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Adaptive estimators for the endpoint and high quantiles of a probability distribution

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Abstract

One of the major aims of one-dimensional extreme-value theory is to estimate quantiles outside the sample or at the boundary of the sample. The underlying idea of any method to do this is to estimate a quantile well inside the sample but near the boundary and then to shift it somehow to the right place. The choice of this "anchor quantile" plays a major role in the accuracy of the method. We present a bootstrap method to achieve the optimal choice of sample fraction in the estimation of either high quantile or endpoint estimation which extends earlier results by [11] Hall and Weissman (1997) in the case of high quantile estimation. An alternative way of attacking problems like this one is given in a paper by [8] Drees and Kaufmann (1998).

1 Introduction

In problems of coastal safety, one wants to estimate the 10,000 years return level based on one hundred years of observations ([12] de Haan (1990)). In finance one seeks a "value-at-risk" which is basically also a quantile outside the range of available observations ([15] Jansen and de Vries 1991, [3] Danielsson and de Vries 1997).

The situation is the following: we have a sample $X_1, X_2, \ldots, X_n$ from some unknown distribution function F and want to estimate the quantile corresponding to a probability close to 1 i.e. we want x_p with $1 - F(x_p) = p$ and $p \leq c/n$. This inequality means that, if we want to apply asymptotic theory and if in the limiting process we want to maintain this essential feature, we are forced to assume that in fact p depends on $n, p = p_n$ and $\lim_{n\to\infty} p_n = 0$. Then there are still several possibilities: $np_n \to c \in (0, \infty)$ or $np_n \to 0$ $(n \to \infty)$. In both cases purely non-parametric methods do not work. Only if $np_n \to \infty$ non-parametric methods are successful ([9] Einmahl, 1990). The use of models for the tail suggested by extreme value theory stems from the fact that there is no sensible way of extrapolating from an intermediate quantile to one outside the sample unless one uses one of the Generalized Pareto Distributions (GPD)

$$H_\gamma(x) := 1 - (1 + \gamma x)^{-1/\gamma} \quad \text{for those } x \text{ for which } 1 + \gamma x > 0, \tag{1.1}$$

$(\gamma \in \mathbb{R})$ for modelling the tail of F. The tail condition for F is:

$$\lim_{t\to\infty} t \left\{ 1 - F\left((1-F)^{\leftarrow}(\frac{1}{t}) + xa(t) \right) \right\} = 1 - H_\gamma(x) \tag{1.2}$$

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for all x for which $0 < H_\gamma(x) < 1$ where $a(t)$ is a suitable positive function. This means for the quantile function that for $x > 0$

$$\lim_{t\to\infty} \frac{(1-F)^{\leftarrow}(\frac{1}{tx}) - (1-F)^{\leftarrow}(\frac{1}{t})}{a(t)} = \frac{x^\gamma - 1}{\gamma}.$$

For our problem this means

$$(1-F)^{\leftarrow}(p_n) \approx (1-F)^{\leftarrow}(\frac{k}{n}) + a(\frac{n}{k})\frac{(\frac{k}{np_n})^\gamma - 1}{\gamma},$$

i.e. an extreme quantile is linked to an intermediate quantile (which can be estimated via the empirical distribution function) by using the GPD approximation. The extreme quantile estimator based on this relation is

$$\hat{x}_{p_n}(k) := X_{n-k,n} + \hat{a}(\frac{n}{k})\frac{(\frac{k}{np_n})^{\hat{\gamma}_n(k)} - 1}{\hat{\gamma}_n(k)} \tag{1.3}$$

where $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ are the order statistics and $\hat{a}(n/k)$ and $\hat{\gamma}_n(k)$ suitable estimators for $a(n/k)$ and γ ([18] Weissman 1978, [17] Smith 1984, [2] Boos 1984, [16] Joe 1987 and many others). A boundary case is $\gamma < 0$ and $p = 0$. Then the same expression (with $p_n \to 0$) can be used as an estimator of the right endpoint of the probability distribution, in the same GPD set-up.

The choice of k (or rather $n - k$, the index of the order statistics from where on the GPD approximation is believed to be valid) is crucial for the accuracy of the procedure. The optimal value depends on the underlying distribution and is a result of balancing variance and bias components. In this paper we present a bootstrap procedure to obtain this optimal value adaptively. The method is an extension of what we used for obtaining the optimal number of order statistics in estimating γ ([4] Danielsson, de Haan, Peng and de Vries 1997 and [6] Draisma, de Haan, Peng and Pereira 1998). The paper [11] Hall and Weissman (1997) presents a (similar but different) bootstrap method for solving the same optimality problem, not for the quantile but for the exceeding probability of a high level which is similar. Unlike that paper, we do not assume any of the parameters known. Also our conditions on p_n are much more relaxed. The quantile problem is more common in applications than the inverse problem of exceedance probabilities of a high level.

We restrict ourselves to the range $\gamma > -\frac{1}{2}$. This range is most important in applications and in this range it is most efficient to choose a sequence $k = k(n)$ in (1.3) that goes to infinity with n. Also, since we consider tail properties, we have to limit ourselves to sequence $k(n) = o(n), n \to \infty$. Hence we are dealing with intermediate sequences $k(n)$ (i.e., the corresponding order statistics $X_{n-k,n}$ are intermediate):

$$k(n) \to \infty,\ k(n)/n \to 0\ (n \to \infty). \tag{1.4}$$

The main idea is the following. We seek

$$k_0(n) := \arg\inf_k \text{ as. } E(\hat{x}_{p_n}(k) - x_{p_n})^2 \tag{1.5}$$

where as. E means the asymptotic expectation (according to the limit distribution) and k ranges from, say, $\log n$ to $n/(\log n)$ (this expresses the restriction to intermediate sequences and includes the optimal one). Since we are looking for an adaptive method for optimization and since x_{p_n} and the averaging probability measure in (1.5) are unknown, we replace them with sample analogues. So we consider

$$E_n\left(\hat{x}_{p_n}(k) - \hat{\bar{x}}_{p_n}(k)\right)^2 \mathbf{1}_{(|\hat{x}_{p_n}(k) - \hat{\bar{x}}_{p_n}(k)| \le k^\delta)}, \quad \delta > 0 \tag{1.6}$$

where $\hat{x}_{p_n}(k)$ is as before, E_n denotes averaging with respect to the empirical distribution function and

$$\hat{\hat{x}}_{p_n}(k) := X_{n-k,n} + \hat{\hat{a}}(\frac{n}{k})\frac{(\frac{k}{np_n})^{\hat{\hat{\gamma}}_n(k)} - 1}{\hat{\hat{\gamma}}_n(k)} \tag{1.7}$$

with $\hat{\hat{a}}(n/k)$ and $\hat{\hat{\gamma}}_n(k)$ alternative estimators.

The reason why we put the indicator function $\mathbf{1}_{()}$ in (1.6) is to ensure the convergence of the mean square error. For details see [6] Draisma, de Haan, Peng and Pereira (1998). Since δ is an arbitrary positive number, we ignore the indicator function in (1.6), in our simulation study.

The quantity (1.6) depends on the sample only and can be approximated using a bootstrap procedure where the bootstrap sample size has to be chosen of lower order than n in order to avoid unwanted extra randomness. Solving the optimization problem for (1.6) makes sense since the value k_1^* minimizing (1.6) is asymptotically related to the value k_0 from (1.5) and in fact with the help of a second bootstrap we can get k_0 from k_1^*.

The procedure for quantile and endpoint estimation is explained in section 2 which also contains the main results. The most general setting is accounted in section 2.1. We also consider two special cases separately. In quantile estimation, if one restricts to the case γ positive, the asymptotic results may be simplified and become more efficient. This is analysed in section 2.2. All these results use the moment estimator ([5] Dekkers, Einmahl and de Haan, 1989) or simplified versions of it to estimate γ. In section 2.3 we use instead a shift-scale invariant estimator of γ in endpoint estimation. In section 3 we present some simulation results and an application. Finally in section 4 are the proofs of the results of section 2.

2 Main results

2.1 Results for high quantile and endpoint estimation

We start by explaining the method in detail. Then we shall state the precise conditions and present the formal results.

We shall use explicit estimators for $a(\frac{n}{k})$ and γ which are as follows. Define for $j = 1, 2, 3$

$$M_n^{(j)} := \frac{1}{k}\sum_{i=0}^{k-1}(\log\ X_{n-i,n} - \log\ X_{n-k,n})^j, \tag{2.1}$$

$$\hat{\gamma}_n(k) := M_n^{(1)} + 1 - \frac{1}{2}(1 - \frac{(M_n^{(1)})^2}{M_n^{(2)}})^{-1}, \tag{2.2}$$

$$\hat{\hat{\gamma}}_n(k) := \sqrt{M_n^{(2)}/2} + 1 - \frac{2}{3}(1 - \frac{M_n^{(1)}M_n^{(2)}}{M_n^{(3)}})^{-1}, \tag{2.3}$$

$$\hat{a}(\frac{n}{k}) := X_{n-k,n}M_n^{(1)}/\rho_1(\hat{\gamma}_n(k)) \tag{2.4}$$

$$\hat{\hat{a}}(\frac{n}{k}) := X_{n-k,n}M_n^{(1)}/\rho_1(\hat{\hat{\gamma}}_n(k)) \tag{2.5}$$

where $\hat{\gamma}_n(k)$ and $\hat{a}(\frac{n}{k})$ are the estimators in (1.3) and $\hat{\hat{\gamma}}_n(k)$ and $\hat{\hat{a}}(\frac{n}{k})$ the alternative estimators in (1.7), and $\rho_1(\gamma) = (1-\gamma_-)^{-1}$. We denote $\min(\gamma, 0)$ by γ_- and $\max(\gamma, 0)$ by γ_+.

Step 1 Select randomly and independently n_1 times ($n_1 << n$) a member from the set $\{X_1, X_2, \ldots, X_n\}$. Indicate the result by $X_1^*, X_2^*, \ldots X_{n_1}^*$. Form the order statistics $X_{1,n_1}^* \leq X_{2,n_1}^* \leq \cdots \leq X_{n_1,n_1}^*$ and compute the quantities (1.3) and (1.7) from (2.1-2.5) on the basis of these order statistics.

We denote the resulting quantities by $\hat{\gamma}_{n_1}^*(k)$, $\hat{\hat{\gamma}}_{n_1}^*(k)$, $\hat{a}^*(n_1/k)$ and $\hat{\hat{a}}^*(n_1/k)$, $\hat{x}_{p_n}^*(k)$, $\hat{\hat{x}}_{p_n}^*(k)$ for $k = 1, 2, \ldots, n_1 - 1$. Form

$$q_{n_1,k}^* = (\hat{x}_{p_n}^*(k) - \hat{\hat{x}}_{p_n}^*(k))^2$$

on the basis of these bootstrap estimators.

Step 2 Repeat step 1 r times independently. This results in a sequence $q^*_{n_1,k,s}$, $k = 1, 2, \ldots, n_1 - 1$ and $s = 1, 2, \ldots, r$. Calculate

$$\frac{1}{r}\sum_{s=1}^{r} q^*_{n_1,k,s}.$$

Step 3 Minimize $\frac{1}{r}\sum_{s=1}^{r} q^*_{n_1,k,s}$ with respect to k but reject values which are very small or very near to n_1. Denote the value of k where the minimum is obtained by $k_0^*(n_1)$.

Step 4 Repeat step 1 up to 3 independently with the number n_1 replaced by $n_2 = (n_1)^2/n$. So n_2 is smaller then n_1. This results in $k_0^*(n_2)$.

Step 5 Calculate

$$\hat{k}_0(n) := \frac{(k_0^*(n_1))^2}{k_0^*(n_2)} \frac{h(\hat{\gamma}_n^+(k), \hat{\gamma}_n^-(k), \hat{\rho}'_{n_1}(k_0^*))}{\overline{h}(\hat{\gamma}_n^+(k), \hat{\gamma}_n^-(k), \hat{\rho}'_{n_1}(k_0^*))}$$

with $\hat{\gamma}_n^+(k)$ and $\hat{\gamma}_n^-(k)$ any consistent estimators of γ_+ and γ_-,

$$\hat{\rho}'_{n_1}(k_0^*) := \frac{\log k_0^*(n_1)}{-2\log n_1 + 2\log k_0^*(n_1)}$$

a consistent estimator of ρ' and the functions h and $\overline{h}$ from Propositions 4.12 and 4.13 below respectively.

This $\hat{k}_0(n)$, which is obtained adaptively, is asymptotically as good as the optimal number of order statistics in (1.5).

Now in order to be able to present our main result we have to state the conditions.

Suppose that the underlying distribution function F is in the domain of attraction of an extreme value distribution (or equivalently that the observations above a large threshold have an asymptotic GPD distribution). We formulate this condition analytically in terms of the quantile-type function $U := (\frac{1}{1-F})^{\leftarrow}$:

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma} \tag{2.6}$$

for all positive x, where $a(t)$ is a suitable positive function. We shall need a second order refinement of this relation which reads as follows: there is a function $A(t) \to 0$ with constant sign near infinity such that for all $x > 0$

$$\lim_{t\to\infty} \frac{\frac{U(tx)-U(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma}}{A(t)} = \frac{1}{\rho}\left[\frac{x^{\gamma+\rho} - 1}{\gamma+\rho} - \frac{x^\gamma - 1}{\gamma}\right] \tag{2.7}$$

with $\rho \le 0$. For the final result we shall have to require $\rho < 0$, $a(t) \sim c_1 t^\gamma$ and $A(t) \sim \overline{c}_2 t^\rho (t \to \infty)$ and in this case (2.7) is equivalent to

$$U(t) = c_0 + c_1 \frac{t^\gamma - 1}{\gamma} + c_2 t^{\gamma+\rho} + o(t^{\gamma+\rho}) \quad \text{with} \quad c_1 > 0, \quad c_2 \neq 0 \qquad (t \to \infty). \tag{2.8}$$

Theorem 2.1. *Suppose $U := \left(\frac{1}{1-F}\right)^{\leftarrow}$ satisfies (2.8). If $\rho < 0$, $\gamma > -1/2$, $\gamma \neq 0$, $\gamma \neq \rho$, $np_n \to c$ (finite, ≥ 0) and $\log p_n = o\left(n^{\frac{-\rho'}{1-2\rho'}}\right)$ $(n \to \infty)$ where ρ' is defined in Lemma 4.1 below. Then for $k_0(n)$ as in (1.5)*

$$\lim_{n\to\infty} \frac{\hat{k}_0(n)}{k_0(n)} = 1$$

in probability, where

$$\hat{k}_0(n) = \frac{(k_0^*(n_1))^2}{k_0^*(n_2)} \frac{h(\hat{\gamma}_n^+(k), \hat{\gamma}_n^-(k), \hat{\rho}'_{n_1}(k_0^*))}{\overline{h}(\hat{\gamma}_n^+(k), \hat{\gamma}_n^-(k), \hat{\rho}'_{n_1}(k_0^*))} \tag{2.9}$$

with $\hat{\gamma}_n^+(k), \hat{\gamma}_n^-(k)$ any consistent estimators of γ_+ and γ_-,

$$\hat{\rho}'_{n_1}(k_0^*) = \frac{\log k_0^*(n_1)}{-2\log n_1 + 2\log k_0^*(n_1)} \tag{2.10}$$

and the functions h and $\overline{h}$ from Propositions 4.12 and 4.13 below respectively.

Hence the asymptotic second moment of the estimator $\hat{x}_{p_n}(k)$ is asymptotically the same whether it is based on $\hat{k}_0(n)$ upper order statistics or on $k_0(n)$ upper order statistics.

Remark 2.2. Since ρ is not known, one could alternatively require $\log p_n = o(n^\varepsilon)$ for all $\varepsilon > 0$.

Theorem 2.3. *Under the conditions of Theorem 2.1, the value $k_0(n)$ of k minimizing the asymptotic second moment of $\hat{x}_{p_n}(k) - x_{p_n}$ satisfies*

$$k_0(n) \sim h(\gamma_+, \gamma_-, \rho')\, n^{\frac{-2\rho'}{1-2\rho'}} \qquad (n \to \infty). \tag{2.11}$$

Remark 2.4. Since the order of magnitude is the same as in the case of minimizing the mean square error of the moment estimator $\hat{\gamma}_n(k)$ (only the constant differs), we could use the bootstrap procedure for one of them in order to get the optimal value for the other.

Next we turn our attention to the estimation of the right endpoint x_0 of the probability distribution when $\gamma < 0$. Define (cf. [5] Dekkers, Einmahl and de Haan, 1989)

$$\hat{x}_0(k) := X_{n-k,n} - \frac{\hat{a}(\frac{n}{k})}{\hat{\gamma}_n^-(k)} \tag{2.12}$$

where

$$\hat{\gamma}_n^-(k) := 1 - \frac{1}{2}(1 - \frac{(M_n^{(1)})^2}{M_n^{(2)}})^{-1}. \tag{2.13}$$

We seek

$$k_0(n) := \arg\inf_k \text{ as. } E(\hat{x}_0(k) - x_0)^2. \tag{2.14}$$

In order to construct an adaptive estimator for $k_0(n)$ we consider an alternative estimator for x_0, namely

$$\hat{\hat{x}}_0(k) := X_{n-k,n} - \frac{\hat{\hat{a}}(\frac{n}{k})}{\hat{\hat{\gamma}}_n^-(k)} \tag{2.15}$$

where

$$\hat{\hat{\gamma}}_n^-(k) := 1 - \frac{2}{3}(1 - \frac{M_n^{(1)} M_n^{(2)}}{M_n^{(3)}})^{-1}. \tag{2.16}$$

Now for $\hat{\hat{x}}_0(k)$ we apply the same bootstrap procedure as described before for $\hat{x}_{p_n}(k)$, but with the constants $h(\hat{\gamma}_n^+(k), \hat{\gamma}_n^-(k), \hat{\rho}'_{n_1}(k_0^*))$ and $\overline{h}(\hat{\gamma}_n^+(k), \hat{\gamma}_n^-(k), \hat{\rho}'_{n_1}(k_0^*))$ replaced by $g(\hat{\gamma}_n^-(k), \hat{\rho}'_{n_1}(k_0^*))$ and $\overline{g}(\hat{\gamma}_n^-(k), \hat{\rho}'_{n_1}(k_0^*))$ respectively.

Theorem 2.5. *Suppose* $U := (\frac{1}{1-F})^{\leftarrow}$ *satisfies (2.8). If* $\rho < 0, -1/2 < \gamma < 0$ *and* $\gamma \neq \rho$*, then for* $k_0(n)$ *as in (2.14)*

$$\lim_{n\to\infty} \frac{\hat{k}_0(n)}{k_0(n)} = 1$$

in probability, where

$$\hat{k}_0(n) = \frac{(k_0^*(n_1)^2)}{k_0^*(n_2)} \; \frac{g(\hat{\gamma}_n^-(k), \hat{\rho}'_{n_1}(k_0^*))}{\bar{g}(\hat{\gamma}_n^-(k), \hat{\rho}'_{n_1}(k_0^*))} \tag{2.17}$$

with $\hat{\gamma}_n^-(k)$ *any consistent estimate of* γ_-*,*

$$\hat{\rho}'_{n_1}(k_0^*) = \frac{\log\ k_0^*(n_1)}{-2\log\ n_1 + \log\ k_0^*(n_1)}$$

and the functions g *and* $\bar{g}$ *from Propositions 4.14 and 4.15 below respectively.*

Hence the asymptotic second moment of the estimator $\hat{x}_0(k)$ is asymptotically the same whether it is based on $\hat{k}_0(n)$ upper order statistics or on $k_0(n)$ upper order statistics.

Theorem 2.6. *Under the conditions of Theorem 2.5, the value* $k_0(n)$ *of* k *minimizing the asymptotic second moment of* $\hat{x}_0(k) - x_0$ *satisfies*

$$k_0(n) \sim g(\gamma_-, \rho')\, n^{\frac{-2\rho'}{1-2\rho'}} \qquad (n \to \infty). \tag{2.18}$$

2.2 Results for quantile, positive γ

Suppose we know, or assume, $\gamma > 0$ and want to estimate a high quantile. Confined to this situation, in this section we present the required asymptotic results to apply the bootstrap procedure as described in the last section. To estimate the quantile we use

$$\hat{x}_{p_n}^+(k) := X_{n-k,n} \left(\frac{k}{np_n}\right)^{\hat{\gamma}_n^+(k)} \qquad \text{where} \qquad \hat{\gamma}_n^+(k) := M_n^{(1)}\,. \tag{2.19}$$

Let

$$\hat{\hat{x}}_{p_n}^+(k) := X_{n-k,n} \left(\frac{k}{np_n}\right)^{\hat{\hat{\gamma}}_n^+(k)} \qquad \text{where} \qquad \hat{\hat{\gamma}}_n^+(k) := \sqrt{\frac{M_n^{(2)}}{2}} \tag{2.20}$$

be a first option to the alternative quantile estimator and

$$\hat{\hat{\hat{x}}}_{p_n}^+(k) := X_{n-k,n} \left(\frac{k}{np_n}\right)^{\hat{\hat{\hat{\gamma}}}_n^+(k)} \qquad \text{where} \qquad \hat{\hat{\hat{\gamma}}}_n^+(k) := \frac{M_n^{(2)}}{2M_n^{(1)}}\,. \tag{2.21}$$

be a second option to the alternative quantile estimator.

Theorem 2.7. *Suppose the second order condition (2.8) holds for* $\gamma > 0$ *and* $\rho < 0$*. Assume* $\gamma \neq \rho$*,* $np_n \to c$ *(finite,* ≥ 0*) and* $\log p_n = o(n^{\varepsilon})$ *for* $\varepsilon > 0$*, as* $n \to \infty$*. Then*

$$k_0(n) \sim l(\gamma_+, \rho\prime)\, n^{\frac{-2\rho\prime}{1-2\rho\prime}} \quad as \quad n \to \infty\,, \tag{2.22}$$

where $k_0(n) := \arg\min_k$ *as.* $E\left(\hat{x}_{p_n}^+(k) - x_{p_n}\right)^2$ *and the function* l *from Proposition 4.16 below.*

Theorem 2.8. *Suppose the second order condition (2.8) holds for $\gamma > 0$ and $\rho < 0$. Assume $\gamma \neq \rho$, $np_n \to c$ (finite, ≥ 0) and $\log p_n = o(n^{\varepsilon})$ for $\varepsilon > 0$, as $n \to \infty$. Then*

$$\bar{k}_0(n) \sim \bar{l}(\gamma_+, \rho') \, n^{\frac{-2\rho'}{1-2\rho'}} \qquad as \qquad n \to \infty \,, \tag{2.23}$$

where $\bar{k}_0(n) := \arg\min_k$ as. $E\left(\hat{x}^+_{p_n}(k) - \hat{\bar{x}}^+_{p_n}(k)\right)^2$ and the function $\bar{l}$ from Proposition 4.17 below.

Theorem 2.9. *Suppose the second order condition (2.8) holds for $\gamma > 0$ and $\rho < 0$. Assume $\gamma \neq \rho$, $np_n \to c$ (finite, ≥ 0) and $\log p_n = o(n^{\varepsilon})$ for $\varepsilon > 0$, as $n \to \infty$. Then*

$$\bar{\bar{k}}_0(n) \sim \bar{l}(\gamma_+, \rho') \, n^{\frac{-2\rho'}{1-2\rho'}} \qquad as \qquad n \to \infty \,, \tag{2.24}$$

where $\bar{\bar{k}}_0(n) := \arg\min_k$ as. $E\left(\hat{x}^+_{p_n}(k) - \hat{\bar{\bar{x}}}^+_{p_n}(k)\right)^2$ and the function $\bar{l}$ from Proposition 4.18 below.

Remark 2.10. As discussed in section 3 it is advantageous to have a small ratio of the function multiplying $n^{-2\rho'/(1-2\rho')}$ in, for example, (2.23) to the function multiplying $n^{-2\rho'/(1-2\rho')}$ in (2.22). Note that in quantile estimation for positive γ we got the same function (cf. (2.23) and (2.24)) whether using $\hat{\bar{x}}_{p_n}(k)$ or $\hat{\bar{\bar{x}}}_{p_n}(k)$ as alternative estimator. However the asymptotic mean square error in Theorem 2.9 is four times the asymptotic mean square error in Theorem 2.8 (cf. proof of these Theorems).

2.3 Results for endpoint with a shift-scale invariant estimator of γ

Here the endpoint estimator itself, as motivated earlier remains the same i.e., we still use as in (2.12)

$$\hat{x}'_0(k) := X_{n-k,n} - \frac{\hat{a}'(\frac{n}{k})}{\hat{\gamma}_n^{-'}(k)}. \tag{2.25}$$

The main difference lies in the quantities $M_n^{(j)}$ (2.1) that change to the following

$$N_n^{(j)} \;:=\; \frac{1}{k}\sum_{i=0}^{k-1}(X_{n-i,n} - X_{n-k,n})^j, \quad j = 1, 2, 3. \tag{2.26}$$

Since γ is negative we shall use

$$\hat{\gamma}_n^{-'}(k) \;:=\; 1 - \frac{1}{2}(1 - \frac{(N_n^{(1)})^2}{N_n^{(2)}})^{-1} \tag{2.27}$$

to estimate the extreme value index. Note that (2.27) is shift and scale invariant whilst the extreme value index estimators used in the previous sections are just scale invariant. In what concerns the estimation of $a(\frac{n}{k})$ it changes to

$$\hat{a}'(\frac{n}{k}) \;:=\; N_n^{(1)}/\rho_1(\hat{\gamma}_n^{-'}(k))\,. \tag{2.28}$$

In what regards the alternative estimators necessary for the bootstrap procedure just apply the same scheme as in section 2.1 for the endpoint. Substitute in (2.16) $M_n^{(j)}$, $j = 1, 2, 3$ by $N_n^{(j)}$, $j = 1, 2, 3$, respectively, to get $\hat{\bar{\gamma}}_n^{-'}(k)$. Substitute in (2.28) $\hat{\gamma}_n^{-'}(k)$ by $\hat{\bar{\gamma}}_n^{-'}(k)$ to get $\hat{\bar{a}}'(\frac{n}{k})$, to finally obtain $\hat{\bar{x}}'_0(k)$.

We now state the main result. Note the resemblance with Theorem 2.6.

Theorem 2.11. *Suppose the second order condition (2.8) holds. If $\rho < 0$ and $-1/2 < \gamma < 0$, then the value $k_0(n)$ of k minimizing the asymptotic second moment of $\hat{x}_0'(k) - x_0$ satisfies*

$$k_0(n) \sim g(\gamma_-, \rho)\, n^{\frac{-2\rho}{1-2\rho}} \qquad (n \to \infty). \tag{2.29}$$

Therefore, Theorem 2.5 still applies with the functions g and $\bar{g}$ from Propositions 4.14 and 4.15 below respectively, but in the case of g take always

$$c_8 := \frac{(2\gamma_- - 6\gamma_-^2 + 4\gamma_-^3 + \rho - 5\gamma_-\rho + 6\gamma_-^2\rho + 2\gamma_-\rho^2)^2}{\gamma_-^4(1-\gamma_- - \rho)^2(\gamma_- + \rho)^2(1 - 2\gamma_- - \rho)^2}$$

and in g and $\bar{g}$ replace $\tilde{c}_2$ by $\bar{c}_2$ (cf. (2.7)-(2.8)).

3 Applications to simulated and real data

3.1 Simulation results

The simulations are based on the following three types of distribution functions.

3.1.1 Generalized Extreme Value distribution (in accordance with theory let $\gamma \neq 0, -1$)

Let $G_\gamma(x) = \exp\{-(1+\gamma x)^{-1/\gamma}\}$, $1 + \gamma x > 0$. The function $U(t) = F^{\leftarrow}(1 - 1/t)$ is given by $U(t) = ((-\log(1-1/t))^{-\gamma} - 1)/\gamma$, $t > 1$, where $\lim_{t\to\infty} U(t) = U(\infty) = -1/\gamma$ if $\gamma < 0$ and $U(\infty) = \infty$ if $\gamma > 0$. Expanding the function $U(t)$, if $\gamma \neq 1$,

$$U(t) = \frac{t^\gamma - 1}{\gamma} - \frac{1}{2}t^{\gamma-1} + o\left(t^{\gamma-1}\right) \quad \text{as } t \to \infty\,, \tag{3.1}$$

and if $\gamma = 1$,

$$U(t) = -\frac{1}{2} + (t-1) - \frac{t^{-1}}{12} + o\left(t^{-1}\right) \quad \text{as } t \to \infty\,, \tag{3.2}$$

and so (2.8) holds with (ρ, c_0, c_1, c_2) equal to $(-1, 0, 1, -1/2)$ if $\gamma \neq 1$ and $(-2, -1/2, 1, -1/12)$ if $\gamma = 1$.

In case of a sample with negative data it has to be translated in order to get just positive data, so that the estimators may be applied (remember the condition $U(\infty) > 0$). Since $U(\infty) = c_0 - c_1/\gamma$, for $\gamma < 0$, the effect of a translation, say adding a positive constant a to the data, changes c_0 to $c_0 + a$. If $\gamma > 0$ the translation has no effect on the asymptotic behaviour of $U(t)$. Hence the functions required in the first and second order conditions in terms of $U(t)$ (see (2.7)) may be taken as $a(t) = c_1 t^\gamma = t^\gamma$ and $A(t) = \rho(\gamma+\rho)c_2 t^\rho / c_1 = (\gamma - 1)t^{-1}/2$ if $\gamma \neq 1$ and $t^{-2}/6$ if $\gamma = 1$, as $t \to \infty$. The function required in the second order condition for $\log U(t)$ (cf. Lemma 4.1) may be taken as $(t \to \infty)$

$$\widetilde{A}(t) = \begin{cases} A(t) = \frac{\gamma-1}{2}\, t^{-1} & ,\gamma < -1 \\ \frac{a(t)}{U(\infty)+a} = \frac{\gamma}{\gamma a - 1}\, t^\gamma & ,-1 < \gamma < 0 \\ \gamma - \frac{a(t)}{U(t)} \sim \frac{c_0 - c_1/\gamma}{c_1/\gamma}\, t^{-\gamma} = t^{-\gamma} & ,0 < \gamma < 1 \\ \gamma - \frac{a(t)}{U(t)} \sim \frac{3}{2}\, t^{-1} & ,\gamma = 1 \\ \frac{\rho A(t)}{\gamma+\rho} = \frac{t^{-1}}{2} & ,\gamma > 1\ . \end{cases} \tag{3.3}$$

Note that $\lim_{t\to\infty}(U(t) - a(t)/\gamma) = -1/\gamma + a$ if $\gamma < 0$, $-1/\gamma$ if $0 < \gamma < 1$, -3/2 if $\gamma = 1$ and $-\infty$ if $\gamma > 1$.

3.1.2 Reversed Burr distribution (in accordance with theory let $\tau \neq 1$)

A random variable (r.v.) Y is said to have Burr distribution function (d.f.) with parameters β, λ and τ if $F_Y(y) = 1 - \beta^\lambda/(\beta + y^\tau)^\lambda$, $y > 0$, $\beta, \lambda, \tau > 0$. Let $X = -Y^{-1}$. Then X is said to have a Reversed Burr distribution, say $RB_{\beta,\lambda,\tau}$, with d.f. given by $F_X(x) = 1 - \beta^\lambda/(\beta + (-x)^{-\tau})^\lambda$, $x < 0 = x_0$, $\beta, \lambda, \tau > 0$. This d.f. is being used in financial applications. In order to properly use simulated data from this model it must then be shifted by a positive constant, say a, so that $x_0 = a$. Therefore $U(t) = a - \beta^{-1/\tau}(t^{1/\lambda} - 1)^{-1/\tau}$, $t > 1$, and $\lim_{t\to\infty} U(t) = U(\infty) = a$. Expanding this function we get

$$\begin{aligned} U(t) &= a - \beta^{-1/\tau} + \frac{\beta^{-1/\tau}}{\lambda\tau}\frac{t^{-1/\lambda\tau} - 1}{-1/\lambda\tau} - \frac{\beta^{-1/\tau}}{\tau}t^{-1/\lambda\tau - 1/\lambda} \\ &\quad + o\left(t^{-1/\lambda\tau - 1/\lambda}\right) \quad \text{as } t \to \infty\,, \end{aligned} \tag{3.4}$$

and so (2.8) holds with $(\gamma, \rho, c_0, c_1, c_2)$ equal to $(-1/\lambda\tau,\ -1/\lambda,\ a - \beta^{-1/\tau},\ \beta^{-1/\tau}/\lambda\tau,\ -\beta^{-1/\tau}/\tau)$. The functions required in the first and second order conditions in terms of $U(t)$ may be taken as $a(t) = \beta^{-1/\tau}t^{-1/\lambda\tau}/\lambda\tau$ and $A(t) = (1+\tau)t^{-1/\lambda}/\lambda\tau$, as $t \to \infty$. The function required in the second order condition in terms of $\log U(t)$ may be taken as $(t \to \infty)$

$$\widetilde{A}(t) = \begin{cases} \frac{1+\tau}{\lambda\tau}\, t^{-1/\lambda} & ,\tau < 1 \\ \frac{\beta^{-1/\tau}}{a\lambda\tau}\, t^{-1/\lambda\tau} & ,\tau > 1\ . \end{cases} \tag{3.5}$$

3.1.3 Cauchy distribution.

Let X with d.f. $F_X(x) = (\arctan x + \pi/2)/\pi$, $x \in \mathbb{R}$. Then $U(t) = \tan(\pi/2 - \pi/t)$, $t > 1$ and $\lim_{t\to\infty} U(t) = U(\infty) = \infty$. Expanding this function we get

$$U(t) = 1/\pi + \frac{1}{\pi}(t-1) - \frac{\pi}{3}t^{-1} + o\left(t^{-1}\right) \quad \text{as } t \to \infty\,, \tag{3.6}$$

and so (2.8) holds with $(\gamma, \rho, c_0, c_1, c_2)$ equal to $(1, -2, 1/\pi, 1/\pi, -\pi/3)$. The functions required in the first and second order conditions in terms of $U(t)$ may be taken as $a(t) = t/\pi$, $A(t) = 2\pi^2 t^{-2}/3$, as $t \to \infty$, and the function required in the second order condition in terms of $\log U(t)$ may be taken as $\widetilde{A}(t) = 4\pi^2 t^{-2}/3$, as $t \to \infty$. Note that $\lim_{t\to\infty}(U(t) - a(t)/\gamma) = 0$ (if γ is positive it holds whenever $0 < \gamma < -\rho$ and $c_0 - c_1/\gamma = 0$).

3.1.4 Simulation results

Two collections of simulation results are presented. The first, based on samples of moderate size and on endpoint estimation, intends to discuss briefly with an example the choice of some parameters required when using the bootstrap, namely n_1 - the size of the first bootstrap resample - and r - the number of bootstrap resamples. The second concerns endpoint and high quantiles estimations from samples of larger size and from several d.f.s with various first and second order parameters γ and ρ.

Thus we start by discussing the influence of varying n_1 with n fixed. Table 1 summarizes the results on endpoint estimation of 100 simulations with independent samples of size $n = 2000$ from $G_{-.25}$ for each $n_1 = 500(250)1750$, where the final bootstrap estimates of k_0 and z_F are presented. For instance there is no clear trend along n_1 in terms of the means of $\hat{x}_0$ but instead a fluctuation around its true value. Also the estimated mean square error (mse, in the table is the square root of it) do not reveal any tendency for increasing or decreasing with n_1. Nonetheless when n_1 (or n_2) is small the minimum of the estimated mse is often unclear and shows a tendency to be attained near n_1 (respectively n_2). Also due to a slight bias on this kind of estimators it is found advisable not to take too small values of n_1 (respectively n_2). On the other hand as n_1 (respectively n_2) increases the results become more unstable in the sense that the number of abortions increases with n_1 (respectively

n_1 (Interv. to look for k_1)	n_2 (Interv. to look for k_2)	$\hat{k}_0$ mean	st. dev.	$\hat{x}_0$ ($x_0 = 4$) mean	rootmse	Abort. Simul.
500 (10,400)	125 (10,100)	50.5	32.1	4.06	2.22	7
750 (10,600)	281 (10,220)	58.6	42.5	3.75	1.01	9
1000 (10,880)	500 (10,400)	60.7	46.5	3.82	.87	10
1250 (10,1000)	781 (10,620)	53.1	36.0	4.03	3.07	12
1500 (10,1200)	1125 (10,900)	55.3	41.9	3.81	.69	24
1750 (10,1400)	1531 (10,1220)	54.1	31.6	4.00	2.00	17

Table 1: Simulation results, bootstrap endpoint estimation with 100 independent samples of size 2000 from $G_{-.25}$ and $r = 300$.

n_2). Hence our advise is to take approximately $n_1 = n/2$ (which corresponds to ε to be approximately equal to $\log 2/\log n$ in $n_1 = 0(n^{1-\varepsilon})$; cf. [6] Draisma, de Haan, Peng and Pereira, 1998).

In table 1 by each n_i is the range within which k_i, $i = 1, 2$, minimizing the estimated mse was obtained. Due to the asymptotic properties of k_i with respect to n_i it makes no sense to look for k_i within a range of very small values (in applications we cut it at 10) and of very large values. In fact when k_i is near n_i the estimated mse frequently shows a sudden downturn to zero. Therefore it is advisable to have a look at the estimated mse to avoid *nonsense* minima.

In what concerns the number of bootstrap resamples, in practice for each n_i the consecutive solutions of k_i along the bootstrap resamples start stabilizing so that 300 replications (denoted by $r = 300$) seem fairly enough in all cases.

To end this first analysis, as it might be seen in table 1, not every simulations work well. The reasons for aborting are the following: k_1 is less or equal to k_2; the consistent estimate of γ is greater than zero; due to bad estimates of k_1 and/or k_2 or of the consistent estimate of γ, $\hat{k}_0$ is 0 or 1; the bootstrap estimate of γ is positive.

In table 2 are summarized some results from both endpoint and high quantiles estimation. Each simulation result is based on 30 independent samples of size 10000 from the three d.f.s presented earlier. In all cases $n_1 = 5000$ and $r = 300$. Below each bootstrap estimated mean of $\rho\prime$, γ and $\hat{x}_0$ or $\hat{x}_{10^{-5}}$ is the correspondent true value.

In what concerns the endpoint estimates they are close to the true value on average with very reasonable mse. Comparing with the classical estimation taking simply $k_0 = \sqrt{n}$ (see figure 1) the reduction in variance is clear and in the positive asymmetry of the sample of the estimates when using the bootstrap procedure. We note however that we have just presented results for distributions verifying $\rho < \gamma < 0$ and in the algorithm right before the calculation of $\hat{k}_0$ after calculating the consistent estimates of γ and $\rho\prime$ we make the following choice: if the ratio of γ over ρ is not greater than one then we assume $\rho < \gamma < 0$ and just use for estimating ρ' the consistent estimate of γ (since in this case $\rho\prime = \gamma$). In that way we avoid the bad estimation of the second order parameter. Indeed the usual models verify $\rho < \gamma < 0$.

In what concerns quantile estimation the results are rather irregular. In fact recall that we had to deal with a wider range of theoretical conditions than on endpoint estimation. The conditions are $\gamma < \rho$, $\rho < \gamma < 0$, $0 < \gamma < -\rho$ & $\lim_{t\to\infty}(U(t) - a(t)/\gamma) \neq 0$, $0 < \gamma < -\rho$ & $\lim_{t\to\infty}(U(t) - a(t)/\gamma) = 0$ or $\gamma > -\rho$ which affect the estimation of the bias functions involved in the algorithm.

Regarding to all the simulation results one sees that the estimates of $\rho\prime$ are in general not good. Moreover theoretically $|\rho\prime/\gamma| \leq 1$ must hold if $\gamma < 0$ or $\gamma > 0$ & $\lim_{t\to\infty}(U(t) - a(t)/\gamma) \neq 0$. Therefore in order to deal with it and to make simulations valuable the following procedure was adopted. After getting the consistent estimates of γ and $\rho\prime$, say $\hat{\gamma}$ and $\hat{\rho}\prime$: *(i)* if $\hat{\gamma} < 0$ check whether $\hat{\rho}\prime/\hat{\gamma} < 1$. *(i-a)* If not then assume $\rho < \gamma < 0$ and use only $\hat{\gamma}$ that is, assume $\rho\prime = \gamma$ and take the same estimate to both. *(ii)* If $\hat{\gamma} > 0$ and if $\lim_{t\to\infty}(U(t) - a(t)/\gamma) \neq 0$ (we assume this known in the simulations) proceed as under *(i)*. *(iii)* If $\lim_{t\to\infty}(U(t) - a(t)/\gamma) = 0$ then there is no possible improvement for the estimates of $\rho\prime$ and then $\hat{\rho}\prime$ is used for the estimation of the bias. In simulations almost always

ENDPOINT	$\hat{k}_0$ mean	st.dev.	$\hat{\rho\prime}$ mean (true)	st.dev.	$\hat{\gamma}$ mean (true)	st.dev.	$\hat{x}_0$ mean (true)	rootmse	st.dev./ mean
$GEV_{-.25}$	112.0	76.8	-2.69 (-1)	.64	-.31 (-.25)	.10	3.88 (4)	.39	.10
$RB_{4,4,2}$	73.2	97.0	-1.93 (-.25)	1.29	-.36 (-.125)	.45	-.09 (0)	.20	-1.22

QUANTILE	$\hat{k}_0$ mean	st.dev.	$\hat{\rho\prime}$ mean (true)	st.dev.	$\hat{\gamma}$ mean (true)	st.dev.	$\hat{x}_{10^{-5}}$ mean (true)	rootmse	st.dev./ mean
$GEV_{-.25}$	6367.2	2341.1	-7.41 (-1)	2.12	-.65 (-.25)	.30	2.64 (3.78)	1.34	.28
$RB_{4,4,2}$	2395.3	1829.6	-1.47 (-.25)	.43	-.64 (-.125)	.31	-.26 (-.12)	.17	-.38
$GEV_{.5}$	2397.9	999.6	-24.27 (-1)	18.3	.50 (.5)	.02	648. (631.)	113.	.17
$GEV_{.5}$*	7488.0	1372.0	-20.64 (-1)	14.4	.46 (.5)	.03	565. (631.)	147.	.23
Cauchy(1)	6181.7	328.4	-9.30 (-2)	1.15	.50 (1)	.01	1401. (31831.)	30432.	.21
$GEV_{1.5}$	2976.5	1417.3	-5.27 (-1)	2.69	1.53 (1.5)	.12	$.31 \times 10^8$ $(.21 \times 10^8)$	$.18 \times 10^8$	.47

(1) These do not include the severe outlier shown on Cauchy boxplot, fig. 2.

Table 2: Summary of bootstrap simulation results with $n = 10000$, $r = 300$ and 30 simulations of each; see table 3 for more details.

ENDPOINT	n_1 (Inter. to look for k_1)	n_2 (Interv. to look for k_2)	a	Abort. Simul.
$GEV_{-.25}$	5000 (10,4000)	2500 (10,2000)	4	4
$RB_{4,4,2}$	5000 (10,4000)	2500 (10,2000)	2359	4

QUANTILE	n_1 (Interv. to look for k_1)	n_2 (Interv. to look for k_2)	a	Abort. Simul.
$GEV_{-.25}$	5000 (10,4000)	2500 (10,2000)	4	16
$RB_{4,4,2}$	5000 (10,4000)	2500 (10,2000)	513	8
$GEV_{.5}$	5000 (10,4750)	2500 (10,2375)	2	1
$GEV_{.5}$*	5000 (10,4750)	2500 (10,2375)	2	0
Cauchy	5000 (10,4900)	2500 (10,2475)	5573	3
$GEV_{1.5}$	5000 (10,4999)	2500 (10,2499)	1	17

Table 3: Simulation parameters, shift (a) and number of abortions.

$GEV_{-.25}$ and $RB_{4,4,2}$ verify *(i)* and *(i-a)*, $GEV_{.5}$ verifies *(ii)*, Cauchy verifies *(iii)* and $GEV_{1.5}$ is on the same pratical situation as in *(iii)*.

However in applications it is not clear how to get $\lim_{t\to\infty}(U(t)-a(t)/\gamma)$. Indeed that decision may be avoided but then $\hat{\rho\prime}$ must be always considered as in *(iii)* (see Remark 4.19). The $GEV_{.5}$* results exemplify this situation.

We will now comment on the quantile simulation results resumed on table 2 and figure 2. We exemplify with $x_{10^{-5}}$ which corresponds to $p = 10^{-5} \approx 1/(n \log n)$ for $n = 10000$.

Starting with $GEV_{-.25}$ and $RB_{4,4,2}$ one may see large means and standard deviations of $\hat{k}_0$ and also large simulated mse of quantile bootstrap estimates. In fact in figure 2 we see that the classical procedure ($k_0 = \sqrt{n}$) performs better than the bootstrap one. Comparing quantile and endpoint estimation, the different outcomes may be explained by mainly the following two reasons. On one hand, the quantile estimator has smaller bias than endpoint estimator and so one must expect beforehand a good behaviour of classical results. Also it corresponds to larger k_0 minimizing asymptotic mse which may be a contribution to instability on the results when it comes to calculate $\hat{k}_0$ since it involves k_i, $i = 1, 2$ that must also be expected larger and with larger variance (see figure 3). This effect is strengthened by the fact that the function h (cf. Theorem 2.1) is also much larger in the quantile

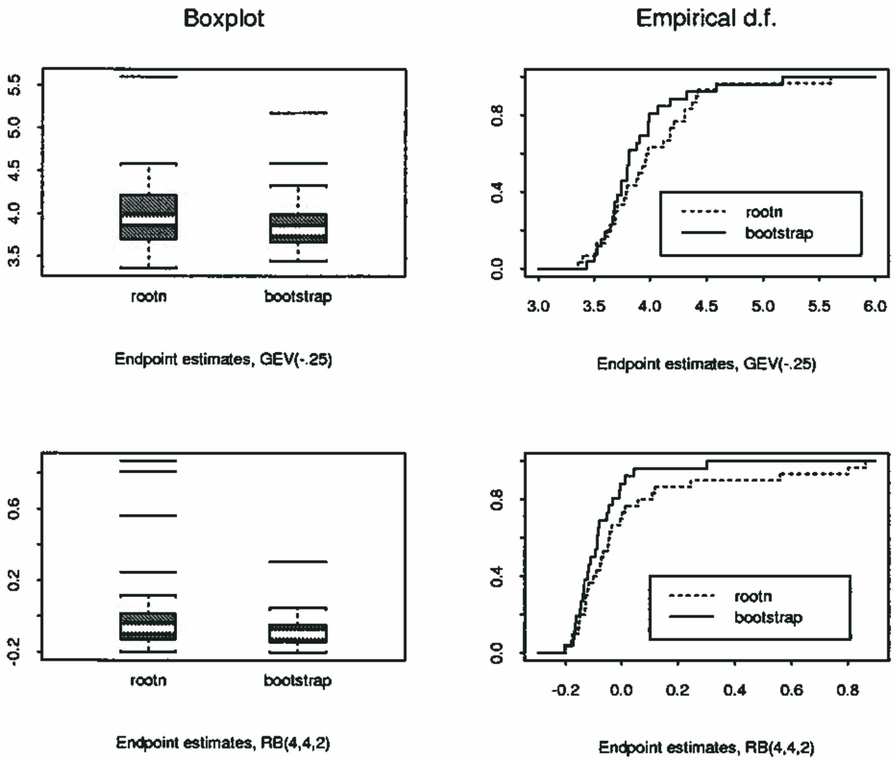

Figure 1: Comparison of bootstrap endpoint estimates with classical ones ($k_0 = \sqrt{n}$).

case. Hence a main problem here is in the 'asymptotic ratio' (Variance(of first estimator)×Bias(of alternative estimator)^2)/ (Variance(of alternative estimator)×Bias(of first estimator)^2) (let us denote it simply by $\mathrm{Var}_1\mathrm{Bias}_2^2)/(\mathrm{Var}_2\mathrm{Bias}_1^2)$).

In what concerns $GEV_{.5}$, $GEV_{.5}{}^{*}$ and $GEV_{1.5}$ the results are quite good, similar to those obtained on endpoint. This is remarkable given that in $GEV_{.5}{}^{*}$ and $GEV_{1.5}$ when estimating the bias $\hat{\rho}\prime$ was used. Even though the results are better for $GEV_{.5}$ than for $GEV_{.5}{}^{*}$. In what concerns Cauchy d.f. the results are definitely not good. But notice that it happens regardless wether one uses bootstrap or one simply takes $k_0 = \sqrt{n}$. One explanation may be the huge shift of 5573 applied to the data. We have adopted here for each distribution function a common shift for any data set in order to have any occasional shift influence under control.

Simulation results regarding quantile estimation, positive gamma, are omitted since they follow a similar trend.

3.1.5 Application

Given the previous discussion is was found enough to give an application on endpoint estimation. The data consists of the total life span (in days) of the people who died as residents in the Netherlands, which were born between the years 1877 - 1881 (included) and were still alive on January 1, 1971. Evidence has been given to support the statment that the underlying distribution of the population under study has a finite endpoint and the extreme value index is between -1/2 and 0; for a brief discussion we reffer to [1] Aarssen and de Haan (1994), where the same sample is analyzed after suitable preparation for statistical analysis. The sample size is 10391. Results are also displayed for women and men separately with sample sizes of 6260 and 4131, respectively.

In table 4 are results on bootstrap endpoint estimation. Below each bootstrap resample size, n_1 and n_2, in round brackets, is the range taken in looking for the optimal k_1 and k_2, respectively. In

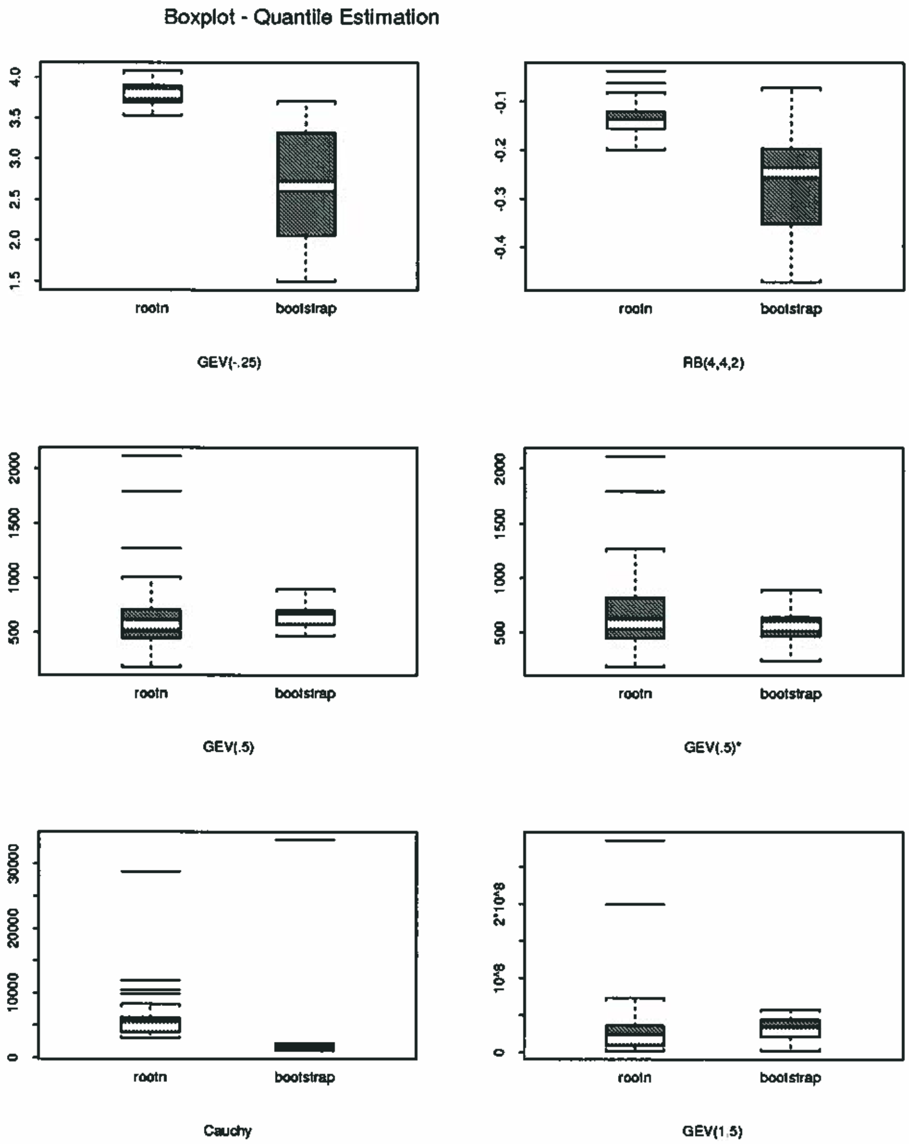

Figure 2: Comparison of bootstrap quantile estimates with classical ones ($k_0 = \sqrt{n}$).

what concerns the number of bootstrap resamples, following the arguments discussed previously it was found convenient to consider larger values: we took 3000 resamples for men+women data, 1500 for women data and 500 for men data.

Note that large values of $k_{1,0}$ related to n_1 and $k_{2,0}$ related to n_2 were obtained, compared to the simulated data on endpoint estimation presented earlier. Results are shown for several options of n_1. Note that the consistent estimate of γ in each bootstrap intermediate result is always the same within each sample of size n, since it is calculated simply by taking $k = \sqrt{n}$. The bootstrap estimates of endpoint for life span data are quite stable. Only for the men data a positive bootstrap estimate of γ was obtained which is inconsistent with the existence of endpoint. It is believed that it is due to having a small sample size, regarding the kind of data.

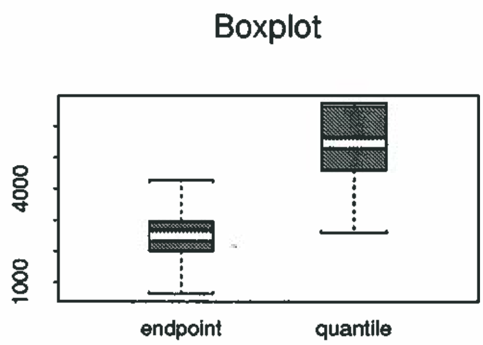

Figure 3: Comparison of $\hat{k}_1^2/\hat{k}_2$, $GEV_{-.25}$.

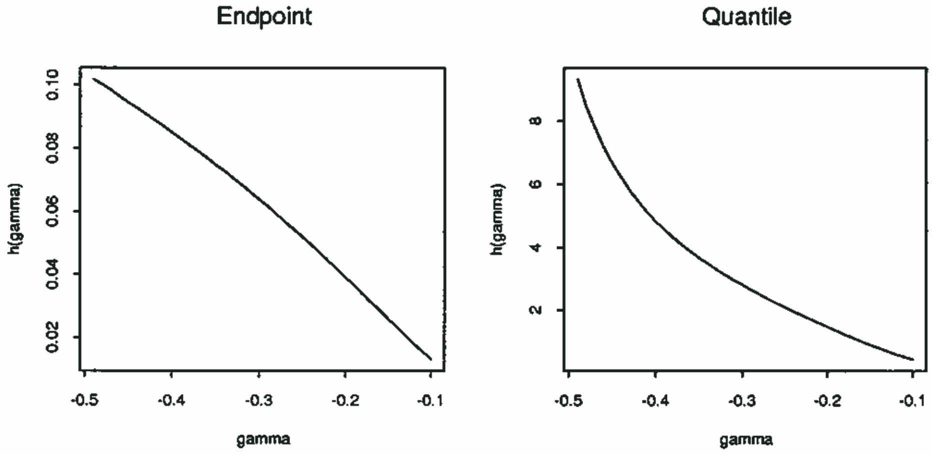

Figure 4: $h/\overline{h}$ and $g/\overline{g}$ functions (i.e. $((\mathrm{Var}_1\mathrm{Bias}_2^2)/\ (\mathrm{Var}_2\mathrm{Bias}_1^2))^{1/(1-2\gamma)}$), $\rho < \gamma < 0$.

4 Proofs

The proof of Theorem 2.3 will be given first and of Theorem 2.1 afterwards. The same reversal happens with the proofs of Theorems 2.6 and 2.5.

We start with a number of auxiliary results. The first one has been taken from [6] Draisma, de Haan, Peng and Pereira (1998).

Lemma 4.1. *Assume* $U(\infty) > 0$ *and there exist functions* $a(t) > 0$ *and* $A(t) \to 0$ *such that*

$$\frac{\frac{U(tx)-U(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma}}{A(t)} \to H_{\gamma,\rho}(x)$$

where

$$H_{\gamma,\rho}(x) = \frac{1}{\rho}\left[\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma - 1}{\gamma}\right] \quad (\rho \le 0).$$

size of the bootstrap resamples	bootstrap intermediate results $k_{1,0}$ $(k_{1,0}/n_1)$	$k_{2,0}$ $(k_{2,0}/n_2)$	$\hat{\gamma}$	bootstrap final results $\hat{k}_0$ $(\hat{k}_0/n)$	$\hat{\rho}\prime$	$\hat{\gamma}$	$\hat{x}_0$
men+women sample $n = 10391$							
$n_1 = 4000; n_2 = 1539$ (10,3200) ; (10,1231)	2536 (.63)	1195 (.78)	-.2940	335 (.03)	-8.60	-.1625	114.8 years
$n_1 = 5000; n_2 = 2405$ (10,4000); (10,1924)	3130 (.62)	1679 (.70)	-.2940	363 (.03)	-8.59	-.1643	114.7 years
$n_1 = 6000; n_2 = 3464$ (10,4500) ; (10,2700)	3752 (.62)	2415 (.70)	-.2940	363 (.03)	-8.76	-.1643	114.7 years
$n_1 = 7000; n_2 = 4715$ (10,5600) ; (10,3772)	4382 (.63)	2950 (.62)	-.2940	405 (.04)	-8.95	-.1452	115.9 years
women sample $n = 6260$							
$n_1 = 3000; n_2 = 1437$ (10,2400) ; (10,1149)	2119 (.71)	1149 (.80)	-.2753	226 (.04)	-11.01	-.1510	115.5 years
$n_1 = 4000; n_2 = 2555$ (10,3200) ; (10,2044)	2826 (.71)	1818 (.71)	-.2753	254 (.04)	-11.44	-.1382	116.4 years
men sample $n = 4131$							
$n_1 = 2000; n_2 = 968$ (10,1600) ; (10,774)	1554(.78)	760(.78)	-.0419	7(-)	-14.56	.0770	–
$n_1 = 3000; n_2 = 2178$ (10,2400) ; (10,1742)	2332(.78)	1685(.77)	-.0419	7(-)	-15.39	.0770	–
$n_1 = 4000; n_2 = 3873$ (10,3200) ; (10,3098)	3000(.75)	2905(.75)	-.0419	7(-)	-13.92	.0770	–

Table 4: Results of bootstrap in endpoint estimation of life span of men and women.

Suppose that $\gamma \neq \rho$. Then

$$\lim_{t\to\infty} \frac{\frac{a(t)}{U(t)} - \gamma_+}{A(t)} = c \in [-\infty, \infty]$$

where

$$c = \begin{cases} 0 & \text{if } \gamma < \rho \\ \frac{\gamma}{\gamma+\rho} & \text{if } \gamma > -\rho \\ \frac{\gamma}{\gamma+\rho} & \text{if } 0 < \gamma < -\rho \text{ and } \lim_{t\to\infty}(U(t) - a(t)/\gamma) = 0 \\ \pm\infty & \text{if } \rho < \gamma \leq 0 \\ \pm\infty & \text{if } 0 < \gamma < -\rho \text{ and } \lim_{t\to\infty}(U(t) - a(t)/\gamma) \neq 0 \\ \pm\infty & \text{if } \gamma = -\rho. \end{cases}$$

Furthermore

$$\frac{\frac{\log U(tx) - \log U(t)}{a(t)/U(t)} - \frac{x^{\gamma_-} - 1}{\gamma_-}}{\tilde{A}(t)} \to H_{\gamma_-, \rho'}(x)$$

where

$$\tilde{A}(t) = \begin{cases} A(t) & \text{if } c = 0 \\ \gamma_+ - \frac{a(t)}{U(t)} & \text{if } c = \pm\infty \\ \rho A(t)/(\gamma + \rho) & \text{if } c = \gamma/(\gamma + \rho), \end{cases}$$

$\tilde{A}(t) \in RV_{\rho'}$,

$$\rho' = \begin{cases} -\gamma & \text{if } (0 < \gamma < -\rho \text{ and } \lim_{t\to\infty}(U(t) - a(t)/\gamma) \neq 0) \\ \gamma & \text{if } \rho < \gamma \leq 0 \\ \rho & \text{if } (0 < \gamma < -\rho \text{ and } \lim_{t\to\infty}(U(t) - a(t)/\gamma) = 0) \\ & \quad \text{or } \gamma < \rho \text{ or } \gamma \geq -\rho. \end{cases}$$

Remark 4.2. *Hence $\rho' = 0$ if $\gamma = 0$.*

Lemma 4.3. *Suppose for some function $a(t) > 0$ and function $A(t)$ not changing sign, $\lim_{t\to\infty} A(t) = 0$,*

$$\lim_{t\to\infty} \frac{\frac{U(tx)-U(t)}{a(t)} - \frac{x^\gamma-1}{\gamma}}{A(t)} = \frac{1}{\rho}\left[\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma-1}{\gamma}\right] = H_{\gamma,\rho}(x)$$

for all $x > 0$, with $\rho < 0$. Then

$$\lim_{\substack{t\to\infty\\ x\to\infty}} \frac{\frac{U(tx)-U(t)}{a(t)}\frac{\gamma}{x^\gamma-1} - 1}{A(t)} = \frac{-1}{\rho+\gamma_-}.$$

The same holds with $\rho = 0$ and $\gamma < 0$.
Moreover, for $\gamma < 0$,

$$\lim_{t\to\infty} \frac{\frac{U(\infty)-U(t)}{a(t)} + \frac{1}{\gamma}}{A(t)} = \frac{-1}{\gamma_-(\gamma_- + \rho)}.$$

Proof. From [7] Drees' inequality (1998) it follows that

$$\lim_{t\to\infty} \sup_{x\geq 1} x^{-\gamma+\rho+\varepsilon}\left[\frac{\frac{U(tx)-U(t)}{a(t)} - \frac{x^\gamma-1}{\gamma}}{A(t)} - H_{\gamma,\rho}(x)\right] = 0.$$

for negative ρ and each positive ε. The first result follows by considering the cases $\gamma > 0$, $\gamma = 0$ and $\gamma < 0$ separately.

As to the second result, relation (2.11) and Remark 2(i) from [14] de Haan and Stadtmüller (1996) imply: $lim_{t\to\infty}(U(t) - a(t)/\gamma) = U(\infty)$ and

$$\lim_{t\to\infty} \frac{U(\infty) - U(t) + a(t)/\gamma}{a(t)A(t)/\gamma} = \frac{-1}{\gamma+\rho}.$$

The result follows. □

Remark 4.4. *If $\{U(tx)/U(t) - x^\gamma\}/\alpha(t) \to x^\gamma(x^\rho - 1)/\rho$, with $\gamma > 0$ and $\rho < 0, t \to \infty$, for all $x > 0$, then*

$$\lim_{\substack{t\to\infty\\ x\to\infty}} \{x^{-\gamma}U(tx)/U(t) - 1\}/\alpha(t) = -1/\rho.$$

Take random variables $Y_1, Y_2, \ldots$ i.i.d. with distribution function $1 - 1/y$, $y > 1$. Then $U(Y_1)$, $U(Y_2)$, ... are i.i.d. F.

Lemma 4.5. *Write*

$$M_j := \frac{M_n^{(j)} U^j(Y_{n-k,n})}{a^j(Y_{n-k,n})} - l_j$$

for $j = 1, 2, 3$ with

$$M_n^{(j)} : = \frac{1}{k}\sum_{i=0}^{k-1}\{\log U(Y_{n-i,n}) - \log U(Y_{n-k,n})\}^j,$$

$$1/l_1 : = 1 - \gamma_-$$
$$1/l_2 : = (1-\gamma_-)(1-2\gamma_-)/2$$
$$1/l_3 : = (1-\gamma_-)(1-2\gamma_-)(1-3\gamma_-)/6.$$

Then under the conditions of Lemma 4.1, for $k = k(n) \to \infty$ and $k(n)/n \to 0$ $(n \to \infty)$

$$M_1 = \frac{P_1}{\sqrt{k}} + d_1\tilde{A}(\frac{n}{k}) + o_p(\frac{1}{\sqrt{k}}) + o_p(\tilde{A}(\frac{n}{k}))$$
$$M_2 = \frac{P_2}{\sqrt{k}} + d_2\tilde{A}(\frac{n}{k}) + o_p(\frac{1}{\sqrt{k}}) + o_p(\tilde{A}(\frac{n}{k}))$$
$$M_3 = \frac{P_3}{\sqrt{k}} + d_3\tilde{A}(\frac{n}{k}) + o_p(\frac{1}{\sqrt{k}}) + o_p(\tilde{A}(\frac{n}{k}))$$

where (P_1, P_2, P_3) is normally distributed with mean vector zero and covariance matrix

$$\begin{cases} EP_1^2 = \frac{1}{(1-\gamma_-)^2(1-2\gamma_-)} \\ EP_2^2 = \frac{4(5-11\gamma_-)}{(1-\gamma_-)^2(1-2\gamma_-)^2(1-3\gamma_-)(1-4\gamma_-)} \\ EP_3^2 = \frac{36(19-105\gamma_-+146\gamma_-^2)}{(1-\gamma_-)^2(1-2\gamma_-)^2(1-3\gamma_-)^2(1-4\gamma_-)(1-5\gamma_-)(1-6\gamma_-)} \\ E(P_1P_2) = \frac{4}{(1-\gamma_-)^2(1-2\gamma_-)(1-3\gamma_-)} \\ E(P_1P_3) = \frac{18}{(1-\gamma_-)^2(1-2\gamma_-)(1-3\gamma_-)(1-4\gamma_-)} \\ E(P_2P_3) = \frac{12(9-21\gamma_-)}{(1-\gamma_-)^2(1-2\gamma_-)^2(1-3\gamma_-)(1-4\gamma_-)(1-5\gamma_-)} \end{cases}$$

and

$$\begin{cases} d_1 = \frac{1}{(1-\gamma_-)(1-\rho'-\gamma_-)} \\ d_2 = \frac{2(3-2\rho'-4\gamma_-)}{(1-\gamma_-)(1-2\gamma_-)(1-\rho'-\gamma_-)(1-\rho'-2\gamma_-)} \\ d_3 = \frac{6(18\gamma_-^2-22\gamma_-+15\rho'\gamma_-+3\rho'^2-8\rho'+6)}{(1-\gamma_-)(1-2\gamma_-)(1-3\gamma_-)(1-\rho'-\gamma_-)(1-\rho'-2\gamma_-)(1-\rho'-3\gamma_-)}. \end{cases}$$

Proof. By Lemma 4.1

$$\{\frac{1}{k}\sum_{i=o}^{k-1} \log U(Y_{n-i,n}) - \log U(Y_{n-k,n})\}\ U(Y_{n-k,n})/a(Y_{n-k,n})$$

$$= \frac{1}{k}\sum_{i=o}^{k-1} \frac{\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right)^{\gamma_-} - 1}{\gamma_-} + \tilde{A}(Y_{n-k,n})\frac{1}{k}\sum_{i=o}^{k-1} H_{\gamma_-,\rho'}\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right) + o_p(\tilde{A}(\frac{n}{k}))$$

$$\stackrel{d}{=} \frac{1}{k}\sum_{i=0}^{k-1}(Y_i^{\gamma_-} - 1)/\gamma_- + \tilde{A}(\frac{n}{k})\frac{1}{k}\sum_{i=0}^{k-1} H_{\gamma_-,\rho'}(Y_i) + o_p(\tilde{A}(\frac{n}{k}))$$

$$= E(Y^{\gamma_-} - 1)/\gamma_- + \frac{P_1}{\sqrt{k}} + \tilde{A}(\frac{n}{k})EH_{\gamma_-,\rho'}(Y) + o_p(\tilde{A}(\frac{n}{k})) + o_p(\frac{1}{\sqrt{k}})$$

with $Y, Y_1, Y_2, \ldots$ i.i.d. with distribution function $1 - 1/y$, $y > 1$, and P_1 the normal limit random variable of

$$\sqrt{k}[\frac{1}{k}\sum_{i=1}^{k}(Y_i^{\gamma_-} - 1)/\gamma_- - E(Y^{\gamma_-} - 1)/\gamma_-].$$

Similarly for $M_n^{(j)}, j = 2, 3$; note that by Lemma 4.1

$$\left(\frac{\log U(tx) - \log U(t)}{a(t)/U(t)}\right)^j = \left(\frac{x^{\gamma_-} - 1}{\gamma_-}\right)^j + j\tilde{A}(t)\left(\frac{x^{\gamma_-} - 1}{\gamma_-}\right)^{j-1} H_{\gamma_-,\rho'}(x) + o(\tilde{A}(t)),$$

hence

$$M_n^{(j)}\{U(Y_{n-k,n})/a(Y_{n-k,n})\}^j$$

$$\stackrel{d}{=} \frac{1}{k}\sum_{i=0}^{k-1}\left\{\frac{Y_i^{\gamma_-}-1}{\gamma_-}\right\}^j + j\tilde{A}(\frac{n}{k})\frac{1}{k}\sum_{i=0}^{k-1}\left\{\frac{Y_i^{\gamma_-}-1}{\gamma_-}\right\}^{j-1} H_{\gamma_-,\rho'}(Y_i) + o_p(\tilde{A}(\frac{n}{k}))$$

$$= E\left\{\frac{Y^{\gamma_-}-1}{\gamma_-}\right\}^j + j\tilde{A}(\frac{n}{k})E\left\{\left(\frac{Y^{\gamma_-}-1}{\gamma_-}\right)^{j-1} H_{\gamma_-,\rho'}(Y)\right\} + \frac{P_j}{\sqrt{k}} + o_p(\tilde{A}(\frac{n}{k})) + o_p(\frac{1}{\sqrt{k}}).$$

□

Lemma 4.6. *Under the given conditions*

$$\hat{\gamma}_n^+(k) = M_n^{(1)} = \gamma_+ + \gamma_+ M_1 + q_{\gamma,\rho} l_1 \tilde{A}(\frac{n}{k})$$

$$\hat{\hat{\gamma}}_n^+(k) = (M_n^{(2)}/2)^{1/2} = \gamma_+ + \frac{\gamma_+}{4}M_2 + q_{\gamma,\rho}(l_2/2)^{1/2}\tilde{A}(\frac{n}{k})$$

with

$$q_{\gamma,\rho} = \lim_{t\to\infty}\frac{a(t)/U(t)-\gamma_+}{\tilde{A}(t)} = \begin{cases} 0 & \text{if } \gamma<\rho \\ \gamma/\rho & \text{if } (\lim_{t\to\infty} U(t) - a(t)/\gamma_+ = 0 \\ & \quad \text{and } 0<\gamma<-\rho) \text{ or } \gamma>-\rho \\ -1 & \text{if } (\lim_{t\to\infty} U(t) - a(t)/\gamma_+ \neq 0 \text{ and } 0<\gamma<-\rho) \\ & \quad \text{or } \rho<\gamma\le 0 \text{ or } \gamma=-\rho \end{cases}$$

Proof.

$$M_n^{(1)} = a(Y_{n-k,n})/U(Y_{n-k,n})\{l_1+M_1\}$$

$$= \frac{\frac{a(Y_{n-k,n})}{U(Y_{n-k,n})}-\gamma_+}{\tilde{A}(Y_{n-k,n})}\frac{\tilde{A}(Y_{n-k,n})}{\tilde{A}(\frac{n}{k})}\tilde{A}(\frac{n}{k})\{l_1+M_1\} + \gamma_+\{l_1+M_1\}$$

$$= q_{\gamma,\rho} l_1 \tilde{A}(\frac{n}{k}) + \gamma_+ + \gamma_+ M_1 + o_p(\tilde{A}(\frac{n}{k})).$$

$$\{M_n^{(2)}/2\}^{1/2} = \frac{a(Y_{n-k,n})}{U(Y_{n-k,n})}\{l_2/2 + M_2/2\}^{1/2}$$

$$= \frac{\frac{a(Y_{n-k,n})}{U(Y_{n-k,n})}-\gamma_+}{\tilde{A}(Y_{n-k,n})}\frac{\tilde{A}(Y_{n-k,n})}{\tilde{A}(\frac{n}{k})}\tilde{A}(\frac{n}{k})\{(l_2/2)^{1/2} + \frac{1}{4}\frac{M_2}{(l_2/2)^{1/2}}\}$$

$$+\gamma_+\{(l_2/2)^{1/2} + \frac{1}{4}\frac{M_2}{(l_2/2)^{1/2}}\} + o_p(\tilde{A}(\frac{n}{k})) + o_p(\frac{1}{\sqrt{k}})$$

$$= q_{\gamma,\rho}(l_2/2)^{1/2}\tilde{A}(\frac{n}{k}) + \gamma_+ + \gamma_+ M_2/4 + o_p(\tilde{A}(\frac{n}{k})) + o_p(\frac{1}{\sqrt{k}}).$$

□

Lemma 4.7. *Under the given conditions*

$$\begin{aligned}
\hat{\gamma}_n^-(k) &= 1-\frac{1}{2}\{1-(M_n^{(1)})^2/M_n^{(2)}\}^{-1} = \gamma_- - \frac{4}{l_1 l_2}M_1 + \frac{2}{l_2^2}M_2 \\
&= \gamma_- + \frac{1}{2}(1-\gamma_-)^2(1-2\gamma_-)\{-4M_1+(1-2\gamma_-)M_2\}. \\
\hat{\bar{\gamma}}_n^-(k) &= 1-\frac{2}{3}\left\{1-\frac{M_n^{(1)}M_n^{(2)}}{M_n^{(3)}}\right\}^{-1} \\
&= \gamma_- - \frac{3l_2}{2l_1^2 l_3}M_1 - \frac{3}{2l_1 l_3}M_2 + \frac{3l_2}{2l_1 l_3^2}M_3 \\
&= \gamma_- + \frac{(1-\gamma_-)^2(1-3\gamma_-)}{12}\{-6M_1-3(1-2\gamma_-)M_2+(1-2\gamma_-)(1-3\gamma_-)M_3\}
\end{aligned}$$

Remark 4.8. *Hence for $\gamma > 0$*

$$\hat{\gamma}_n^-(k) = -2M_1 + \frac{1}{2}M_2 + o(\frac{1}{\sqrt{k}}) + o(\tilde{A}(\frac{n}{k}))$$

$$\hat{\bar{\gamma}}_n^-(k) = -\frac{1}{2}M_1 - \frac{1}{4}M_2 + \frac{1}{12}M_3 + o(\frac{1}{\sqrt{k}}) + o(\tilde{A}(\frac{n}{k})).$$

Proof (of Lemma 4.7). For the expansion of $\hat{\gamma}_n^-(k)$ see [5] Dekkers, Einmahl and de Haan (1989), proof of Corollary 3.2. Next we consider $\hat{\bar{\gamma}}_n^-(k)$:

$$\frac{M_n^{(1)}M_n^{(2)}}{M_n^{(3)}} - \frac{1-3\gamma_-}{3(1-\gamma_-)} = \frac{(l_1+M_1)(l_2+M_2)}{l_3+M_3} - \frac{l_1 l_2}{l_2}$$

$$= \frac{l_2}{l_3}M_1 + \frac{l_1}{l_3}M_2 - \frac{l_1 l_2}{l_3^2}M_3 \ + \ \text{terms of lower order.}$$

Write $\pi := M_n^{(1)}M_n^{(2)}/M_n^{(3)}$ and $\lambda := (1-3\gamma_-)/\{3(1-\gamma_-)\}$.

$$\hat{\bar{\gamma}}_n^-(k) - \gamma_- = 1 - \frac{2}{3}\frac{1}{1-\pi} - 1 + \frac{2}{3}\frac{1}{1-\lambda} = \frac{2}{3}\frac{\lambda-\pi}{(1-\lambda)(1-\pi)}$$

Hence, disregarding terms of lower order,

$$\begin{aligned}
\hat{\bar{\gamma}}_n^-(k) - \gamma_- &= \frac{2}{3}(\lambda-\pi)/(1-\lambda)^2 \\
&= -\frac{2}{3}\{\frac{3}{2}(1-\gamma_-)\}^2\left[\frac{l_2}{l_3}M_1 + \frac{l_1}{l_3}M_2 - \frac{l_1 l_2}{l_3^2}M_3\right] \\
&= -\frac{3}{2}\left[\frac{l_2}{l_1^2 l_3}M_1 + \frac{1}{l_1 l_3}M_2 - \frac{l_2}{l_1 l_3^2}M_3\right]
\end{aligned}$$

□

Lemma 4.9. *Let $\hat{b}(n/k) = U(Y_{n-k,n})$. Under the given conditions*

$$\frac{\hat{b}(\frac{n}{k}) - U(\frac{n}{k})}{a(\frac{n}{k})} = \frac{B}{\sqrt{k}} + o_p(\frac{1}{\sqrt{k}}) + o_p(A(\frac{n}{k}))$$

with B a standard normal random variable, independent of P_1, P_2 and P_3.

Proof. We use the second order conditions for U.

$$\begin{aligned}
\frac{U(Y_{n-k,n})-U(\frac{n}{k})}{a(\frac{n}{k})} &= \\
&= \frac{(\frac{k}{n}Y_{n-k,n})^{\gamma}-1}{\gamma}+A(\frac{n}{k})H_{\gamma,\rho}(\frac{k}{n}Y_{n-k,n})+o(A(\frac{n}{k})) \\
&= (\frac{k}{n}Y_{n-k,n}-1)+o_p(\frac{k}{n}Y_{n-k,n}-1)+A(\frac{n}{k})o_p(1)+o(A(\frac{n}{k})) \\
&= \frac{B}{\sqrt{k}}+o_p(\frac{1}{\sqrt{k}})+o_p(A(\frac{n}{k}))
\end{aligned}$$

□

Remark 4.10. *No bias term comes into play.*

Lemma 4.11. *Under the given conditions*

$$\begin{aligned}
\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})}-1 &= \frac{l_2+4l_1}{l_1l_2}M_1-\frac{2l_1}{l_2^2}M_2+\gamma\frac{B}{\sqrt{k}}+o_p(A(\frac{n}{k})) \\
&= (1-\gamma_-)(3-4\gamma_-)M_1-\frac{1}{2}(1-\gamma_-)(1-2\gamma_-)^2M_2+\gamma\frac{B}{\sqrt{k}}+o_p(A(\frac{n}{k}))
\end{aligned}$$

and

$$\begin{aligned}
\frac{\hat{\hat{a}}(\frac{n}{k})}{a(\frac{n}{k})}-1 &= \frac{2l_3+3l_2}{2l_1l_3}M_1+\frac{3}{2l_3}M_2-\frac{3l_2}{2l_3^2}M_3+\gamma\frac{B}{\sqrt{k}}+o_p(A(\frac{n}{k})) \\
&= \frac{3}{2}(1-\gamma_-)^2M_1+\frac{1}{4}(1-\gamma_-)(1-2\gamma_-)(1-3\gamma_-)M_2 \\
&\quad -\frac{1}{12}(1-\gamma_-)(1-2\gamma_-)(1-3\gamma_-)^2M_3+\gamma\frac{B}{\sqrt{k}}+o_p(A(\frac{n}{k})).
\end{aligned}$$

Proof.

$$\begin{aligned}
\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} &= \frac{X_{n-k,n}M_n^{(1)}(1-\hat{\gamma}_n^-(k))}{a(\frac{n}{k})} \\
&= \frac{(1-\gamma_-)M_n^{(1)}U(Y_{n-k,n})}{a(Y_{n-k,n})}\,\frac{a(Y_{n-k,n})}{a(\frac{n}{k})}\,\frac{1-\hat{\gamma}_n^-(k)}{1-\gamma_-}.
\end{aligned}$$

Now by the second order conditions for U

$$\lim_{t\to\infty}\frac{\frac{a(tx)}{a(t)}-x^{\gamma}}{A(t)}=x^{\gamma}\,\frac{x^{\rho}-1}{\rho}$$

locally uniformly for $x>0$, hence

$$\frac{a(Y_{n-k,n})}{a(\frac{n}{k})}-1=$$

$$=\left(\frac{k}{n}Y_{n-k,n}\right)^{\gamma}-1+A(\frac{n}{k})(\frac{k}{n}Y_{n-k,n})^{\gamma}\frac{(\frac{k}{n}Y_{n-k,n})^{\rho}-1}{\rho}+o(A(\frac{n}{k}))$$

$$=\gamma(\frac{k}{n}Y_{n-k,n}-1)+o(\frac{k}{n}Y_{n-k,n}-1)+A(\frac{n}{k})o_p(1)+o(A(\frac{n}{k}))$$

$$= \gamma \frac{B}{\sqrt{k}} + o_p(\frac{1}{\sqrt{k}}) + o_p(A(\frac{n}{k})).$$

Consequently

$$\begin{aligned} \frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} &= (1 + (1-\gamma_-)M_1)(1 + \gamma\frac{B}{\sqrt{k}} + o_p(\frac{1}{\sqrt{k}}) + o_p(A(\frac{n}{k}))(1 - \frac{\hat{\gamma}_n^-(k) - \gamma_-}{1-\gamma_-}) && (4.1) \\ &= 1 + (1-\gamma_-)M_1 + \gamma\frac{B}{\sqrt{k}} + \frac{4l_1}{l_1 l_2}M_1 - \frac{2l_1}{l_2^2}M_2 + o_p(\frac{1}{\sqrt{k}}) + o_p(A(\frac{n}{k})). && (4.2) \end{aligned}$$

Similarly for

$$\begin{aligned} \frac{\hat{\bar{a}}(\frac{n}{k})}{a(\frac{n}{k})} &= \frac{X_{n-k,n}M_n^{(1)}(1 - \hat{\bar{\gamma}}_n^-(k))}{a(\frac{n}{k})} \\ &= (1 + (1-\gamma_-)M_1)(1 + \gamma\frac{B}{\sqrt{k}} + o_p(\frac{1}{\sqrt{k}}) + o_p(A(\frac{n}{k})))(1 - \frac{\hat{\bar{\gamma}}_n^-(k) - \gamma_-}{1-\gamma_-}). \end{aligned}$$

Hence

$$\frac{\hat{\bar{a}}(\frac{n}{k})}{a(\frac{n}{k})} - 1 = (1-\gamma_-)M_1 + \gamma\frac{B}{\sqrt{k}} + \frac{3l_2}{2l_1 l_3}M_1 + \frac{3}{2l_3}M_2 - \frac{3l_2}{2l_3^2}M_3 + o_p(\frac{1}{\sqrt{k}}) + o_p(A(\frac{n}{k})).$$

□

Proposition 4.12. *Under the conditions of Theorem 2.1, as $n \to \infty$,*

$$k_0(n) \sim \begin{cases} \left(\frac{c_3}{c_4 \tilde{c}_2(-2\rho')}\right)^{\frac{1}{1-2\rho'}} n^{\frac{-2\rho'}{1-2\rho'}} & \text{for } \gamma > 0 \\ \left(\frac{c_5}{c_6 \tilde{c}_2}\,\frac{1+2\gamma}{-2\rho'-2\gamma}\right)^{\frac{1}{1-2\rho'}} n^{\frac{-2\rho'}{1-2\rho'}} & \text{for } \gamma < 0\,, \end{cases}$$

$$=: h(\gamma_+, \gamma_-, \rho')\, n^{\frac{-2\rho'}{1-2\rho'}}$$

where $k_0(n) := \arg\min_k$ as. $E(\hat{x}_{p_n}(k) - x_{p_n})^2$, $\tilde{c}_2$ from $\tilde{A}(t) \sim \tilde{c}_2 t^{\rho'} (t \to \infty)$ and

$$\begin{aligned} c_3 &:= \frac{\gamma^2}{c_1^2} c_3(\gamma_+) := (\gamma_+^2 + 1) \\ c_4 &:= \frac{\gamma^2}{c_1^2} c_4(\gamma_+, \rho') := \begin{cases} \frac{(\gamma_+ + \rho' - \gamma_+ \rho')^2}{\rho'^2 (1-\rho')^4} & \text{if } q_{\gamma,\rho} = \frac{\gamma}{\rho} \\ \frac{(\gamma_+ + 2\rho' - \gamma_+ \rho' - \rho'^2)^2}{(1-\rho')^4} & \text{if } q_{\gamma,\rho} = -1 \end{cases} \\ c_5 &:= c_5(\gamma_-) := \frac{(1-\gamma_-)^2(1 - 3\gamma_- + 4\gamma_-^2)}{\gamma_-^4(1-2\gamma_-)(1-3\gamma_-)(1-4\gamma_-)} \\ c_6 &:= c_6(\gamma_-, \rho') := \begin{cases} \frac{(1-\gamma_-)^2 \rho'^2}{\gamma_-^4 (1-\gamma_- - \rho')^2 (\gamma_- + \rho')^2 (1 - 2\gamma_- - \rho')^2} & \text{if } q_{\gamma,\rho} = 0 \\ \frac{(3\gamma_-^2 - \gamma_- - 2\gamma_-^3 + 2\rho' - 2\gamma_- \rho' - \gamma_-^2 \rho' - \rho'^2)^2}{\gamma_-^4 (1-\gamma_-)^2 (1-\gamma_- - \rho')^2 (1-2\gamma_- - \rho')^2} & \text{if } q_{\gamma,\rho} = -1\,. \end{cases} \end{aligned}$$

Proof. Write $a_n := k/(np_n)$. As in [13] de Haan and Rootzén (1993, p.7) we write

$$\begin{aligned} \hat{x}_{p_n}(k) - x_{p_n} &= \frac{a_n^{\hat{\gamma}_n(k)} - 1}{\hat{\gamma}_n(k)} \hat{a}(\frac{n}{k}) + \hat{b}(\frac{n}{k}) - U(\frac{1}{p_n}) \\ &= \left(\frac{a_n^{\hat{\gamma}_n(k)} - 1}{\hat{\gamma}_n(k)} - \frac{a_n^{\gamma} - 1}{\gamma}\right) \frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} a(\frac{n}{k}) + \frac{a_n^{\gamma} - 1}{\gamma} \left(\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} - 1\right) a(\frac{n}{k}) \\ &\quad + \frac{\hat{b}(\frac{n}{k}) - U(\frac{n}{k})}{a(\frac{n}{k})} a(\frac{n}{k}) - \left\{\frac{U(\frac{1}{p_n}) - U(\frac{n}{k})}{a(\frac{n}{k})} - \frac{a_n^{\gamma} - 1}{\gamma}\right\} a(\frac{n}{k}). \end{aligned}$$

We have asymptotic expansions for $\hat{\gamma}_n(k), \hat{a}(n/k), \hat{b}(n/k)$ and also for the last term (the bias term) but not for $(a_n^{\hat{\gamma}_n(k)} - 1)/\hat{\gamma}_n(k)$. So we want to simplify the expression (as in [13] de Haan and Rootzén, 1993). Since we are dealing with the *asymptotic* second moment it makes sense to first consider the limit behaviour in distribution rather than in L_2.

First suppose $\gamma > 0$. Note that $a(\frac{n}{k})a_n^\gamma \sim c_1 p_n^{-\gamma}$. Hence

$$\begin{aligned}\hat{x}_{p_n}(k) - x_{p_n} \quad &\sim \quad c_1 p_n^{-\gamma} \Bigg[\left\{ (1 - a_n^{-\gamma})(\frac{1}{\hat{\gamma}_n(k)} - \frac{1}{\gamma}) + \frac{a_n^{\hat{\gamma}_n(k)-\gamma} - 1}{\hat{\gamma}_n(k)} \right\} \frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} \\ &+ \frac{1 - a_n^{-\gamma}}{\gamma}(\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} - 1) + a_n^{-\gamma} \frac{\hat{b}(\frac{n}{k}) - U(\frac{n}{k})}{a(\frac{n}{k})} \\ &- a_n^{-\gamma} \left\{ \frac{U(\frac{1}{p_n}) - U(\frac{n}{k})}{a(\frac{n}{k})} - \frac{a_n^\gamma - 1}{\gamma} \right\} \Bigg]\end{aligned}$$

$$\sim c_1 p_n^{-\gamma} \left[\frac{1}{\hat{\gamma}_n(k)} - \frac{1}{\gamma} + \frac{a_n^{\hat{\gamma}_n(k)-\gamma} - 1}{\hat{\gamma}_n(k)} + \frac{1}{\gamma}(\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} - 1) - \frac{1}{\gamma} \frac{-1}{\rho + \gamma_-} A(\frac{n}{k}) \right]$$

plus terms of lower order by the Lemmas above for any intermediate sequence $k(n)$ and $n \to \infty$. Consider this expression for the sequence $\tilde{k}(n) = [n^{-2\rho'/(1-2\rho')}]$. Then by the expressions of Lemmas 4.6 and 4.7 we have $\hat{\gamma}_n(k) - \gamma = o((\tilde{k}(n))^{-1/2})$ (see also [6] Draisma, de Haan, Peng and Pereira, 1998). Hence, since $\log p_n = o(\sqrt{\tilde{k}(n)})$, $(\hat{\gamma}_n(k) - \gamma)\log a_n$ converges to zero for the sequence $\tilde{k}(n)$, and in fact the entire expression in square brackets tends to zero. This must then also be the case for the as yet unknown optimal sequence. Hence we may replace $(a_n^{\hat{\gamma}_n(k)-\gamma} - 1)/\hat{\gamma}_n(k)$ by $(\log a_n)(\hat{\gamma}_n(k) - \gamma)/\hat{\gamma}_n(k)$ in the minimization procedure. We get

$$\hat{x}_{p_n}(k) - x_{p_n} \quad \sim \quad c_1 p_n^{-\gamma} \left[\frac{1}{\hat{\gamma}_n(k)} - \frac{1}{\gamma} + (\log\ a_n)\frac{\hat{\gamma}_n(k) - \gamma}{\hat{\gamma}_n(k)} + \frac{1}{\gamma}\left(\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} - 1 \right) + \frac{1}{\gamma(\rho + \gamma_-)} A(\frac{n}{k}) \right]$$

plus terms of lower order. Since $(\log\ a_n)(\hat{\gamma}_n(k) - \gamma)/\hat{\gamma}_n(k)$ dominates all the other terms we find $(n \to \infty)$

$$\inf_k \text{ as. } E(\hat{x}_{p_n}(k) - x_{p_n})^2 \sim \left(\frac{c_1 p_n^{-\gamma}}{\gamma} \right)^2 \inf_k \text{ as. } E(\log a_n)^2 (\hat{\gamma}_n(k) - \gamma)^2. \tag{4.3}$$

Next suppose $\gamma < 0$. Note that

$$\begin{aligned}\hat{x}_{p_n}(k) - x_{p_n} \quad &= \quad a(\frac{n}{k}) \Bigg[(a_n^\gamma - 1)(\frac{1}{\hat{\gamma}_n(k)} - \frac{1}{\gamma}) + \frac{a_n^{\hat{\gamma}_n(k)} - a_n^\gamma}{\hat{\gamma}_n(k)} \frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} \\ &+ \frac{a_n^\gamma - 1}{\gamma} \left(\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} - 1 \right) + \frac{\hat{b}(\frac{n}{k}) - U(\frac{n}{k})}{a(\frac{n}{k})} - \frac{1}{\gamma} \frac{-1}{\rho + \gamma_-} A(\frac{n}{k}) \Bigg]\end{aligned}$$

$$= a(\frac{n}{k}) \left[\frac{1}{\gamma} - \frac{1}{\hat{\gamma}_n(k)} + \frac{a_n^{\hat{\gamma}_n(k)} - a_n^\gamma}{\hat{\gamma}_n(k)} - \frac{1}{\gamma} \left(\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} - 1 \right) \frac{\hat{b}(\frac{n}{k}) - U(\frac{n}{k})}{a(\frac{n}{k})} - \frac{1}{\gamma(\rho + \gamma_-)} A(\frac{n}{k}) \right]$$

plus terms of lower order, for any intermediate sequence $k(n)$.

Now

$$\frac{a_n^{\hat{\gamma}_n(k)} - a_n^\gamma}{\hat{\gamma}_n(k) - \gamma} = \frac{\log\ a_n}{\hat{\gamma}_n(k) - \gamma} \int_\gamma^{\hat{\gamma}_n(k)} a_n^s ds \leq (\log\ a_n) a_n^{\max(\hat{\gamma}_n(k), \gamma)} \to 0$$

$(n \to \infty)$. Hence the second term $(a_n^{\hat\gamma_n(k)} - a_n^{\gamma})/\hat\gamma_n(k)$ is of smaller order than the first term $1/\gamma - 1/\hat\gamma_n(k)$. We find $(n \to \infty)$

$$
\begin{aligned}
&\inf_k \text{ as. } E(\hat x_{p_n}(k) - x_{p_n})^2 \sim \\
&\sim \inf_k \text{ as. } .E\left\{a^2(\frac{n}{k})\left[\frac{\hat\gamma_n(k)-\gamma}{\gamma^2} - \frac{1}{\gamma}\left(\frac{\hat a(\frac{n}{k})}{a(\frac{n}{k})}-1\right) + \frac{\hat b(\frac{n}{k}) - U(\frac{n}{k})}{a(\frac{n}{k})} - \frac{1}{\gamma(\rho+\gamma_-)}A(\frac{n}{k})\right]^2\right\} \\
&\sim \inf_k \text{ as. } E\left\{a^2(\frac{n}{k})\left[\frac{\hat\gamma_n(k)-\gamma}{\gamma^2} - \frac{1}{\gamma}\left(\frac{\hat a(\frac{n}{k})}{a(\frac{n}{k})}-1\right) + \frac{\hat b(\frac{n}{k}) - U(\frac{n}{k})}{a(\frac{n}{k})} - \frac{1_{\{\gamma<\rho\}}}{\gamma(\rho+\gamma_-)}\tilde A(\frac{n}{k})\right]^2\right\}
\end{aligned}
$$

by Lemma 4.1.

Next we consider as. $E(\hat x_{p_n}(k) - x_{p_n})^2$ for $\gamma > 0$: by Lemmas 4.6 and 4.7, disregarding terms that are $o(\frac{1}{\sqrt{k}})$ or $o(\tilde A(\frac{n}{k}))$,

$$
\begin{aligned}
\text{as. } E(\hat\gamma_n(k)-\gamma)^2 &= E\{(\gamma_+ - 2)M_1 + \frac{1}{2}M_2 + q_{\gamma,\rho}\tilde A(\frac{n}{k})\}^2 \\
&= E\{(\gamma_+ - 2)(\frac{P_1}{\sqrt{k}} + d_1\tilde A(\frac{n}{k})) + \frac{1}{2}(\frac{P_2}{\sqrt{k}} + d_2\tilde A(\frac{n}{k})) + q_{\gamma,\rho}\tilde A(\frac{n}{k})\}^2 \\
&= (\gamma_+ - 2)^2\frac{EP_1^2}{k} + \frac{1}{4}\frac{EP_2^2}{k} + (\gamma_+ - 2)\frac{EP_1P_2}{k} + \{(\gamma_+ - 2)d_1 + \frac{1}{2}d_2 + q_{\gamma,\rho}\}^2\tilde A^2(\frac{n}{k}) \\
&=: \frac{\gamma^2 c_3(\gamma_+)}{c_1^2 k} + \frac{\gamma^2 c_4(\gamma_+,\rho')}{c_1^2}\tilde A^2(\frac{n}{k}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\gamma^2}{c_1^2}as.\ E(\hat x_{p_n}(k) - x_{p_n})^2 &\sim (\log\ a_n)^2 p_n^{-2\gamma}\left\{\frac{c_3(\gamma_+)}{k} + c_4(\gamma_+,\rho')\tilde A^2(\frac{n}{k})\right\} \\
&\sim p_n^{-2\gamma}(\log(\frac{k}{np_n}))^2\left\{\frac{c_3(\gamma_+)p_n^{-1}}{(k/(np_n))}\frac{1}{n} + c_4(\gamma_+,\rho')\tilde c_2(\frac{n}{k})^{2\rho'}\right\} \\
&\sim p_n^{-2\gamma-2\rho'}(\log(\frac{k}{np_n}))^2\left\{\frac{c_3(\gamma_+)p_n^{2\rho'-1}}{(k/(np_n))}\frac{1}{n} + c_4(\gamma_+,\rho')\tilde c_2(\frac{np_n}{k})^{2\rho'}\right\}.
\end{aligned}
$$

So we are looking for

$$
\arg\min_u p_n^{-2\gamma-2\rho'}\left\{(\log u)^2\frac{c_3 p_n^{2\rho'-1}}{u}\frac{1}{n} + c_4\tilde c_2(\log u)^2 u^{-2\rho'}\right\}.
$$

Write $s := (\log u)^2/u$. Then $u \sim s^{-1}(\log s)^2 (u \to \infty)$ and we are dealing with

$$
\arg\min_s p_n^{-2\gamma-2\rho'}\left\{\frac{c_3 p_n^{2\rho'-1}}{n}s + c_4\tilde c_2 s^{2\rho'}(\log s)^{2(1-2\rho')}\right\}.
$$

This can be minimized by equating the derivative to zero. The result is

$$
\begin{aligned}
\frac{c_3}{\tilde c_2 c_4(-2\rho')}\frac{p_n^{2\rho'-1}}{n} &= s^{2\rho'-1}(\log s)^{2(1-2\rho')} + \frac{1-2\rho'}{\rho'}s^{2\rho'-1}(\log\ s)^{2(1-2\rho')-1} \\
&\sim s^{2\rho'-1}(\log s)^{2(1-2\rho')}.
\end{aligned}
$$

That is,

$$
\frac{1}{u} \sim \frac{s}{(\log\ s)^2} = \left(\frac{c_4\tilde c_2(-2\rho')}{c_3}\right)^{\frac{1}{1-2\rho'}} p_n n^{\frac{1}{1-2\rho'}}.
$$

Note that the right hand side tends to zero since $np_n \to 0(n \to \infty)$. Now, replacing u by $k/(np_n)$, we get

$$\frac{k}{np_n} \sim \left(\frac{c_4\tilde{c}_2(-2\rho')}{c_3}\right)^{\frac{-1}{1-2\rho'}} p_n^{-1} n^{\frac{-1}{1-2\rho'}}$$

or

$$k_0(n) \sim \left(\frac{c_3}{c_4\tilde{c}_2(-2\rho')}\right)^{\frac{1}{1-2\rho'}} n^{\frac{-2\rho'}{1-2\rho'}}.$$

Note that $k_0(n)$ does not depend on p_n.

Finally we consider as. $E(\hat{x}_{p_n}(k) - x_{p_n})^2$ for $\gamma < 0$: by the preceding Lemmas, disregarding terms which are $o(\frac{1}{\sqrt{k}})$ or $o(\tilde{A}(\frac{n}{k}))$,

$$\begin{aligned}
&\hat{x}_{p_n}(k) - x_{p_n} = \\
&= a(\frac{n}{k})\left\{\frac{\hat{\gamma}_n(k)-\gamma}{\gamma^2} - \frac{1}{\gamma}\left(\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} - 1\right) + \frac{\hat{b}(\frac{n}{k}) - U(\frac{n}{k})}{a(\frac{n}{k})} + \frac{1_{\{\gamma<\rho\}}}{\gamma(\rho+\gamma_-)}\tilde{A}(\frac{n}{k})\right\} \\
&= a(\frac{n}{k})\left[\frac{q_{\gamma,\rho}l_1}{\gamma^2}\tilde{A}(\frac{n}{k}) - \frac{4}{\gamma^2 l_1 l_2}M_1 + \frac{2}{\gamma^2 l_2^2}M_2\right. \\
&\quad \left. - \frac{1}{\gamma}\left\{\left(\frac{1}{l_1} + \frac{4}{l_2}\right)M_1 - \frac{2l_1}{l_2^2}M_2\right\} - \frac{B}{\sqrt{k}} + \frac{B}{\sqrt{k}} + \frac{1_{\{\gamma<\rho\}}}{\gamma(\rho+\gamma_-)}\tilde{A}(\frac{n}{k})\right] \\
&= a(\frac{n}{k})\left(\frac{-4}{\gamma^2 l_1 l_2} - \frac{1}{\gamma l_1} - \frac{4}{\gamma l_2}\right)M_1 + \left(\frac{2}{\gamma^2 l_2^2} + \frac{2l_1}{\gamma l_2^2}\right)M_2 \\
&\quad + \left(\frac{q_{\gamma,\rho}l_1}{\gamma^2} + \frac{1_{\{\gamma<\rho\}}}{\gamma(\rho+\gamma_-)}\right)\tilde{A}(\frac{n}{k}).
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{\text{as. } E(\hat{x}_{p_n}(k) - x_{p_n})^2}{a^2(\frac{n}{k})} = \\
&= \left(-\frac{4}{\gamma^2 l_1 l_2} - \frac{1}{\gamma l_1} - \frac{4}{\gamma l_2}\right)^2 \frac{EP_1^2}{k} + \left(\frac{2}{\gamma^2 l_2^2} + \frac{2l_1}{\gamma l_2^2}\right)^2 \frac{EP_2^2}{k} \\
&\quad + \left(-\frac{4}{\gamma^2 l_1 l_2} - \frac{1}{\gamma l_1} - \frac{4}{\gamma l_2}\right)\left(\frac{2}{\gamma^2 l_2^2} + \frac{2l_1}{\gamma l_2^2}\right)\frac{EP_1P_2}{k} \\
&\quad + \left\{\left(-\frac{4}{\gamma^2 l_1 l_2} - \frac{1}{\gamma l_1} - \frac{4}{\gamma l_2}\right)d_1 + \left(\frac{2}{\gamma^2 l_2^2} + \frac{2l_1}{\gamma l_2^2}\right)d_2 + \left(\frac{q_{\gamma,\rho}l_1}{\gamma^2} + \frac{1_{\{\gamma<\rho\}}}{\gamma(\rho+\gamma_-)}\right)\right\}^2 \tilde{A}^2(\frac{n}{k}) \\
&=: \frac{c_5(\gamma_-)}{k} + c_6(\gamma_-,\rho')\tilde{A}^2(\frac{n}{k}) = \frac{c_5(\gamma_-)}{k} + \tilde{c}_2^2 c_6(\gamma_-,\rho')(\frac{n}{k})^{2\rho'}.
\end{aligned}$$

Hence

$$\text{as. } E(\hat{x}_{p_n}(k) - x_{p_n})^2 = c_1^2(\frac{n}{k})^{2\gamma}\left\{\frac{c_5}{k} + c_6\tilde{c}_2^2(\frac{n}{k})^{2\rho'}\right\} c_1^2 n^{2\gamma}\left\{\frac{c_5}{k^{1+2\gamma}} + c_6\tilde{c}_2^2\frac{n^{2\rho'}}{k^{2\gamma+2\rho'}}\right\}.$$

By assumption $1+2\gamma > 0$. Write $t := k^{-(1+2\gamma)}$. We want to minimize

$$tc_5 + \tilde{c}_2^2 c_6 n^{2\rho'} t^{\frac{2\rho'+2\gamma}{1+2\gamma}}.$$

Equating the derivative to zero yields

$$k^{1-2\rho'} = t^{\frac{2\rho'+2\gamma}{1+2\gamma}-1} = \frac{c_5}{\tilde{c}_2^2 c_6} n^{-2\rho'} \frac{1+2\gamma}{-2\rho'-2\gamma},$$

i.e. $(n \to \infty)$

$$k_0(n) \sim \left(\frac{1+2\gamma}{-2\rho' - 2\gamma}\,\frac{c_5}{\tilde{c}_2^2 c_6}\right)^{\frac{1}{1-2\rho'}} n^{\frac{-2\rho'}{1-2\rho'}}.$$

□

Proof of Theorem 2.1. Immediate consequence of Proposition 4.12. □

Proposition 4.13. *Under the conditions of Theorem 2.3, as $n \to \infty$,*

$$\overline{k}_0(n) \sim \begin{cases} \left(\frac{\overline{c}_3}{\overline{c}_4 \tilde{c}_2(-2\rho')}\right)^{\frac{1}{1-2\rho'}} n^{\frac{-2\rho'}{1-2\rho'}} & \text{for } \gamma > 0 \\ \left(\frac{1+2\gamma}{-2\rho'-2\gamma}\,\frac{\overline{c}_5}{\tilde{c}_2 \overline{c}_6}\right)^{\frac{1}{1-2\rho'}} n^{\frac{-2\rho'}{1-2\rho'}} & \text{for } \gamma < 0 \end{cases}$$

$$=: \overline{h}(\gamma_+, \gamma_-, \rho')\, n^{\frac{-2\rho'}{1-2\rho'}}$$

where $\overline{k}_0(n) := \arg\min_k$ *as.* $E(\hat{x}_{p_n}(k) - \hat{\overline{x}}_{p_n}(k))^2$,

$$\hat{\overline{x}}_{p_n}(k) := X_{n-k,n} + \hat{\overline{a}}(\frac{n}{k})\frac{(\frac{k}{np_n})^{\hat{\overline{\gamma}}_n(k)} - 1}{\hat{\overline{\gamma}}_n(k)}, \quad \tilde{c}_2 \text{ from } \tilde{A}(t) \sim \tilde{c}_2 t^{\rho'} (t \to \infty) \text{ and}$$

$$\begin{aligned}
\overline{c}_3 &:= \overline{c}_3(\gamma_+) := \frac{1}{4}(1+\gamma_+^2) \\
\overline{c}_4 &:= \overline{c}_4(\gamma_+, \rho') := \frac{(\rho' + \gamma_+ - \gamma_+\rho')^2}{4(1-\rho')^6} \\
\overline{c}_5 &:= \overline{c}_5(\gamma_-) := \frac{(1-\gamma_-)^2(1 - 6\gamma_- + 35\gamma_-^2 - 78\gamma_-^3 + 72\gamma_-^4)}{4\gamma_-^4(1-2\gamma_-)(1-3\gamma_-)(1-4\gamma_-)(1-5\gamma_-)(1-6\gamma_-)} \\
\overline{c}_6 &:= \overline{c}_6(\gamma_-, \rho') :=
\end{aligned}$$

$$\frac{(1-\gamma_-)^2\rho'^2}{4\gamma_-^4(1-\gamma_- - \rho')^2(1-2\gamma_- - \rho')^2(1-3\gamma_- - \rho')^2} \qquad if\ q_{\gamma,\rho} = 0\,,$$

$$\Bigg[\frac{-2+12\gamma_- -22\gamma_-^2+12\gamma_-^3+5\rho' -22\gamma_-\rho'+21\gamma_-^2\rho'-6\rho'^2+12\gamma_-\rho'^2+2\rho'^3}{2\gamma_-^2(1-\gamma_-)(1-\gamma_- -\rho')(1-2\gamma_- -\rho')(1-3\gamma_- -\rho')}$$

$$+\frac{2-14\gamma_- +34\gamma_-^2-34\gamma_-^3+12\gamma_-^4-6\rho'+30\gamma_-\rho'-46\gamma_-^2\rho'+22\gamma_-^3\rho'+6\rho'^2-18\gamma_-\rho'^2+12\gamma_-^2\rho'^2-2\rho'^3+2\gamma_-\rho'^3}{2\gamma_-^2(1-\gamma_-)(1-\gamma_- -\rho')(1-2\gamma_- -\rho')(1-3\gamma_- -\rho')\sqrt{(1-\gamma_-)(1-2\gamma_-)}}\Bigg]^2 \quad if\ q_{\gamma,\rho} = -1\,.$$

Proof. For $\gamma > 0$, neglecting terms which are $o(\frac{1}{\sqrt{k}})$ or $o(\tilde{A}(\frac{n}{k}))$, and by similar arguments as in the proof of Proposition 4.12, the dominant term in the expansion of $\hat{x}_{p_n}(k) - \hat{\overline{x}}_{p_n}(k)$ turns out to be

$$\hat{x}_{p_n}(k) - \hat{\overline{x}}_{p_n}(k) \sim c_1 p_n^{-\gamma}\gamma^{-1}(\log a_n)(\hat{\gamma}_n(k) - \hat{\overline{\gamma}}_n(k))$$

$$\sim c_1 p_n^{-\gamma}\gamma^{-1}\log a_n \left[(\gamma_+ - \frac{3}{2})M_1 + \frac{1}{4}(3-\gamma_+)M_2 - \frac{1}{12}M_3\right]$$

$$\sim c_1 p_n^{-\gamma}\gamma^{-1}\log\, a_n \left[(\gamma_+ - \frac{3}{2})(\frac{P_1}{\sqrt{k}} + d_1\tilde{A}(\frac{n}{k})) + \frac{3-\gamma_+}{4}\left(\frac{P_2}{\sqrt{k}} + d_2\tilde{A}(\frac{n}{k})\right) - \frac{1}{12}(\frac{P_3}{\sqrt{k}} + d_3\tilde{A}(\frac{n}{k}))\right]$$

.

Hence

$$\begin{aligned}
\text{as. } E(\hat{x}_{p_n}(k) - \hat{\bar{x}}_{p_n}(k))^2 &= (\log a_n)^2 c_1^2 p_n^{-2\gamma}\gamma^{-2} \\
&\Big[(\gamma_+ - \frac{3}{2})^2\frac{EP_1^2}{k} + (\frac{3-\gamma_+}{4})^2\frac{EP_2^2}{k} + (\frac{1}{12})^2\frac{EP_3^2}{k} \\
&+\frac{1}{2}(\gamma_+ - \frac{3}{2})(3-\gamma_+)EP_1P_2 - \frac{1}{6}(\gamma_+ - \frac{3}{2})\frac{EP_1P_3}{k} - \frac{1}{24}(3-\gamma_+)\frac{EP_2P_3}{k} \\
&+\left\{(\gamma_+ - \frac{3}{2})d_1 + \frac{1}{4}(3-\gamma_+)d_2 - \frac{1}{12}d_3\right\}^2 \tilde{A}^2(\frac{n}{k})\Big] \\
=: \quad & c_1^2 p_n^{-2\gamma}\gamma^{-2}\left[\bar{c}_3(\gamma_+)\frac{(\log a_n)^2}{k} + \bar{c}_4(\gamma_+,\rho')(\log a_n)^2\tilde{A}^2(\frac{n}{k})\right].
\end{aligned}$$

Minimizing this over k as in the proof of Proposition 4.12 yields $(n \to \infty)$

$$\bar{k}_0(n) \sim \left(\frac{\bar{c}_3}{\bar{c}_4\tilde{c}_2(-2\rho')}\right)^{\frac{1}{1-2\rho'}} n^{\frac{-2\rho'}{1-2\rho'}}.$$

Next suppose $\gamma < 0$. Then, neglecting terms which are $o(\frac{1}{\sqrt{k}})$ or $o(\tilde{A}(\frac{n}{k}))$, as in the proof of Proposition 4.12,

$$\begin{aligned}
&\hat{x}_{p_n}(k) - \hat{\bar{x}}_{p_n}(k) = \\
= \quad & a(\frac{n}{k})\left\{\frac{\hat{\gamma}_n(k) - \hat{\bar{\gamma}}_n(k)}{\gamma^2} - \frac{1}{\gamma}\left(\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} - 1\right) + \frac{1}{\gamma}\left(\frac{\hat{\bar{a}}(\frac{n}{k})}{a(\frac{n}{k})} - 1\right)\right\} \\
= \quad & a(\frac{n}{k})\left[\frac{1}{\gamma^2}\left\{q_{\gamma,\rho}l_1\tilde{A}(\frac{n}{k}) - q_{\gamma,\rho}(\frac{l_2}{2})^{1/2}\tilde{A}(\frac{n}{k}) + \left(-\frac{4}{l_1l_2} + \frac{3l_2}{2l_1^2l_3}\right)M_1\right.\right. \\
&\left.\left. + \left(\frac{2}{l_2^2} + \frac{3}{2l_1l_3}\right)M_2 - \frac{3l_2}{2l_1l_3^2}M_3\right\} + \frac{1}{\gamma}\left\{\left(-\frac{4}{l_2} + \frac{3l_2}{2l_1l_3}\right)M_1\left(\frac{2l_1}{l_2^2} + \frac{3}{2l_3}\right)M_2 - \frac{3l_2}{2l_3^2}M_3\right\}\right] \\
= \quad & a(\frac{n}{k})\left[\left\{\frac{1}{\gamma^2}\left(-\frac{4}{l_1l_2} + \frac{3l_2}{2l_1^2l_3}\right) + \frac{1}{\gamma}\left(-\frac{4}{l_2} + \frac{3l_2}{2l_1l_3}\right)\right\}M_1\right. \\
&+\left\{\frac{1}{\gamma^2}\left(\frac{2}{l_2^2} + \frac{3}{2l_1l_3}\right) + \frac{1}{\gamma}\left(\frac{2l_1}{l_2^2} + \frac{3}{2l_3}\right)\right\}M_2 \\
&\left.+\left(-\frac{1}{\gamma^2}\frac{3l_2}{2l_1l_3^2} - \frac{1}{\gamma}\frac{3l_2}{2l_3^2}\right)M_3 + \frac{q_{\gamma,\rho}}{\gamma^2}\left(l_1 - (\frac{l_2}{2})^{1/2}\right)\tilde{A}(\frac{n}{k})\right] \\
=: \quad & a(\frac{n}{k})\left(g_1M_1 + g_2M_2 + g_3M_3 + g_0\tilde{A}(\frac{n}{k})\right) \\
= \quad & a(\frac{n}{k})\left(g_1\left(\frac{P_1}{\sqrt{k}} + d_1\tilde{A}(\frac{n}{k})\right) + g_2\left(\frac{P_2}{\sqrt{k}} + d_2\tilde{A}(\frac{n}{k})\right) + g_3\left(\frac{P_3}{\sqrt{k}} + d_3\tilde{A}(\frac{n}{k})\right) + g_0\tilde{A}(\frac{n}{k})\right) \\
= \quad & a(\frac{n}{k})\left(g_1\frac{P_1}{\sqrt{k}} + g_2\frac{P_2}{\sqrt{k}} + g_3\frac{P_3}{\sqrt{k}} + (g_1d_1 + g_2d_2 + g_3d_3 + g_0)\tilde{A}(\frac{n}{k})\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\text{as. } E(\hat{x}_{p_n}(k) - \hat{\bar{x}}_{p_n})^2 &= a^2(\frac{n}{k}) \\
&\left(g_1^2\frac{EP_1^2}{k} + g_2^2\frac{EP_2^2}{k} + g_3^2\frac{EP_3^2}{k} + 2g_1g_2\frac{EP_1P_2}{k} + 2g_1g_3\frac{EP_1P_3}{k}\right. \\
&\left.+2g_2g_3\frac{EP_2P_3}{k} + (g_1d_1 + g_2d_2 + g_3d_3 + g_0)^2\tilde{A}^2(\frac{n}{k})\right) \\
=: \quad & a^2(\frac{n}{k})\left(\bar{c}_5(\gamma_-)\frac{1}{k} + \bar{c}_6(\gamma_-,\rho')\tilde{A}^2(\frac{n}{k})\right) = c_1^2(\frac{n}{k})^{2\gamma}\left(\frac{\bar{c}_5}{k} + \bar{c}_6\tilde{c}_2^2(\frac{n}{k})^{2\rho'}\right).
\end{aligned}$$

As in the proof of Proposition 4.12 we find $(n \to \infty)$

$$\bar{k}_0(n) \sim \left(\frac{1+2\gamma}{-2\rho'-2\gamma}\frac{\bar{c}_5}{\tilde{c}_2^2\bar{c}_6}\right)^{\frac{1}{1-2\rho'}} n^{\frac{-2\rho'}{1-2\rho'}}.$$

□

Proof of Theorem 2.1. Cf. [6] Draisma, de Haan, Peng and Pereira (1998). □

Proposition 4.14. *Under the conditions of Theorem 2.6, as $n \to \infty$,*

$$k_0(n) \sim \left(\frac{1+2\gamma_-}{-2\rho'-2\gamma_-}\frac{c_7}{\tilde{c}_2^2 c_8}\right)^{\frac{1}{1-2\rho'}} n^{\frac{-2\rho'}{1-2\rho'}} =: g(\gamma_-,\rho')\, n^{\frac{-2\rho'}{1-2\rho'}}$$

where $k_0(n) := \arg\min_k$ *as.* $E(\hat{x}_0(k)-x_0)^2$, $\tilde{c}_2$ *from* $\tilde{A}(t) \sim \tilde{c}_2 t^{\rho'} (t \to \infty)$ *and*

$$c_7 := c_7(\gamma_-) := \frac{(1-\gamma_-)^2(1-3\gamma_-+4\gamma_-^2)}{\gamma_-^4(1-2\gamma_-)(1-3\gamma_-)(1-4\gamma_-)}$$

$$c_8 := c_8(\gamma_-,\rho') := \begin{cases} \frac{(2\gamma_- - 6\gamma_-^2 + 4\gamma_-^3 + \rho' - 5\gamma_-\rho' + 6\gamma_-^2\rho' + 2\gamma_-\rho'^2)^2}{\gamma_-^4(1-\gamma_--\rho')^2(\gamma_-+\rho')^2(1-2\gamma_--\rho')^2} & \text{if } q_{\gamma,\rho} = 0, \\ \frac{(1-3\gamma_-+2\gamma_-^2+\gamma_-\rho')^2}{\gamma_-^4(1-\gamma_--\rho')^2(1-2\gamma_--\rho')^2} & \text{if } q_{\gamma,\rho} = -1. \end{cases}$$

Proof. By the preceding Lemmas, apart from terms $o(\frac{1}{\sqrt{k}})$ and $o(\tilde{A}(\frac{n}{k}))$,

$$\begin{aligned}
\hat{x}_0(k) - x_0 &= \hat{b}(\tfrac{n}{k}) - \hat{a}(\tfrac{n}{k})\,\frac{1}{\hat{\gamma}_n^-(k)} - U(\infty) \\
&= a(\tfrac{n}{k})\left[\frac{\hat{b}(\frac{n}{k})-b(\frac{n}{k})}{a(\frac{n}{k})} - \left(\frac{1}{\hat{\gamma}_n^-(k)} - \frac{1}{\gamma_-}\right)\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} - \frac{1}{\gamma_-}\left(\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})}-1\right) + \frac{b(\frac{n}{k})-U(\infty)}{a(\frac{n}{k})} - \frac{1}{\gamma_-}\right] \\
&= a(\tfrac{n}{k})\left[\frac{\hat{b}(\frac{n}{k})-b(\frac{n}{k})}{a(\frac{n}{k})} + \frac{\hat{\gamma}_n^-(k)-\gamma_-}{\gamma_-^2} - \frac{1}{\gamma_-}\left(\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})}-1\right) + \frac{1}{\gamma_-(\gamma_-+\rho)}A(\tfrac{n}{k})\right] \\
&= a(\tfrac{n}{k})\left[\frac{B}{\sqrt{k}} + \frac{1}{\gamma_-^2}\left\{-\frac{4}{l_1 l_2}M_1 + \frac{2}{l_2^2}M_2\right\} - \frac{1}{\gamma_-}\left\{(\frac{1}{l_1}+\frac{4}{l_2})M_1\right.\right. \\
&\qquad \left.\left. -\frac{2l_1}{l_2^2}M_2 + \gamma_-\frac{B}{\sqrt{k}}\right\} + \frac{1_{\{\gamma<\rho\}}}{\gamma_-(\gamma_-+\rho)}\tilde{A}(\tfrac{n}{k})\right] \\
&= a(\tfrac{n}{k})\left[\left(-\frac{4}{\gamma_-^2 l_1 l_2} - \frac{1}{\gamma_-}\frac{l_2+4l_1}{l_1 l_2}\right)M_1 + \left(\frac{2}{\gamma_-^2 l_2^2} + \frac{2l_1}{\gamma_- l_2^2}\right)M_2 + \frac{1_{\{\gamma<\rho\}}}{\gamma_-(\gamma_-+\rho)}\tilde{A}(\tfrac{n}{k})\right] \\
&= a(\tfrac{n}{k})\left[\left(-\frac{4}{\gamma_-^2 l_1 l_2} - \frac{1}{\gamma_-}\frac{l_2+4l_1}{l_1 l_2}\right)\left(\frac{P_1}{\sqrt{k}} + d_1\tilde{A}(\tfrac{n}{k})\right)\right. \\
&\qquad \left. + \left(\frac{2}{\gamma_-^2 l_2^2} + \frac{2l_1}{\gamma_- l_2^2}\right)\left(\frac{P_2}{\sqrt{k}} + d_2\tilde{A}(\tfrac{n}{k})\right) + \frac{1_{\{\gamma<\rho\}}}{\gamma_-(\gamma_-+\rho)}\tilde{A}(\tfrac{n}{k})\right].
\end{aligned}$$

Hence

$$
\begin{aligned}
&\text{as. } E(\hat{x}_0(k)-x_0)^2 = \\
&= \quad a^2(\frac{n}{k})\Bigg[\Big(-\frac{4}{\gamma_-^2 l_1 l_2}-\frac{1}{\gamma_-}\frac{l_2+4l_1}{l_1 l_2}\Big)^2\frac{EP_1^2}{k}+\Big(\frac{2}{\gamma_-^2 l_2^2}+\frac{2l_1}{\gamma_- l_2^2}\Big)^2\frac{EP_2^2}{k} \\
&\quad +2\Big(-\frac{4}{\gamma_-^2 l_1 l_2}-\frac{1}{\gamma_-}\frac{l_2+4l_1}{l_1 l_2}\Big)\Big(\frac{2}{\gamma_-^2 l_2^2}+\frac{2l_1}{\gamma_- l_2^2}\Big)\frac{EP_1P_2}{k}+ \\
&\quad \Big\{\Big(-\frac{4}{\gamma_-^2 l_1 l_2}-\frac{1}{\gamma_-}\frac{l_2+4l_1}{l_1 l_2}\Big)d_1+\Big(\frac{2}{\gamma_-^2 l_2^2}+\frac{2l_1}{\gamma_- l_2^2}\Big)d_2+\frac{1_{\{\gamma<\rho\}}}{\gamma_-(\gamma_-+\rho)}\Big\}^2\tilde{A}^2(\frac{n}{k})\Bigg] \\
&=: \quad a^2(\frac{n}{k})\Big\{\frac{c_7(\gamma_-)}{k}+c_8(\gamma_-,\rho')\tilde{A}^2(\frac{n}{k})\Big\} = c_1^2(\frac{n}{k})^{2\gamma_-}\Big\{\frac{c_7}{k}+c_8\tilde{c}_2^2(\frac{n}{k})^{2\rho'}\Big\}.
\end{aligned}
$$

Minimizing with respect to k as in Proposition 4.12 yields

$$k_0(n) \sim \left(\frac{1+2\gamma_-}{-2\rho'-2\gamma_-}\frac{c_7}{\tilde{c}_2^2 c_8}\right)^{\frac{1}{1-2\rho'}} n^{\frac{-2\rho'}{1-2\rho'}}.$$

□

Proof of Theorem 2.6. Immediate consequence of Proposition 4.14. □

Proposition 4.15. *Under the conditions of Theorem 2.5, as $n \to \infty$,*

$$\overline{k}_0(n) \sim \left(\frac{1+2\gamma_-}{-2\rho'-2\gamma_-}\frac{\overline{c}_7}{\tilde{c}_2^2\overline{c}_8}\right)^{\frac{1}{1-2\rho'}} n^{\frac{-2\rho'}{1-2\rho'}} =: \overline{g}(\gamma_-,\rho')\, n^{\frac{-2\rho'}{1-2\rho'}}$$

where $\overline{k}_0(n) := \arg\min_k$ as. $E(\hat{x}_0(k)-\hat{\overline{x}}_0(k))^2$, $\tilde{c}_2$ from $\tilde{A}(t) \sim \tilde{c}_2 t^{\rho'} (t \to \infty)$ and

$$
\begin{aligned}
\overline{c}_7 &:= \overline{c}_7(\gamma_-) := \frac{(1-\gamma_-)^2(1-6\gamma_-+35\gamma_-^2-78\gamma_-^3+72\gamma_-^4)}{4\gamma_-^4(1-2\gamma_-)(1-3\gamma_-)(1-4\gamma_-)(1-5\gamma_-)(1-6\gamma_-)} \\
\overline{c}_8 &:= \overline{c}_8(\gamma_-,\rho') := \frac{((\gamma_--1)\rho')^2}{4\gamma_-^4(1-\gamma_--\rho')^2(1-2\gamma_--\rho')^2(1-3\gamma_--\rho')^2}.
\end{aligned}
$$

Proof.

$$\hat{x}_0(k)-\hat{\overline{x}}_0(k) = X_{n-k,n}-\frac{\hat{a}(\frac{n}{k})}{\hat{\gamma}_n^-(k)}-\left\{X_{n-k,n}-\frac{\hat{\overline{a}}(\frac{n}{k})}{\hat{\overline{\gamma}}_n^-(k)}\right\}$$

$$= a(\frac{n}{k})\left[-\frac{1}{\hat{\gamma}_n^-(k)}\left(\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})}-1\right)+\frac{1}{\hat{\overline{\gamma}}_n^-(k)}\left(\frac{\hat{\overline{a}}(\frac{n}{k})}{a(\frac{n}{k})}-1\right)-\left(\frac{1}{\hat{\gamma}_n^-(k)}-\frac{1}{\hat{\overline{\gamma}}_n^-(k)}\right)\right].$$

So, disregarding terms of $o(\frac{1}{\sqrt{k}})$ or $o(\tilde{A}(\frac{n}{k}))$, from the preceding Lemmas

$$\begin{aligned}
&\hat{x}_0(k) - \hat{\bar{x}}_0(k) = \\
&= a(\frac{n}{k})\left[-\frac{1}{\gamma_-}\left(\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})}-1\right)+\frac{1}{\gamma_-}\left(\frac{\hat{\bar{a}}(\frac{n}{k})}{a(\frac{n}{k})}-1\right)+\frac{1}{\gamma_-^2}(\hat{\gamma}_n^-(k)-\hat{\bar{\gamma}}_n^-(k))\right] \\
&= a(\frac{n}{k})\left[-\frac{1}{\gamma_-}\left(\frac{l_2+4l_1}{l_1l_2}M_1-\frac{2l_1}{l_2^2}M_2+\gamma_-\frac{B}{\sqrt{k}}\right)\right. \\
&\quad +\frac{1}{\gamma_-}\left(\frac{2l_3+3l_2}{2l_1l_3}M_1+\frac{3}{2l_3}M_2-\frac{3l_2}{2l_3^2}M_3+\gamma_-\frac{B}{\sqrt{k}}\right)+\frac{1}{\gamma_-^2}\left(-\frac{4}{l_1l_2}M_1+\frac{2}{l_2^2}M_2\right) \\
&\quad \left.-\frac{1}{\gamma_-^2}\left(\frac{-3l_2}{2l_2^2l_3}M_1-\frac{3}{2l_1l_3}M_2+\frac{3l_2}{2l_1l_3^2}M_3\right)\right] \\
&= a(\frac{n}{k})\left[\left(-\frac{l_2+4l_1}{\gamma_-l_1l_2}+\frac{2l_3+3l_2}{2\gamma_-l_1l_3}-\frac{4}{\gamma_-^2l_1l_2}+\frac{3l_2}{2\gamma_-^2l_1^2l_3}\right)M_1\right. \\
&\quad \left.+\left(\frac{2l_1}{\gamma_-l_2^2}+\frac{3}{2\gamma_-l_3}+\frac{2}{\gamma_-^2l_2^2}+\frac{3}{2\gamma_-^2l_1l_3}\right)M_2+\left(-\frac{3l_2}{2\gamma_-l_3^2}-\frac{3l_2}{2\gamma_-^2l_1l_3^2}\right)M_3\right] \\
&=: a(\frac{n}{k})(g_1M_1+g_2M_2+g_3M_3) \\
&= a(\frac{n}{k})\left\{g_1\left(\frac{P_1}{\sqrt{k}}+d_1\tilde{A}(\frac{n}{k})\right)+g_2\left(\frac{P_2}{\sqrt{k}}+d_2\tilde{A}(\frac{n}{k})\right)+g_3(\frac{P_3}{\sqrt{k}}+d_3\tilde{A}(\frac{n}{k})\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\text{as. } E(\hat{x}_0(k)-\hat{\bar{x}}_0(k))^2 = \\
&= a^2(\frac{n}{k})\left[\frac{1}{k}\{g_1^2EP_1^2+g_2^2EP_2^2+g_3^2EP_3^2+2g_1g_2EP_1P_2\right. \\
&\quad \left.+2g_1g_3EP_1P_3+2g_2g_3EP_2P_3\}+(g_1d_1+g_2d_2+g_3d_3)^2\tilde{A}^2(\frac{n}{k})\right] \\
&=: a^2(\frac{n}{k})\left\{\frac{\bar{c}_7(\gamma_-)}{k}+\bar{c}_8(\gamma_-,\rho')\tilde{A}^2(\frac{n}{k})\right\}=c_1^2(\frac{n}{k})^{2\gamma_-}\left\{\frac{\bar{c}_7}{k}+\bar{c}_8\tilde{c}_2^2(\frac{n}{k})^{2\rho'}\right\}.
\end{aligned}$$

Minimizing with respect to k as before yields ($n\to\infty$)

$$\bar{k}_0(n)\sim\left(\frac{1+2\gamma_-}{-2\rho'-2\gamma_-}\frac{\bar{c}_7}{\tilde{c}_2^2\bar{c}_8}\right)^{\frac{1}{1-2\rho'}}n^{\frac{-2\rho'}{1-2\rho'}}.$$

□

Proof of Theorem 2.5. Cf. [6] Draisma, de Haan, Peng and Pereira (1998). □

Proposition 4.16. *Under the conditions of Theorem 2.7, as $n\to\infty$,*

$$k_0(n)\sim\left(\frac{-\rho'(1-\rho')^2}{2\tilde{c}_2}\right)^{1/(1-2\rho')}n^{\frac{-2\rho'}{1-2\rho'}}=:\ l(\gamma_+,\rho')\,n^{\frac{-2\rho'}{1-2\rho'}} \tag{4.4}$$

where $k_0(n) := \arg\min_k$ as. $E\left(\hat{x}^+_{p_n}(k)-x_{p_n}\right)^2$ and $\tilde{c}_2$ from $\tilde{A}(t)\sim\tilde{c}_2t^{\rho'}(t\to\infty)$.

Proof. Set $a_n = k/(np_n)$ and note that $a_n\to\infty$ as $n\to\infty$. From Lemma 4.3

$$U(tx)=U(t)+a(t)\frac{x^\gamma-1}{\gamma}\left\{1-\frac{1}{\rho}A\left(\frac{n}{k}\right)+o\left(A\left(\frac{n}{k}\right)\right)\right\},\ t\to\infty,x\to\infty\,.$$

Also from [6] Draisma, de Haan, Peng and Pereira (1998) (or from $q_{\gamma,\rho}$ in Lemma 4.6)

$$\frac{a(t)/U(t)-\gamma_+}{\tilde{A}(t)} = \frac{\gamma_+}{\rho'}(1+o(1)) \Leftrightarrow \frac{U(t)}{a(t)} = \frac{1}{\gamma_+} - \frac{1}{\rho'}\tilde{A}(t) + o\left(\tilde{A}(t)\right) . \tag{4.5}$$

Hence, still using the asymptotic expansion of $a(Y_{n-k,n})/a(n/k)$ in the proof of Lemma 4.11,

$$\begin{aligned}
\hat{x}^+_{p_n}(k) - x_{p_n} &= U(Y_{n-k,n})a_n^{\hat{\gamma}_n^+(k)} - U\left(\frac{1}{p_n}\right) \\
&= a\left(\frac{n}{k}\right)\left\{\frac{a(Y_{n-k,n})}{a\left(\frac{n}{k}\right)}\frac{U(Y_{n-k,n})}{a(Y_{n-k,n})}a_n^{\hat{\gamma}_n^+(k)} - \frac{U\left(\frac{n}{k}\right)}{a\left(\frac{n}{k}\right)} - \frac{a_n^{\gamma_+}-1}{\gamma_+}\left[1-\frac{1}{\rho}A\left(\frac{n}{k}\right)+o\left(A\left(\frac{n}{k}\right)\right)\right]\right\} \\
&= a\left(\frac{n}{k}\right)\left\{\left[\frac{1}{\gamma_+}+\frac{B}{\sqrt{k}}-\frac{1}{\rho'}\tilde{A}\left(\frac{n}{k}\right)+o\left(\tilde{A}\left(\frac{n}{k}\right)\right)+o\left(\frac{1}{\sqrt{k}}\right)\right]a_n^{\hat{\gamma}_n^+(k)}\right. \\
&\quad \left. -\left[\frac{1}{\gamma_+}-\frac{1}{\rho'}\tilde{A}\left(\frac{n}{k}\right)+o\left(\tilde{A}\left(\frac{n}{k}\right)\right)\right] - \frac{a_n^{\gamma_+}-1}{\gamma_+}\left[1-\frac{1}{\rho}A\left(\frac{n}{k}\right)+o\left(A\left(\frac{n}{k}\right)\right)\right]\right\} \\
&= a\left(\frac{n}{k}\right)a_n^{\gamma_+}\left\{\frac{a_n^{\hat{\gamma}_n^+(k)-\gamma_+}-1}{\gamma_+}(1+o_p(1)) + \frac{B}{\sqrt{k}} - \frac{1}{\rho'}\tilde{A}\left(\frac{n}{k}\right) + \frac{1}{\gamma_+\rho}A\left(\frac{n}{k}\right)\right. \\
&\quad \left. +o\left(\tilde{A}\left(\frac{n}{k}\right)\right)+o\left(A\left(\frac{n}{k}\right)\right)+o\left(\frac{1}{\sqrt{k}}\right)\right\} .
\end{aligned}$$

Therefore following the same arguments as in Proposition 4.12 for positive γ, noticing that

$$A(t) = \begin{cases} o\left(\tilde{A}(t)\right) & \text{if} \quad 0<\gamma<-\rho \;\&\; \lim_{t\to\infty} U(t)-a(t)/\gamma \neq 0 \quad \text{or} \qquad \gamma = -\rho \\ O\left(\tilde{A}(t)\right) & \text{otherwise} , \end{cases}$$

and using $a(t) \sim c_1 t^{\gamma_+} \Rightarrow a(n/k)\,a_n^{\gamma_+} \sim c_1 p_n^{-\gamma_+}$, as $t\to\infty$, we get for the optimal sequence $k_0(n)$

$$\hat{x}^+_{p_n}(k_0(n)) - x_{p_n} \sim \frac{c_1 p_n^{-\gamma_+}}{\gamma_+}\log a_n\left(\hat{\gamma}_n^+(k_0(n))-\gamma_+\right)$$

that is,

$$\inf_k \text{as. } \mathrm{E}\left(\hat{x}^+_{p_n}(k)-x_{p_n}\right)^2 \sim \left(\frac{c_1 p_n^{-\gamma_+}}{\gamma_+}\right)^2 \inf_k \text{as. } \mathrm{E}\left[(\log a_n)^2\left(\hat{\gamma}_n^+(k)-\gamma_+\right)^2\right] .$$

Therefore from Lemma 4.6

$$\mathrm{E}\left(\hat{\gamma}_n^+(k)-\gamma_+\right)^2 \sim \frac{\gamma_+^2}{k} + \frac{\gamma_+^2}{\rho'^2(1-\rho')^2}\tilde{A}^2\left(\frac{n}{k}\right) \tag{4.6}$$

and the result follows. The rest of the proof is similar to the proof of Proposition 4.12 □

Proof of Theorem 2.7. Immediate consequence of Proposition 4.16. □

Proposition 4.17. *Under the conditions of Theorem 2.8, as $n\to\infty$,*

$$\bar{k}_0(n) \sim \left(\frac{(1-\rho')^4}{-2\rho'\tilde{c}_2}\right)^{1/(1-2\rho')} n^{\frac{-2\rho'}{1-2\rho'}} =: \bar{l}(\gamma_+,\rho')\, n^{\frac{-2\rho'}{1-2\rho'}} \tag{4.7}$$

where $\bar{k}_0(n) := \arg\min_k$ as. $E\left(\hat{x}^+_{p_n}(k)-\hat{\bar{x}}^+_{p_n}(k)\right)^2$ and $\tilde{c}_2$ from $\tilde{A}(t)\sim\tilde{c}_2 t^{\rho'}\,(t\to\infty)$.

Proof. Following similar arguments as before, for the optimal sequence $k_0(n)$ we have

$$\hat{x}^+_{p_n}(k_0(n)) - \hat{\hat{x}}^+_{p_n}(k_0(n)) \sim \frac{c_1 p_n^{-\gamma_+}}{\gamma_+} \log a_n \left(\hat{\gamma}^+_n(k_0(n)) - \hat{\hat{\gamma}}^+_n(k_0(n))\right)$$

where from Lemma 4.6, neglecting terms which are $o(\frac{1}{\sqrt{k}})$ or $o(\tilde{A}(\frac{n}{k}))$,

$$\hat{\gamma}^+_n(k) - \hat{\hat{\gamma}}^+_n(k) = \frac{\gamma_+ P}{\sqrt{k}} - \frac{\gamma_+}{4}\frac{Q}{\sqrt{k}} - \frac{\gamma_+}{2\,(1-\rho\prime)^2}\tilde{A}\left(\frac{n}{k}\right) \tag{4.8}$$

so that

$$\mathrm{E}\left(\hat{\gamma}^+_n(k) - \hat{\hat{\gamma}}^+_n(k)\right)^2 \sim \frac{\gamma_+^2}{4k} + \frac{\gamma_+^2}{4(1-\rho\prime)^4}\tilde{A}^2\left(\frac{n}{k}\right) \qquad \text{as} \qquad n \to \infty\,.$$

The result follows. □

Proof of Theorem 2.8. Immediate consequence of Proposition 4.17. □

Proposition 4.18. *Under the conditions of Theorem 2.9, as $n \to \infty$,*

$$\bar{\bar{k}}_0(n) \sim \left(\frac{(1-\rho\prime)^4}{-2\rho\prime\tilde{c}_2}\right)^{\frac{1}{1-2\rho\prime}} n^{\frac{-2\rho\prime}{1-2\rho\prime}} =: \bar{l}(\gamma_+, \rho\prime)\, n^{\frac{-2\rho\prime}{1-2\rho\prime}} \tag{4.9}$$

where $\bar{\bar{k}}_0(n) := \arg\min_k$ as. $E\left(\hat{x}^+_{p_n}(k) - \hat{\hat{\hat{x}}}^+_{p_n}(k)\right)^2$ and $\tilde{c}_2$ from $\tilde{A}(t) \sim \tilde{c}_2 t^{\rho'}(t \to \infty)$.

Proof. By Lemma 4.5 and (4.5), neglecting terms which are $o(\frac{1}{\sqrt{k}})$ or $o(\tilde{A}(\frac{n}{k}))$,

$$\hat{\hat{\hat{\gamma}}}^+_n(k) = \frac{M_n^{(2)}}{2M_n^{(1)}} = \frac{1}{2}\frac{a\,(Y_{n-k,n})}{U\,(Y_{n-k,n})}\frac{M_2 + l_2}{M_1 + l_1} = \gamma_+ + \frac{\gamma_+}{2}\frac{P_2}{\sqrt{k}} - \frac{\gamma_+ P_1}{\sqrt{k}} + \frac{\gamma_+}{(1-\rho\prime)^2\,\rho\prime}\tilde{A}\left(\frac{n}{k}\right)\,.$$

Hence

$$\hat{\gamma}^+_n(k) - \hat{\hat{\hat{\gamma}}}^+_n(k) = \frac{2\gamma_+ P_1}{\sqrt{k}} - \frac{\gamma_+}{2}\frac{P_2}{\sqrt{k}} - \frac{\gamma_+}{(1-\rho\prime)^2}\tilde{A}\left(\frac{n}{k}\right)\,.$$

Therefore following the same arguments as before

$$\hat{x}^+_{p_n}(k_0(n)) - \hat{\hat{\hat{x}}}^+_{p_n}(k_0(n)) \sim \frac{c_1 p_n^{-\gamma_+}}{\gamma_+} \log a_n \left(\hat{\gamma}^+_n(k_0(n)) - \hat{\hat{\hat{\gamma}}}^+_n(k_0(n))\right)$$

and since

$$\mathrm{E}\left(\hat{\gamma}^+_n(k) - \hat{\hat{\hat{\gamma}}}^+_n(k)\right)^2 \sim \frac{\gamma_+^2}{k} + \frac{\gamma_+^2}{(1-\rho\prime)^4}\tilde{A}^2\left(\frac{n}{k}\right) \qquad \text{as} \qquad n \to \infty$$

the result follows. □

Proof of Theorem 2.9. Immediate consequence of Proposition 4.18. □

Remark 4.19. *Compared to the quantile results when not restricting γ, in Theorems 2.7 2.8 and 2.9 a slightly different scheme was adopted, not separating results when $((0 < \gamma < -\rho$ and $\lim_{t\to\infty}(U(t) - a(t)/\gamma) \neq 0)$ or $(\gamma = -\rho))$ or not. Note that in applications it is not evident to know about $\lim_{t\to\infty}(U(t) - a(t)/\gamma)$. In the proofs the main difference relies on having taken $q_{\gamma,\rho'} = \gamma/\rho'$ in (4.5) and in the expansions of the γ estimators (4.6), (4.8) and (4) instead of $q_{\gamma,\rho}$ as in Lemma 4.6, which is not but a unified way of writing $q_{\gamma,\rho}$ when γ is positive.*

Proof of Theorem 2.11. Since we are dealing with $N_n^{(j)}$, $j = 1, 2, 3$, using the second order condition (2.8) we have

$$\left(\frac{U(tx) - U(t)}{a(t)}\right)^j = \left(\frac{x^{\gamma_-} - 1}{\gamma_-}\right)^j + j\, A(t) \left(\frac{x^{\gamma_-} - 1}{\gamma_-}\right)^{j-1} H_{\gamma_-,\rho}(x) + o(A(t)).$$

Therefore

$$M_j = \frac{N_n^{(j)}}{a^j(Y_{n-k,n})} - l_j$$

for $j = 1, 2, 3$ with M_j and l_j as in Lemma 4.5, just replacing ρ' by ρ and $\tilde{A}$ by A. The rest of the proof is the same as before. □

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