambda,\, t_0+\lambda]} \frac{n}{\log n} \log \frac{\log n}{nh_n} \Big\| \Xi_n(t)\eta_n^{(1)}(h_n, t; I)$$
$$- \frac{\Xi(t_0)}{1-G(t_0)} \Big\{ U_n(H^{(1)}(t+Ih_n)) - U_n(H^{(1)}(t)) \Big\} \Big\| = 0 \quad \text{a.s.}$$

Therefore it suffices to show (4.64) with $\mathscr{L}^{[s]}(\Xi_n)^{[\epsilon]}$ replaced by

$$\left\{\frac{\Xi(t_0)}{1-G(t_0)}\frac{n}{\log n}\log\frac{\log n}{nh_n}\Big\{U_n(H^{(1)}(t+Ih_n)) - U_n(H^{(1)}(t))\Big\}: t\in[a,b]\cap[t_0-\lambda, t_0+\lambda]\right\}^{[\varepsilon/2]},$$

for some properly chosen small $\lambda > 0$. This follows however from observing (by inspection of the proof) that the corresponding result for the uniform-(0,1) distribution, that is, Theorem 1 in Deheuvels (1996), immediately generalizes to a distribution with a continuous density bounded away from 0 and ∞ on a fixed closed interval. Because of (F2) and $G(b) < 1$ we indeed have that $f(1-G)$ satisfies this condition on the interval $[a,b]\cap[t_0-\lambda, t_0+\lambda]$. □

Acknowledgment. We thank the referee for a careful reading of our manuscript and insightful comments.

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