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Supporting points processes

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SUPPORTING POINTS PROCESSES AND SOME OF THEIR APPLICATIONS

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ABSTRACT. We introduce a stochastic point process of $\mathcal{S}$-supporting points and prove that upon rescaling it converges to a Gaussian field. The notion of $\mathcal{S}$-supporting points specializes (for adequately chosen $\mathcal{S}$) to Pareto (or, more generally, cone) extremal points or to vertices of convex hulls or to centers of generalized Voronoi tessellations in the models of large scale structure of the Universe based on Burgers equation. The central limit theorems proven here imply i.a. the asymptotic normality for the number of convex hull vertices in large Poisson samples from a simple polyhedron or for the number of Pareto (vector extremal) points in Poisson samples with independent coordinates.

0. INTRODUCTION

0.1. Imagine a sea bottom which someone attempts to measure using a measuring rod. If the rod is thick, the measuring is not ideal, and not all points at the bottom will be touched. In this paper we deal with the model in which the "bottom" is a realization of some Poisson point process, homogeneous in the horizontal direction. The points that *are* touched form a certain new point process and its properties are the subject of this paper. The Figure 1 illustrates the model. We will call the point process loosely described above *the supporting points process.*

This model seems to carry some aesthetic appeal by itself. Our interest in it, however, is motivated by its intimate connections with several much more attended problems of applied and geometric probability theory.

0.2. In this paper we prove central limit theorems for the supporting point process in the situation where the template $\mathcal{S}$ modeling the measuring rod of the informal description above is described as the superset of a function growing at least linearly and at most polynomially at infinity, and the intensity rate of the underlying Poisson point process is bounded by the exponent of the height and is essentially positive at negative heights (exact formulations see in section 1, assumptions A and B). These assumptions are not, of course, necessary, and can be relaxed in a variety of ways; the form used in this paper is a result of efforts to minimize volume of the paper while retaining the scope of applications of results.

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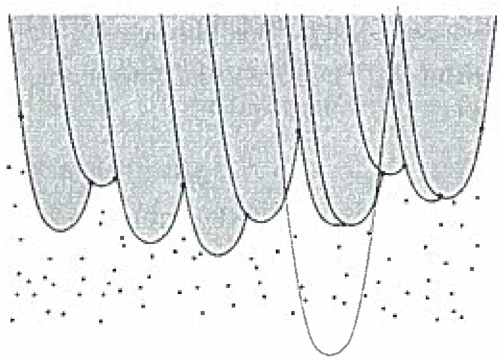

Figure 1. The "measuring rod" process. Supporting points are filled. Template $\mathcal{S}$ is depicted as the dotted line.

0.3. The central limiting theorems we prove here, while apparently new, are not very surprising. The main novelty of this paper are the applications of these results, which include the investigations of the point processes of Pareto (vector) extremal points in a sample with independent coordinates and the processes of the convex hulls vertices for the standard Poisson sample from infinite orthant or from the interior of the paraboloid. These latter processes are well-known to be the main ingredients in the study of the asymptotic behavior of convex hulls of large *iid* samples from uniform distribution in simple polyhedra or strictly convex bodies with smooth boundary, respectively. We derive the central limit theorems for all these processes and deduce, for example, the CLT for the number of Pareto extremal points with independent coordinates, a long standing problem.

Yet another application is the CLT for some point processes associated with asymptotic solutions of the Burgers' equation

$$\partial v/\partial t + v\partial v/\partial x = \epsilon\Delta v$$

in the inviscous limit, $\epsilon \to 0$. These processes describe the spatial distribution of matterless cells in some models of the large scale structure of the Universe.

There are certainly other models of applied probability theory which fit into the general scheme of supporting points processes, for example the crystal growth model of Johnson-Mehl (see[M] for a thorough treatment). They will not be discussed here to save place.

0.4. Plan of the paper. Section 1 contains the basic construction and main results, Theorems 1.8.1 and 1.9.2. The applications of these results to the problems discussed above are given in Section 2. The proofs of all technical results are contained in Section 3 and miscellaneous results and remarks in Section 4. The proofs are rather elementary and use the moments method in the guise of B-mixing and exponential clustering.

1. Constructions and Results

1.1. Basic notations. We consider point processes in Euclidean $N = (n+1)$-dimensional space $W = V \times \mathbb{R}, V \cong \mathbb{R}^n$, with generic point denoted as $w = (x,h), x \in V, h \in \mathbb{R}$. The projections to corresponding factors will be denoted as $x(\cdot)$ and $h(\cdot)$ correspondingly, that is $x(w) = x$ and $h(w) = h$ for $w = (x,h)$. We will also imply that V and $\mathbb{R}$ are embedded into W and will sometimes denote $(x,0)$ simply as x and $(0,h)$ simply as h when there can be no confusion.

It will be assumed throughout the paper, that in the Euclidean metric on W, the V-plane and h-axis are orthogonal. The metric on W as well as the induced metrics on V and h-axis is denoted as $|\cdot|$.

The sums of set are understood always in Minkovski sense; the notations $A+w, A+x, A+h$ for $A \subset W$ are reserved for $A+\{w\}, A+\{x\}, A+\{h\}$ correspondingly.

1.2. Assumption A. The role of the "measuring rod" of informal discussion in the Introduction is played by a fixed subset $\mathcal{S} \subset W$ which we assume to be a supergraph of a function,

$$\mathcal{S} = \{(x,h) : h \geq \phi(x)\},$$

where the function ϕ is assumed to be continuous and to grow at least linearly and at most polynomially at infinity:

$$a_{\mathbf{A}} + b_{\mathbf{A}}|x| \leq \phi(x) \leq A_{\mathbf{A}} + B_{\mathbf{A}}|x|^{\gamma_{\mathbf{A}}} \tag{A}$$

for some positive $b_{\mathbf{A}}, B_{\mathbf{A}}$ and $\gamma_{\mathbf{A}} \geq 1$.
The interior of $\mathcal{S}$ we denote by $\mathcal{S}^o = \{(x,h) : h > \phi(x)\}$, the boundary of $\mathcal{S}$ as $\partial\mathcal{S} = \{(x,h) : h = \phi(x)\}$.

1.3. Poisson point process. Let ρ be a locally integrable nonnegative function on $\mathbb{R}$. We consider the Poisson point process ξ on W with the intensity measure μ given by the density $\rho(h)dhdx$. We assume that the μ-content of the shifted set $\mathcal{S}+h$ is finite for all h, and denote this μ-content as

$$k(h) = \int_{\mathcal{S}+h} \rho(h)dhdx.$$

By construction, $\mu(\mathcal{S}+w) = k(h(w))$.
We will use the same notation ξ for the random measure associated with discrete point process ξ (that is the sum of deltas at points of ξ).

1.4.1. Definition. *A point w is $(\mathcal{S},\xi)$-***supporting** *(or simply ξ-supporting, or just supporting when the context is unambiguous), if there exists a point w' such that $\xi(\mathcal{S}^o+w') = 0$ and $w \in \partial\mathcal{S}+w'$. Such a set $\mathcal{S}+w'$ is called* **supported set**, *and w' its* **apex**.
The set of $(\mathcal{S},\xi)$-supporting points in ξ will be denoted as $\xi_{\mathcal{S}}$.

For $w' = (x', h')$, the "depth" h' is exactly the first instant of hitting the "bottom", that is an element of ξ, by a set in the family of vertical shifts $\mathcal{S}+x'+h, -\infty < h < \infty$, conforming with the intuitive description from the Introduction.
The point process $\xi_{\mathcal{S}}$ is the central object of this paper.

1.5. Assumption B. Throughout the paper we will assume that the intensity rate is bounded by an exponential function of the negated height and is essentially positive when $h \to -\infty$, that is

$$\rho(h) \leq A_{\mathbf{B}} \exp(-C_{\mathbf{B}} h)) \text{ everywhere; } \rho(h) \geq a_{\mathbf{B}} > 0 \text{ for } h \leq 0, \tag{B}$$

where $A_{\mathbf{B}}, a_{\mathbf{B}}$ and $C_{\mathbf{B}}$ are some positive constants.

1.6. Correlation functions. For a point process η in W, the value of the correlation function $r_k^\eta : W^k \to \mathbb{R}$ at the tuple $\{w_1, \ldots, w_k\}$ of pairwise distinct points $w_i \in W, w_i \neq w_j$ can be defined as

$$\lim_{\epsilon_1 \to 0, \ldots \epsilon_k \to 0} \frac{\mathbb{E}_\eta \eta(w_1+\epsilon_1 B) \cdots \eta(w_k+\epsilon_k B)}{\epsilon_1^N \cdots \epsilon_k^N vol(B)^k},$$

where B is (say) the unit ball in W.

In our situation the correlation functions can be calculated as follows. For a point $w \in W$ let $I(w, \xi) = 1$, if w is ξ-supporting and 0 otherwise. For a k-tuple $\{w_1, \dots, w_k\}$ of pairwise distinct points in W, let

$$\bar{r}_k(w_1, \dots, w_k) = \mathbb{E}_\xi \prod_i I(w_i, \xi).$$

The function $\bar{r}_k$ is the probability that all points in the tuple are ξ-supporting. The correlation densities for the supporting point process ξ_S are then equal to

$$r_k = \bar{r}_k \cdot \prod_i \rho(h(w_i))\, dw_i.$$

This follows from the standard properties of the Poisson point processes: indeed, conditioned on $\{w_1, \dots, w_k\} \subset \xi$, the point process $\xi - \{w_1, \dots, w_k\}$ is again Poisson with the same intensity measure.

1.7. Properties of ξ_S. Henceforth both **A** and **B** are assumed.

The distribution of the point process ξ_S is, apparently, invariant with respect to shifts along $V \subset W$. Further, as $\xi_S \subset \xi$, its first moment measure has a density $r \leq \rho$ with respect to the Lebesgue measure on W (one has, $r = r_1$ as defined above). The following Proposition says that ξ_S is essentially concentrated near V:

1.7.1. Proposition. *The correlation functions r_k of ξ_S decrease exponentially with $|h|$:*

$$r_k(w_1, \dots, w_k) \leq A_k \exp\left(-C_k \max_i |h_i|\right)$$

for some positive constants A_k, C_k.

Proof of this and the following Propositions will be given in Section 3.

The next important property of the correlation densities r_k is the exponential clustering.

1.7.2. Proposition. *For any natural k, l there exist positive constants $A_{k,l}, C_{k,l}$ such that*

$$|r_{k+l}(w_1, \dots, w_k, w'_1, \dots, w'_l) - r_k(w_1, \dots, w_k) r_l(w'_1, \dots, w'_l)| \leq A_{k,l} \exp(-C_{k,l} d),$$

where d is the distance between sets $\{w_1, \dots, w_k\}$ and $\{w'_1, \dots, w'_l\}$ (the smallest of the pairwise distances between points in these finite sets).

1.8. These two Propositions imply the main result on the asymptotics of the processes of $\mathcal{S}$-supporting points.

Let $\xi_{S,V} = x(\xi_S)$ be the x-projection of ξ_S. The point process $\xi_{S,V}$ is homogeneous in V (as the distribution of ξ_S is invariant with respect to shifts along V) and has finite intensity density $r_V\, dx$ with $r_V = \int_{-\infty}^{\infty} r(0, h)\, dh$ (the integral converges by 1.7.1). Consider the rescaled random measure given by $\xi_{S,V,\lambda}(A) = \xi_{S,V}(\lambda A)$ for Borel $A \subset V$, and normalize it

$$\nu_\lambda = \frac{\xi_{S,V,\lambda} - \mathbb{E}(\xi_{S,V,\lambda})}{\sqrt{\lambda^n}}. \tag{1.8.1}$$

The following result is our main tool in applications:

1.8.1. Theorem. *Asymptotically, as $\lambda \to \infty$, $\xi_{S,V,\lambda}(A)/\lambda^n$ converges to r vol(A) in probability, and ν_λ converges in law to a generalized Gaussian random field with covariance kernel $C\delta(x-x'), C = (\int_V q_{2,V}(0,y)dy + r_V)^{1/2}$, where $q_{2,V}$ is the second cumulant density for $\xi_{S,V}$.*

Proof. It follows more or less straightforward from the Propositions 1.7.1 and 1.7.2. Proposition 1.7.1 implies that $\xi_{S,V}$ has finite constant (as ξ_S is invariant under shifts along V) intensity $r_V = \int_{-\infty}^{\infty} r(h)dh$. Further, Propositions 1.7.1 and 1.7.2 together imply that $\xi_{S,V}$ clusters exponentially. Indeed, denote by $r_{k,V}$ the k-th correlation function for $\xi_{S,V}$. It is easily seen that $r_{k,V}(x_1, \dots, x_k)$ is just the integral of k-th correlation function for the supporting point process ξ_S along the fiber consisting of all $(w_1, \dots, w_k)$'s projecting to the tuple $\{x_1, \dots, x_k\}$,

$$r_{k,V}(x_1, \dots, x_k) = \int_{h_1, \dots, h_k} r_k((x_1, h_1), \dots, (x_k, h_k))dh_1 \cdots dh_k,$$

whence the estimate follows immediately.

The cumulant densities $q_k(x_1, \dots, x_k)$ for $\xi_{S,V}$ are related to the correlation functions $r_{k,V}$ via logarithmic transformation (see, e.g. **[Ru]**, Ch. 4.4, "algebraic method"), and it is standard that the exponential clustering of $r_{k,V}$ is equivalent to the exponential decreasing of $q_k(x_1, \dots, x_k)$ as a function of the differences $x_i - x_{i+1}$ at infinity (**[Ru]** again[1]). This latter property implies the Brillinger's *B-mixing* for the point process $\xi_{S,V}$ **[Bri]**:

$$\int_{V^{k-1}} |q_k(x_1, \dots, x_k)| dx_2 \cdots dx_k < \infty$$

for all k and, consequently, the central limit theorem for $\xi_{S,V}$ (see, e.g. **[Iv]**). □

1.9. As a corollary, one can deduce the central limit theorem for the ξ_S-content of reasonably behaving large open subsets of W. The condition we need is the following. Let $\Lambda \subset W$ be open and $\Lambda_V = \Lambda \cap V$ be its intersection with V.

1.9.1. Definition. *We say that Λ is* **quasitransversal** *to V if the boundary of Λ_V is the intersection of the boundary of Λ with V.*

1.9.2. Theorem. *For any open bounded $\Lambda \subset W$, let $N(\lambda)$ be the number of points of ξ_S in $\lambda\Lambda$, i.e. $N(\lambda) = \xi_S(\lambda\Lambda)$. If Λ is quasitransversal to V and has nonempty intersection with V, then:*

a) both expectation and variance of $N(\lambda)$ grow as $|\Lambda_V|\lambda^n$ as $\lambda \to \infty$:

$$\frac{\mathbf{E}N(\lambda)}{|\Lambda_V|\lambda^n} \to e; \qquad \frac{\mathbf{V}N(\lambda)}{|\Lambda_V|\lambda^n} \to v$$

for some positive constants e, v (depending on S, ρ only), and

b) the distribution of $N(\lambda)$ is asymptotically normal:

$$\frac{N(\lambda) - |\Lambda_V|\lambda^n e}{\sqrt{|\Lambda_V|\lambda^n v}} \to \mathcal{N}(0,1)$$

[1] Although in this reference just the equivalence of clustering of r and vanishing of q at infinity is shown, the modification of the result to the *exponential* decreasing is immediate.

in distribution.

Proof. The claim of Theorem 1.9.2 apparently follows from Proposition 1.8.2 for cylinders over an open base in V: if $\Lambda = x^{-1}(x(\Lambda))$, then $\xi_S(\Lambda) = \xi_{S,V}(\Lambda \cap V)$. To prove the claim for arbitrary open bounded $\Lambda \subset W$, we make first some intermediate estimate. Let A be the symmetric difference between Λ and the cylinder over Λ_V, $A = (x^{-1}(\Lambda_V) - \Lambda) \cup (\Lambda - x^{-1}(\Lambda_V))$.

1.9.3. Lemma. *If Λ is quasitransversal to V, both the expectation and the variance of ξ_S-content of λA are $o(\lambda^n)$ for $\lambda \to \infty$.*

Proof. This can be deduced as follows. Denote the volume of the intersection of r-tube around V with A as $b(r)$. The n-dimensional Lebesgue measure of $\bar{A}_V = \bar{A} \cap V = \partial\Lambda_V$ vanishes (here $\bar{A}$ is the closure of A), and the condition of quasitransversality implies that $b(r)/r \to 0$ as $r \to 0$. The expectation of ξ_S-content of λA is given by $\mathbb{E}\xi_S(\lambda A) = \int_{\lambda A} r(x)dx$ can be then estimated as (here A_1 and C_1 are the constants provided by Proposition 1.7.1)

$$\int_{\lambda A} r(x)dx \leq A_1 \int_{\lambda A} e^{-C_1|h(x)|}dx = A_1\lambda^{n+1} \int_A e^{-C_1\lambda|h(z)|}dz = A_1\lambda^{n+1} \int_0^\infty e^{-C_1\lambda r}db(r),$$

which is $o(\lambda^n)$ by Tauberian theorem for Laplace transforms [P].
Similarly, let $b_2(r)$ be the volume of $A^2 \subset W^2$ within the distance r to the V-diagonal $\Delta_V = \{(x,x), x \in V\} \subset W^2$. The measure of $A_{\Delta_V} = A^2 \cap \Delta_V$ in V^2 is zero, whence, by quasitransversality condition again, $b_2/r^{n+2} \to 0$ as $r \to 0$ (the exponent $n+2$ here is the codimension of Δ_V in W).
The variance of $\xi_S(\lambda A)$ is given by

$$\int_{(\lambda A)^2} q_2(x,y)dxdy + \int_{\lambda A} r(x)dx.$$

The second term is already known to be $o(\lambda^n)$. The integral in the first term is estimated in absolute value as (here $A_{1,1}$ and $C_{1,1}$ are the constants implied by Proposition 1.7.2)

$$A_{1,1} \int_{(\lambda A)^2} e^{-C_{1,1}|x-y|}dxdy = A_{1,1}\lambda^{2n+2} \int_{A^2} e^{-C_{1,1}\lambda|z-t|}dzdt = A_{1,1} \int_0^\infty e^{-C_{1,1}\lambda r/2}db_2(r),$$

which is $o(\lambda^n)$ by Tauberian theorem again . □

Lemma 1.9.3. implies evidently the part a) of Theorem 1.9.2.
To prove the part b) it is enough to notice that the difference of (centered) ξ_S-contents of $\lambda\Lambda$ and $x^{-1}(\lambda\Lambda_V)$, the cylinder over its intersection with V, is majorized by the (centered) ξ_S-content of λA, which is of smaller order than any of them. □

2. Applications

2.0. In this section we consider the applications of our main results, Theorems 1.8.1 and 1.9.2.

2.1 Pareto extremal points. Consider $N = n+1$-dimensional vector space W and a convex cone $K \subset W$. This cone defines a partial ("vector") order on $W : z >_K z' \iff$

$z - z' \in K$. The case we will be interested here is that of Pareto cone, that is of positive orthant $K_P = \{z : z_i > 0, i = 1, \dots, N\}$.

Given a subset $X \subset W$, one defines K-extremal points in X (or Pareto extremal for K_P) as follows: a point $z \in X$ is K-extremal, if there is no $z' \in X, z \neq z'$, such that $z' >_K z$.

2.1.1. Pareto extremal points with independent coordinates. Let X be a finite *iid* sample of size m *with independent coordinates without atoms*, and $P(m)$ is the number of Pareto extremal points in X. What can be said about $P(m)$?

The pioneering work **[B-NS]** provides with the following information:

- The expected number of Pareto extremal points, explicitly given as "multidimensional harmonic series"

$$\mathbb{E}P(m) = \sum_{1 \leq i_1 \leq \dots \leq i_n \leq m} \frac{1}{i_1 \cdot \dots \cdot i_n},$$

  grows as $\log^n m/n!$;
- In dimensions $N = 2, 3$ the growth of variance is of the same order $\log^n m$ as that of expectation;
- In dimension $N = 2$, the distribution of $P(m)$ is asymptotically normal (actually, the generating function of $P(m)$, closely connected with the symmetric group paraphernalia, has been found explicitly).

The series representation for the variance of $P(m)$, also reminiscent of "multidimensional harmonic" were found in **[In]**. However, it is difficult to extract the asymptotic behavior from them.

2.1.2. Pareto extremal points and supporting point process. The connection between just described model and supporting point processes is rather immediate. First, the monotone increasing continuous coordinate-wise changes (that is changes $z = (z_1, \dots, z_N) \mapsto (f_1(z_1), \dots, f_N(z_N))$ with functions f_i strictly increasing and continuous) do not change the Pareto partial order. Taking f_i to be the distribution function of z_i, we reduce the problem to the case when z is uniformly distributed in the unit cube. For convenience, we shift the cube by $(-1, \dots, -1)$ to arrive at the uniform distribution in the cube $I = \{-1 \leq w_i \leq 0\}, i = 1, \dots, N$.

The intuition suggests that the chances to find an extremal point somehow far from the union of coordinate hyperplanes are slim, whence it is enough to concentrate on the domain close to the coordinate hyperplanes. The restriction of the *iid* sample X of large size m to this domain is nearly Poisson. It is intuitive, therefore, to consider, as an approximation step, a variation of the initial problem, where the sample X is a realization of the Poisson point process with intensity density $m\,dw$ in I, or, equivalently, with Lebesgue intensity measure in $m^{1/N}I$.

One can extend the probability space so as to assume that X is just the intersection of $m^{1/N}I$ with the standard Poisson point process ξ_E (with Lebesgue intensity measure) on $W_- := \{z : z_i \leq 0,\ i = 1, \dots, N\}$. Moreover, it is clear that Pareto extremal points in $\xi_E \cap (m^{1/N}I)$ is just the Pareto extremal points of ξ_E which lie in $m^{1/N}I$.

Now we choose new coordinates on E_-:

$$y_i = -\ln(-z_i), i = 1, \dots, N. \tag{2.1.1}$$

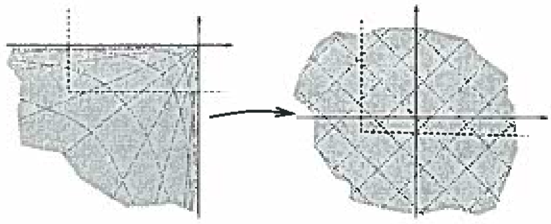

Figure 2. Logarithmic transformation of the negative orthant.

This change takes W_- to $\mathbb{R}^N \cong W$ and is again monotone and coordinate-wise, thus Pareto order preserving.

Let

$$h = \frac{1}{N}\sum_{i=1}^{N} y_i.$$

The hyperplane $V := \{h = 0\}$ has dimension n. We choose some orthonormal coordinates $(x_i), i = 1, \dots, n$ on V and take the h-axis to be spanned by $(1, \dots, 1)$. This makes (x_i, h) an orthogonal coordinate system on W. One expresses the functions y_i in the new basis, $y_i = h - l_i(x), i = 1, \dots, N$, where l_i are some linear functions on V (one has $\sum_i l_i = 0$).

The Pareto partial order on W is given by the conditions

$$(x,h) >_K (x',h') \iff y_i(x,h) > y_i(x',h') \text{ for all } i = 1, \dots, N,$$

or, equivalently, that $(x,h) \in \mathcal{S}^o + (x',h')$ for $\mathcal{S}$ given by

$$\mathcal{S} = \{(x,h) : h \geq l_i \ \text{ for } \ i = 1, \dots, N\}. \tag{2.1.2}$$

In other words, $\mathcal{S}$ is defined as in assumption A with the function $\phi = \max_i l_i$ on V.

By definition, if w is Pareto extremal in X, then $w + \mathcal{S}^o \cap X$ is empty and $w \in w + \mathcal{S}$. It follows that $w = (x,h)$ is Pareto extremal if and only if it is $\mathcal{S}$-supporting for $\xi = X$.

It remains to find the intensity measure in the new coordinates: an immediate calculation gives that its density with respect to Lebesgue measure is e^{-Nh}.

2.1.3 Limit theorems for Pareto extremal points.

Now we are in the position to apply the results of the previous section. Indeed, the assumptions A and B can be checked immediately. The supported set template $\mathcal{S}$ defined by the function (2.1.2) satisfies apparently the conditions of assumption A. That the intensity density of the Poisson point process satisfies assumption B is clear as well.

Hence we arrive at the following result.

2.1.4 Proposition. *Let ξ be the standard Poisson point process in W_- (with Lebesgue intensity measure) and ξ_P be the process of Pareto extremal points in ξ. Let ξ_S be the image of ξ_P under transformation (2.1.1). Then the rescaled process ν_λ defined as in (1.8) converges in distribution to a generalized Gaussian random field supported by V and invariant with respect to shifts along V as $\lambda \to \infty$.*

Proof. This is an immediate corollary of the Proposition 1.8.1. □

The transformation (2.1.1) sends the cube $m^{1/N}I$ to the displaced positive orthant $\Lambda_\lambda = \{y_i \geq -\lambda, i = 1, \dots, N\}$, where $\lambda = (1/N)\ln m$. As $\Lambda_\lambda = \lambda\Lambda_1$, and the set Λ_1 is clearly quasitransversal to V, we are in the situation of Theorem 1.9.2, which implies the central limit theorem for the number of Pareto points in a Poisson sample with independent coordinates:

2.1.5 Corollary. *If X is the Poisson point process in the unit cube with intensity density $m\,dw$. Then the number $P(m)$ of Pareto extremal points in X is asymptotically normal with both expectation and variance growing as $\ln^n m$.*

2.1.6. In fact, one can derive from 2.1.5 similar statements for the number of Pareto extremal points in the *iid* samples of fixed size m from the unit cube I. The estimates for the growth order of the variance for the Poisson sample case can be modified *mutatis mutandis* to the fixed size m case implying the $\log^n m$ growth.

The central limit theorem in this case is also rather straightforward. Indeed, consider the subset $S_m \subset I$ defined as $S_m = \{\prod_i |z_i| \leq \ln^2 m/m\}$. One can show that the probability to find just *one* Pareto extremal point outside this set (estimated from above by the expected number of Pareto points there, which can be derived easily e.g. from results of **[B-NS]**) tends to zero when $m \to \infty$. Analogous estimate is valid for the Poisson sample from I with intensity density $m\,dw$. Further, the standard arguments (e.g. results of Prokhorov on the total variation differences between binomial and Poisson random values) show that one can find coupling of *iid* size m sample from I and of the standard Poisson point process with intensity density $m\,dw$ on I which coincide on S_m with probability converging to 1 as $m \to \infty$. It follows that the centered normalized distributions for the number of Pareto points in the samples of fixed size and in Poisson point processes in I converge to the same limit as the sample size increases indefinitely. Summarizing this sketch of a proof, we claim

2.1.6 Corollary. *The number of Pareto extremal points in an iid size m sample with independent coordinates is asymptotically normal, with expectation and variance both growing as $\ln^n m$ when $m \to \infty$.*

2.2. Convex hull vertices. Let P be a convex body in N-dimensional linear space E and X be an *iid* size m sample from uniform distribution in P. The distribution of the number of vertices of convex hull of X has been discussed in literature many times. Detailed surveys of what is known can be found in **[Sch]** or **[Buc]**, and here I just sketch the results relevant to our situation, restricting my attention to asymptotics.

2.2.1. Asymptotics of the number of convex hull vertices.

Let X be the size m *iid* sample from P and $C(m)$ the number of vertices of the convex hull of X. The following is known about the asymptotics of $C(m)$ for $m \to \infty$:

- Let P be a *simple* polyhedron, which means that near each vertex P is affinely isomorphic to the positive orthant (or, equivalently, each vertex belongs to exactly N faces). Then the expectation of $C(m)$ grows as $f_0(P)c_N \ln^{N-1} m$, where f_0 is the number of vertices of P and c_N is a constant depending only on the dimension **[Dw]**.
- If P is strictly convex with smooth enough boundary, then the expectation of $N(m)$ grows as $m^{(N-1)/(N+1)}$ (Raynaud, Wieacker, see references in **[Sch]**).
- Let the dimension $N = 2$ and P be either a polygon (all plane polygons are simple) or strictly convex plane domain with smooth boundary. Then the growth of the variance of $N(m)$ is of the same order as that of the expectation and the central limit theorem holds **[Gr]**.

The intuition behind these results is that, similarly to the case of vector extremal points, the vertices of the convex hull of large sample concentrate near the boundary, more specifically, in a neighborhood of the boundary where the "floating volume" **[Bi, BL]** — the function which associates to a point the minimal volume of the piece cut from the body by a hyperplane through the point — is small. As in the extremal vertices case, this justifies the replacement of the large fixed size sample by the Poisson sample as a valid approximation for the convex hull. In this paper we concentrate on this Poisson approximation.

2.2.2. Limit case: orthant. Here we deal with P a simple polyhedron. To localize the vertices of the convex hull of the sample "close" to a vertex of P, we associate to P its dual *simplicial fan.* Recall that this is the partition of the space W^* of linear functionals on W into the simplicial cones, one for each facet of P. The cone corresponding to the facet F consists of the linear functionals attaining maximum at a (relative) interior point of F. The cones of maximal dimension N correspond to the vertices of P and their number is therefore $f_0(P)$. We will denote by C_p the cone associated to the vertex p of P.

A point in a closed subset $X \subset W$ is an extremal point of the convex hull of X if it maximizes a linear functional on X. We denote the set of all extremal points in X as $\operatorname{extr}(X)$, so that $C(m) = |extr(X)|$ for X an *iid* sample of size m. If we restrict the linear functional whose maxima on X we consider to a cone $C \subset W^*$, we get a smaller subset of extremal points which we denote as $\operatorname{extr}_C(X) \subset X$. If $C = W^* - \{0\}$, $\operatorname{extr}_C(X)$ is again just all the extremal points of the convex hull of X.

Clearly, $\operatorname{extr}(X) = \cup_{p\in F_0(P)} \operatorname{extr}_{C_p}(X)$. Fix a vertex p of P (and denote C_p just as C to save on typing). The extremal points in $\operatorname{extr}_{C_p}(X)$ flock around p and it is convenient to study those subsets independently.

As the operation of forming the convex hull commutes with affine transformations, one can assume that p is at the origin and that near p the polyhedron coincides with the negative orthant W_-.

Now we focus on the Poisson sample X from P with intensity density $m\,dw$. Equivalently, one can assume that X is the intersection of $m^{1/N}P$ with the standard Poisson sample ξ from the negative orthant W_-, similarly to the construction of 2.1.2. Again, one can define the C-convex hull of the whole (a.s. infinite) Poisson point process ξ. Unlike the Pareto case, however, the set of C-extremal points of the convex hull of $\xi \cap (m^{1/N}P)$ is not equal the set of the C-extremal points of the convex hull of ξ (intersected with $m^{1/N}P$): the latter set is smaller in general. One can show that the difference between these random sets is small enough, so that the main contribution is just the part of the C-convex hull of ξ falling within $m^{1/N}P$. We will give the details elsewhere and refer to this result only to justify the attention to the process C-convex vertices of ξ.

We notice that the C-convex hull of the standard Poisson sample from W_- equals almost surely the plain convex hull (recall that in our assumptions $C = W_+$, the Pareto cone), and that the vertices of $\operatorname{conv}(\xi)$ are all Pareto extremal points in ξ. Somewhat more surprising is that the logarithmic transformation (2.1.1) takes the process of convex hull vertices into the $\mathcal{S}$-supporting point process for an adequate $\mathcal{S}$.

We preserve the notations of 2.1.2 (so that z_i are the coordinates on E and $W_- = \{z_i \leq 0, i = 1, \ldots, N\}$). The condition that w is a vertex of $\operatorname{conv}(\xi)$ is equivalent to the existence of a linear functional $l = \sum a_i z_i, a_i \geq 0$ whose maximum on ξ is attained at w: $a = l(w) \geq l(w')$ for all $w' \in \xi$. Any hyperplane $H = \{\sum a_i z_i = a, \quad a < 0, a_i > 0\}$ can be obtained from a fixed hyperplane, say $H_0 = \{\sum z_i = -1\}$, by coordinate-wise dilations $z_i \mapsto \kappa_i z_i, \kappa_i > 0, i = 1, \ldots, N$.

In (x, h) coordinates the hyperplane H_0 is given by

$$\sum_i e^{h-l_i(x)} = 1, \tag{2.2.1}$$

The equation (2.2.1) describes the hypersurface bounding the set

$$\mathcal{S} = \{(x,h) : h \geq \ln\big(\sum_i e^{l_i(x_i)}\big)\}. \tag{2.2.2}$$

The coordinate-wise dilations are just the shifts by vectors $(\ln \kappa_1, \dots, \ln \kappa_N$'s in y-coordinates. Summarizing, this shows that a point is a vertex of the convex hull of the sample ξ if and only if it is an $\mathcal{S}$-supporting point for $\mathcal{S}$ given by (2.2.2) after the transform (2.1.1). The assumption A is clearly satisfied. The intensity measure is, as in 2.1.2, $e^{-Nh}\, dhdx$, implying B.

Therefore, we immediately obtain the following results.

2.2.3 Proposition. *Let ξ be the standard Poisson point process in W_- (with Lebesgue intensity measure) and ξ_C be the process of convex hull vertices for ξ. Let ξ_S be the image of ξ_C under transformation (2.1.1). Then the rescaled process ν_λ defined as in (1.8) converges in distribution to a generalized Gaussian random field supported by V and invariant with respect to shifts along V as $\lambda \to \infty$.*

Proof. Immediate. □

Analogously, we get the central limit theorem for the number of convex hull vertices for ξ within the inflated polyhedron $m^{1/N}P$.

2.2.4 Corollary. *The number of points of ξ_C within $m^{1/N}P$ is asymptotically normal with both expectation and variance growing as $\ln^n m$ when $m \to \infty$.*

2.2.5 Convex hull of the fixed size sample in P. The results obtained so far form a compelling evidence that the central limit theorem for the number of vertices of convex hull of large fixed size samples holds for any simple polyhedron P. This is indeed the case. The detailed proof will be presented elsewhere, because, while no new ideas are involved, some rather tedious technical estimates should be done. The lacking pieces are the following:

- One has to work out the size of the relevant neighborhood of the boundary of P which contain almost all extremal points of the convex hull for both binomial and Poisson samples and small enough to provide the coupling one needs;
- One has to estimate the difference between the sets of C-extremal points of the convex hull of the Poisson sample from P and the set of C-extremal points of convex hull of ξ falling into P. This difference can be shown to have both expectation and variance of order $\ln^{n-1} m$ so that its contribution is small compared with that of $\operatorname{extr}_C(\xi) \cap P$;
- One has to estimate the overcount of the convex hull vertices caused by the fact that some of them are C_p-extremal for several vertices p. Intersection of different cones $C_{p,p'} = C_p \cap C_{p'}$ is contained in a linear subspace $L \subset W^*$ of positive codimension and each point, that is a vertex which is both C_p- and $C_{p'}$ extremal in the convex hull of X, is a vertex of the convex hull for projection of X along the annulator of L. The number of such convex hull vertices in less dimensional situation can be estimated and is of smaller order than the main contributions of C_p-convex hull vertices.

2.2.6. Limit case: paraboloid. Assume now that the convex body $P \subset W$ has smooth enough boundary and is strictly convex in the sense that the second fundamental form is positive definite everywhere. Repeating the mantra of previous sections one is led to the standard Poisson sample with Lebesgue intensity measure in the inflated body λP. Again,

as the convex hulls formation commutes with the affine transformations, one can always transform λP to a body P_λ of the same volume with origin on the boundary, tangent plane at the origin coinciding with $V \subset W$ and with given second quadratic form at the origin (say, $-|\cdot|^2$). The smoothness of the boundary implies that at arbitrarily large vicinity of the origin, P_λ is arbitrarily close to paraboloid $P = \{h \leq -\sum x_i^2\}$ (we assume that $\{x_i\}$ form a coordinate system on V, as in the setup of section 1). Hence the limiting point process approximating the vertices of the convex hull of a strictly convex body near a point is the process of convex hull vertices for the Poisson sample (with Lebesgue intensity) from P. A point $w \in \xi$ belongs to the convex hull of ξ if and only if there exists a hyperplane through w bounding a halfspace $H_l = \{h \geq l(x)\}$, l linear, without further points of ξ.
The transformation

$$x \mapsto x; h \mapsto h + \sum_i x_i^2 \tag{2.2.3}$$

takes the paraboloid P into the halfspace $\{h \leq 0\}$ and the family of halfspaces H_l into the family of shifts of the set

$$\mathcal{S} = \{h \geq \sum_i x_i^2\}.$$

Therefore, the transformation (2.2.3) takes the convex hull vertices into the $\mathcal{S}$-supporting points of its image. The assumption A is clearly satisfied. The Lebesgue measure is preserved by (2.2.3), and the resulting sample is standard Poisson in the halfspace $\{h \leq 0\}$, whence the assumption B is satisfied too.

2.2.7 Proposition. *Let ξ_S be the transformation of the convex hull vertices process for the standard Poisson sample in the infinite paraboloid P and ν_λ its rescaling defined as in (1.8). Then ν_λ converges to a generalized Gaussian process concentrated on V.*

2.3. Large-scale structure of the Universe. Another area of applications of the $\mathcal{S}$-supporting points processes is related to the asymptotic solutions of the Burgers model for turbulence with random initial data, which is used commonly as a working approximation for the evolution of the large-scale structure of the Universe. The body of literature dedicated to this equation is enormous, and I mention here only the book **[GSS]** and recent papers **[AMS, MSW]** as starting points and collections of references.
In the limit of vanishing viscosity, the solution of the Burgers equation in $\mathbb{R}^n$ (without external forces) and with potential initial velocity $v = \partial S_0/\partial x, S_0 : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\begin{aligned} v_t(x) &= \partial S_t/\partial x; \quad S_t = e^{f_t(x)} \\ f_t(x) &= \sup_{y \in V}\Big[f_0(y) - \frac{(x-y)^2}{2t}\Big]. \end{aligned} \tag{2.3.1}$$

If S_0 is a random function and oscillates strongly enough, then only its local maxima matter. A standard simplifying assumption (valid, e.g. in the zero-range shot-noise model) is that the positions x and heights h of these local maxima form a Poisson process ξ in $\mathbb{R}^n \times \mathbb{R}$. Another situation where the Poisson process of maxima realizes is the case of Gaussian random field f_0. Under some assumptions on the correlation function for f, it is shown in **[MSW]** that the process of relevant local maxima of $\sigma_L f(x/L)$ (where $\sigma_L = L^2 \ln L$ is the standard scaling in the theory of extremal values of Gaussian processes) converges in appropriate sense to the Poisson point process with the intensity $e^{-h}\, dhdx$.

Whenever the Poisson approximation for the local maxima process is valid, one can apply the approach of this paper. The solutions of (2.3.1) are then just the boundary of the union of all ξ-supporting sets, where the template $\mathcal{S}$ is the paraboloid $\{h \geq |x|^2\}$. The supporting points of this process correspond in the physical picture to the matterless voids in the Universe.

The results of section 1 imply the central limit theorem for the number of such areas, if the density of h decreases rapidly enough. Details are straightforward and are omitted.

3. Proofs

3.0. In this section the proofs of the technical results are given.

3.1. We will need some constructions first. Let $\mathcal{B}_{A,B}$ be given by

$$\mathcal{B}_{A,B} = \{(x,h) : h \geq A + B|x|^{\gamma_{\mathbf{A}}}\} \subset W,$$

with $\gamma_{\mathbf{A}}$ the exponent from the assumption A. One can choose A, B large enough and $\delta > 0$ small enough so that the sum of $\mathcal{B}_{A,B}$ with the horizontal δ-disk $K_\delta = \{(x,0) : |x| \leq \delta\} \subset V$ is contained in $\mathcal{S}$:

$$\mathcal{B}_{A,B} + K_\delta \subset \mathcal{S}.$$

We fix these $A; B, \delta$ once and forever and denote $\mathcal{B}_{A,B}$ simply as $\mathcal{B}$. Further, we fix a lattice L in V such that the δ-neighborhood of L in V is the whole of V.

3.1.1 Lemma. *For any $x \in V$ there exists a lattice point $l \in L$ such that $\mathcal{B} + l \subset \mathcal{S} + x$.*

Proof. The δ-neighborhood of x contains a lattice point l, whence $\mathcal{B} + (l - x) \subset \mathcal{B} + K_\delta \subset \mathcal{S}$. □

3.2. Proof of the Proposition 1.7.1. One has, obviously, $\bar{r}_k(w_1, \dots, w_k) \leq \bar{r}_1(w_i)$ for any i.

3.2.1 Lemma. *The function $\bar{r}_1(w)$ is bounded and decreases more rapidly than any exponent of $h(w)$ as $h(w) \to -\infty$:*

$$\bar{r}_1(w) \leq A(C) \exp(C \min(h(w), 0)),$$

for any positive C.

Proof. To estimate $\bar{r}(w)$ we use the following discretization argument. If w is supporting and $\mathcal{S} + (x,h)$ is the supported set, then, by Lemma 3.1.1, there exists a lattice point l within δ-distance to x, such that $\mathcal{B} + (l,h) \subset \mathcal{S} + (x,h)$. Now

$$\begin{aligned} h =& h(w) - \phi(x(w) - x) \\ \leq& h(w) - b_{\mathbf{A}} - a_{\mathbf{A}}|x(w) - x| \\ \leq& (h(w) - b_{\mathbf{A}} + a_{\mathbf{A}}\delta - a_{\mathbf{A}}|x(w) - l|), \end{aligned}$$

(assumption A and associated constants used). Hence, as $\xi(\mathcal{S} + w) = 0$, the ξ content of $\mathcal{B} + l + h(w) + \tilde{h}(l)$, where $\tilde{h}(l) = -b_{\mathbf{A}} + a_{\mathbf{A}}\delta - a_{\mathbf{A}}|x(w) - l|$, vanishes as well. If we denote as $m(h)$ the μ-content of $\mathcal{B} + h$, then the probability that $\xi(\mathcal{B} + h(w) + \tilde{h}) = 0$ is just

$\exp(-m(h(w)+\tilde{h}))$. For $h,h' \le h_0 < 0$ one has $m(h+h') \ge m(h) - ah'$ for a positive constant a, whence the latter probability is at most $ce^{-m(h(w))}e^{-aa_{\mathbf{A}}|x(w)-l|}$ for a positive c. The sum of these probabilities over all $l \in L$ majorizes the probability that w is supporting, which gives the estimate

$$\mathbb{E}I(w,\xi) \le e^{-m(h(w))} \sum_{l\in L} e^{-a_{\mathbf{A}}a|x(w)-l|},$$

where the second multiplier obviously converges to some continuous, L-periodic, and, therefore, bounded function. Now, $m(h) \ge C(A_{\mathbf{A}}-h)^{1+n/\gamma_{\mathbf{A}}}$ for a positive C and $A_{\mathbf{A}}, \gamma_{\mathbf{A}}$ from assumption A. This proves the Lemma. □

Proof of 1.7.1: Final. The rest is simple. Indeed, if $h_- = \min_i(h(w_i))$, $h_+ = \max_i(h(w_i))$, then $\bar{r}_k \le A(C)\exp(C\min(h_-,0))$ (Lemma 3.2.1), and $\prod_i \rho(w_i) \le A^k_{\mathbf{B}}\exp(-kC_{\mathbf{B}}h_+)$. The product of these two functions, for $C \ge 2C_{\mathbf{B}}$, is $O(\exp(-C_{\mathbf{B}}\max(h_+,-h_-)))$; $\max(h_+,-h_-) = \max_i |h(w_i)|$, and the Proposition 1.7.1 follows. □

3.2.2 Remark. Actually, the same reasonings prove that for any $\alpha > 0$, the function $\bar{r}^{\alpha}_k(w_1,\dots,w_k)\prod_i \rho(w_i)$ is bounded. This fact will be used later.

3.3 Next we prove that the apexes of the sets supported by a point, are localized near V.

3.3.1 Proposition. *The probability $P(w,H)$ that a point w supports a set $\mathcal{S}+(x,h)$ with $h < H$ decreases exponentially when $H \to -\infty$:*

$$P(w,H) \le Ah(w)^n \exp(CH)$$

for some positive A, C.

Proof. This follows essentially from the construction of Lemma 3.2.1. The probability in question is majorised by the sum of probabilities $\mathbf{P}\{\xi(\mathcal{B}+l+[h(w)+\tilde{h}]) = 0\}$ taken over only such l, where the $\tilde{h}(l) \le H + |a_{\mathbf{A}}|\delta$. The number of lattice points at distance $\le R$ grows as R^n; the summand decreases as $\exp(-aa_{\mathbf{A}}R)$, and the summation starts at the distance R of order $(h(w)-H)/a_{\mathbf{A}}$. An easy estimate implies that the sum is bounded from above by a constant multiple of $(h(w)-H)^n\exp(aH)$, whence the the desired inequality follows.□

3.4 Proof of Proposition 1.7.2. Consider two tuples of points in W, $\{w_1,\dots,w_k\}$ and $\{w'_1,\dots,w'_l\}$ at the distance d (that is $\min_{i,j}|w_i - w'_j| = d$). We want to estimate the difference

$$r_{k+l}(w_1,\dots,w_k,w'_1,\dots,w'_l) - r_k(w_1,\dots,w_k)r_l(w'_1,\dots,w'_l). \tag{3.4.1}$$

We can assume that all $h(w_i), h(w'_j)$ are at most ad in absolute value for a constant $a > 0$ (unspecified for a while). Indeed, otherwise, by Proposition 1.7.1, all terms in 3.4.1 are bounded by $\exp -$something $\times\, d$ and there is nothing to prove. For our further estimates we will need

$$4a/\sqrt{1-4a^2} < b_{\mathbf{A}}. \tag{3.4.2}$$

Clearly, for a small enough, this is satisfied.

Introduce the cones $K_i = \{(x,h) : h \ge [a_{\mathbf{A}} - 3ad] + b_{\mathbf{A}}|x - x(w_i)|\}, i = 1,\dots,k$ and similarly K'_j for w'_j. The set K_i contains all sets $\mathcal{S}+w$ having w_i on its boundary and such that $h(w) \geq -ad$. Indeed, if $w = (x,h)$ is such a point, then one has $h(w_i) - h \geq a_{\mathbf{A}} + b_{\mathbf{A}}|x - x(w_i)|$. Using $h(w_i) \leq ad, h \geq -ad$ one derives $b_{\mathbf{A}}|x - x(w_i)| \leq 2ad - a_{\mathbf{A}}$, whence $h \geq -ad \geq [a_{\mathbf{A}} - 3ad] + b_{\mathbf{A}}|x - x(w_i)|$.

Denote the intersections of cones K_i, K'_j with the halfspace $\{h \leq ad\}$ as U_i, U'_j correspondingly. Let $\xi_i = \xi \cap U_i$ is the intersection of the point process ξ with U_i. Analogously, define $\xi'_j = \xi \cap U'_j$. Consider the random values $\bar{I}_i(\xi)$ and $\bar{I}'_j(\xi)$ defined as following: $\bar{I}_i(\xi) = 1$, if there exists a set $\mathcal{S} + w \subset K_i$ supported by w_i, and 0 otherwise. The value $\bar{I}'_j(\xi)$ is defined analogously. Clearly, $\bar{I}_i(\xi) \leq I(w_i, \xi), \bar{I}'_j(\xi) \leq I(w'_j, \xi)$.

We will use the shorthand I, I' and $\bar{I}, \bar{I}'$ for $\prod_i I(w_i, \xi), \prod_j I(w'_j, \xi)$ and $\prod_i \bar{I}_i(\xi), \prod_j \bar{I}'_j(\xi)$ correspondingly. Once again, $\bar{I} \leq I, \bar{I}' \leq I'$.

Let $\mathcal{E}$ be the event that there are no points of ξ in any of the sets K_i, K'_j with h-coordinate larger than ad; by $\overline{\mathcal{E}}$ we denote the complement to $\mathcal{E}$.

3.4.1 Lemma. *The probability of $\overline{\mathcal{E}}$ is exponentially small with d:*

$$\mathbf{P}\overline{\mathcal{E}} \leq A \exp(-C(ad)),$$

for some positive A, C independent of $\{w_1, \dots, w_k\}, \{w'_1, \dots, w'_l\}$.

Proof. The integral of ρ over K_i is of order

$$\int_{ad}^{\infty} A_{\mathbf{B}} \exp(-C_{\mathbf{B}} h) \Big(\frac{h + 3ad - a_{\mathbf{A}}}{b_{\mathbf{A}}} \Big)^n dh,$$

and the total probability is majorized by the sum of these integrals over all i, j, whence the claim follows. □

We denote by $I_{\mathcal{E}}$ the indicator function of $\mathcal{E}$. Set

$$D = I - I_{\mathcal{E}} \bar{I}; D' = I' - I_{\mathcal{E}} \bar{I}'.$$

The random values D, D' are, apparently, $\{0,1\}$-valued, and the expectations of both of them are exponentially small with d. Indeed, $I(w_i, \xi_i) \neq I_{\mathcal{E}} \prod_i \bar{I}_i(\xi)$ if either $I_{\mathcal{E}} = 0$, or when for one of i-s one has $I(w_i, \xi) \neq I_i(\xi)$, which implies that w_i is ξ-supporting for a set $\mathcal{S} + w$ with $h(w) < -ad$. Both events have probabilities which are exponentially small with d uniformly in w_i: the former by Lemma 3.4.1, and the latter by Proposition 3.3.1 (where we use $|h(w)| \leq ad$).

The key observation now is that, conditioned on $\mathcal{E}$, the random values $I_{\mathcal{E}} \bar{I}$ and $I_{\mathcal{E}} \bar{I}'$ are independent. Indeed, the event that w_i is ξ-supporting, with a supporting set within K_i and *with no points of ξ in K_i above $\{h = ad\}$* depends only on the intersection of ξ with the set U_i, and the same is valid for w'_js. For a satisfying (3.4.2), the sets U_i and $U'_j, i = 1, \dots, k; j = 1, \dots, l$ do not intersect. Indeed, otherwise one would have point $w = (x, h) \in U_i \cup U'_j$, which would imply $b_{\mathbf{A}}|x - x(w_i)| \leq h - h(w_i); b_{\mathbf{A}}|x - x(w'_j)| \leq h - h(w'_j); h \leq ad; h(w_i), h(w'_j) \leq -ad$ and $|x(w_i) - x(w'_j)|^2 + |h(w_i) - h(w'_j)|^2 \leq d^2$, an incompatible system of inequalities for our choice of a. The restrictions of Poisson point process to non-intersecting parts of W are independent and the independence in question follows.

Hence, one has

$$\mathbb{E}(I_{\mathcal{E}} \bar{I} \times I_{\mathcal{E}} \bar{I}') = \mathbb{E}(I_{\mathcal{E}} \bar{I}) \times \mathbb{E}(I_{\mathcal{E}} \bar{I}') \times \mathbf{P}(\mathcal{E}). \tag{3.4.3}$$

Now,

$$\bar{r}_{k+l}(w_1,\dots,w_k,w'_1,\dots,w'_l) = \mathbb{E}\big[\big(I_{\mathcal{E}}\tilde{I}+D\big)\big(I_{\mathcal{E}}\tilde{I}'+D'\big)\big] \qquad (3.4.4);$$

and

$$\bar{r}_k(w_1,\dots,w_k)\bar{r}_l(w'_1,\dots,w'_l) = \mathbb{E}\big(I_{\mathcal{E}}\tilde{I}+D\big)\mathbb{E}\big(I_{\mathcal{E}}\tilde{I}'+D'\big). \qquad (3.4.5)$$

Expanding (3.4.4) and (3.4.5), subtracting and taking into account (3.4.3), we get the expression

$$\mathbb{E}(\tilde{I}D') + \mathbb{E}(D\tilde{I}') + \mathbb{E}(DD') - \mathbb{E}\tilde{I}\mathbb{E}\tilde{I}'(1-\mathbb{E}I_{\mathcal{E}}) - \mathbb{E}\tilde{I}\mathbb{E}D' - \mathbb{E}D\mathbb{E}\tilde{I}' - \mathbb{E}D\mathbb{E}D' \qquad (3.4.6)$$

for the difference $\bar{r}_{k+l} - \bar{r}_k\bar{r}_l$.

Notice, that for $\{0,1\}$-valued random elements $A, B, C, \dots$ one has, by Cauchy inequality, $\mathbb{E}(AB) \le (\mathbb{E}A\,\mathbb{E}B)^{1/2}, \mathbb{E}(ABC) \le (\mathbb{E}A\,\mathbb{E}B\,\mathbb{E}C)^{1/3}$ and so on. Recalling that $\tilde{I} \le I, \tilde{I}' \le I'$ and therefore $D = DI, D' = D'I'$, we deduce that (3.4.6) is estimated in absolute value by the sum of the absolute values of the summands,

$$\begin{aligned}(\mathbb{E}I\mathbb{E}I'\mathbb{E}D')^{1/3} + (\mathbb{E}I\mathbb{E}I'\mathbb{E}D')^{1/3} + (\mathbb{E}I\mathbb{E}I'\mathbb{E}D\mathbb{E}D')^{1/4} + \mathbb{E}I\mathbb{E}I'(1-\mathbb{E}I_{\mathcal{E}}) +\\ +\mathbb{E}I(\mathbb{E}I'\mathbb{E}D')^{1/2} + (\mathbb{E}D\mathbb{E}I)^{1/2}\mathbb{E}I' + (\mathbb{E}D\mathbb{E}I)^{1/2}(\mathbb{E}D'\mathbb{E}I')^{1/2}.\end{aligned}$$

The difference (3.4.1) is equal to (3.4.6) multiplied by $\prod_i \rho(w_i)\prod_j \rho(w'_j)$. By Remark 3.2.2, the terms $I^\alpha \prod_i \rho(w_i)$ and $(I')^\alpha \prod_j \rho(w'_j)$ are bounded for any $\alpha > 0$. Hence, the difference (3.4.1) is a linear combination with bounded coefficients of terms $D^\alpha, (D')^\alpha$ and $1 - \mathbb{E}I_{\mathcal{E}}\,(\alpha = 1/3$ or $1/4)$, which uniformly exponentially decrease with d. The Proposition 1.7.2 is proved. □

4. Concluding remarks

4.0. In this section some ramifications and unsettled questions are discussed.

4.1. Generalized Delaunay triangulations and Voronoi tessellations. The process of $\mathcal{S}$-supporting points defines implicitly a more rich structure, manifest in the convex hulls, for example. Specifically, one can associate to (almost every realization of) the point process $\xi_{\mathcal{S}}$ the structure of simplicial complex, joining k supporting points by a simplex if and only if there exists a common set they support. If $\mathcal{S}$ is convex, then the resulting simplicial complex can be realized geometrically as a triangulation (with vertices in the points of $\xi_{\mathcal{S},V}$) of the hyperplane V.

If the intensity measure of ξ is concentrated on V, and the set $\mathcal{S}$ is just the cone $\{h \ge |x|\}$, then we get the standard Poisson Delaunay triangulations.

For the processes associated with the convex hulls of Poisson samples, we get just the simplicial faces of resulting polyhedral surface.

Dually, one can define the generalizations of the Voronoi tessellations: for each vertex w of $\xi_{\mathcal{S}}$ consider the V-projection of the set formed by the apexes of sets supported by w (the boundaries of cells of the tessellations by such sets presumably describe the concentration of matter in the Burgers turbulence approach to the large-scale structure of the Universe).

The central limit theorems of section 1 for the process of supporting points (that is of 0-simplices of the Delaunay triangulations) can be extended without much difficulty to simplices of all dimensions, so that, for example, an analogue of the Theorem 1.8.2 holds:

the number of simplices of any dimension k of the generalized Delaunay triangulation within a large body $\lambda\Lambda$ (quasitransversal to V) is asymptotically normal with expectation and variance growing as λ^n. For the expectations and the convex hulls of large samples from convex bodies, a similar result — the growth order of the expectation of the number of faces in all dimensions is the same — was proved in [**Bi**].

4.2. Constants. The results of this paper all deal only with the orders of the asymptotics. The question of constants is quite tricky and I do not know a general approach. There are some special cases in which the exact densities r and r_2 can be calculated (as certain multidimensional integrals) which will be discussed in a separate publication.

4.3. Vector extremal points in generic polyhedra. The results on the asymptotics of the number of vector extremal points discussed above dealt only with the case of independent coordinates and Pareto cone C, in line with tradition (the expectation was calculated in many papers in a variety of contexts). I would like to emphasize here that, their dissemination notwithstanding, the logarithmic asymptotics of growths for the expectation (and the variance) of the number of Pareto extremal points are far from universal or even generic.

Consider the following situation, where the genericity can be treated more or less precisely. Assume that the points of the sample are uniformly distributed in a convex polyhedron P. The independent coordinates case corresponds to parallelepipeds with facets parallel to the coordinate axes.

It turns out that exactly this latter property is responsible for the logarithmic growth. More precisely, if any linear subspace parallel to a facet of P is transversal to the coordinate subspaces, then both expectation and variance grow as $m^{k/N}$, where k is the maximal dimension of a facet of P belonging to the C-convex hull of P. The logarithmic terms appear when this transversality condition fails.

Figure 3. Facets near which extremal points concentrate are marked.

The figure 3 illustrate this claim. On the left picture, the facets of P lying on the boundary of its C-convex hull are two 1-dimensional edges (facing north-east). Hence the number of Pareto extremal points grows as $m^{1/2}$. On the right picture, only the north-easternmost vertex of P lies on the boundary of its C-convex hull. In this case, the number of Pareto extremal points has bounded mean and variance (actually, converges in distribution). In the intermediate case (middle picture), the independent coordinate case, the logarithmic growth edges in.

One can argue that in natural families (for example, in the family $\{Pg\}_{g\in GL(W)}$) of polyhedra, the condition of transversality formulated above is *not* satisfied on a (singular) hypersurface and is therefore not generic.

To finish, under the transversality condition, if the mean grows unboundedly with m, the central limit theorem can be proved.

I will not give here any details, as these result use different methods than those employed in the present work, referring to [**Bv**].

REFERENCES

[AW] Affentranger, F., Wieacker J.A. On the Convex Hulls of Uniform Random Points, *Discr. Comp. Geometry*, **6**, 291-305, 1991.

[AMS] Alberevio, S., Molchanov, S., Surgalis, D. Stratified structure of the Universe and Burger's equation — a probabilistic approach. *Probab. Theory Relat. Fields*, **100**, 457-484, 1994.

[B-NS] Barndorf-Nielsen O., Sobel M. On the distribution of the number of admissible points in a vector sample. *Probab. Th. and Appl.* **11**, n.2, 1966.

[Bi] Barany, I. Intrinsic volumes and f-vectors of random polytopes. *Math. Ann.*, **285**, 671-699, 1989.

[BL] Barany, I., Larman D. Convex bodies, economic gap covering, random polytopes. *Mathematika*, **35**, 274-291, 1988.

[Bv] Baryshnikov Yu. On the distribution of the number of options non-dominated with respect to a binary relation (in Russian). *Izvestija AN SSSR, Tekhn. kibernetika*, n.3, 1986.

[Bri] Brillinger, D. Statistical inference for stationary point processes, *In: Stochastic processes and related topics, Puri, M. (ed.)*, Academic Press, New York, 1975.

[Buc] Buchta, C. Zufällige Polyeder — Eine Übersicht. *In: Zahlentheoretische Analysis, Hlawka, E. (ed.), LN in Math. 1114*, Springer, Heidelberg, 1985.

[CG] Cabo, J., Groenenboom, P. Limit theorems for functionals of convex hulls, *Prob. Theory Relat. Fields*, **100**, 31-55, 1994.

[Dw] Dwyer R.A. On the Convex Hull of Random Points in a Polytope. *J. Appl. Probab.*, **25**, 688-699, 1988.

[G] Groeneboom, P. Limit theorems for convex hulls, *Prob. Theory Relat. Fields*, **79**, 327-368, 1988.

[GMS] Gurbatov, S., Malakhov A., Saichev, A. *Nonlinear random waves and turbulence in nondispersive media: waves, rays, particles.* Manchester UP, Manchester, 1991.

[H] Hueter, I. *The convex hull of n random points and its vertex process. PhD Thesis*, U of Berne, 1992.

[In] Ivanin V.M. Computation of the variance of the number of elements of Pareto set for a vector sample with independent components (In Russian). *In: Problemy Optimizacii*, Naukova Dumka, Kiev, 1976.

[If] Ivanoff, G. Central limit theorems for point processes. *Stoch. Proc. Appl.*, **12**, 171-186, 1982.

[M] Møller, J. *Lectures on random Voronoi tessellations. (Lecture notes in statistics ; 87)* Springer, Heidelberg, 1994.

[MSW] Molchanov, S., Surgulis, D., Woyczinski, W. Hyperbolic asymptotics in Burger's turbulence and extremal processes, *Comm. Math. Physics*, **168**, 209-226, 1995.

[P] Postnikov, A. *Tauberian theory and its applications.* AMS, Providence, 1980.

[Ru] Ruelle, D. *Statistical Mechanics: Rigorous Results.* W. Benjamin, Amsterdam, 1970.

[Sch] Schneider, R. Random Approximations of Convex Sets. *J. of Microscopy*, **151**, 211-227, 1988.

[VW] Van Wel, B. The convex hull of a uniform sample from the interior of a simple d-polytope. *J. Appl. Prob.*, **26**, 259-273, 1989.