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On Exact Group Extensions
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ON EXACT GROUP EXTENSIONS

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ABSTRACT. We give conditions for the exactness of $\mathbb{R}^d$-extensions.

§0 INTRODUCTION

A nonsingular transformation $(X, \mathcal{B}, m, T)$ of a standard probability space is called a *fibred system* if there is a generating measurable partition α such that $T : a \to Ta$ is invertible, nonsingular for $a \in \alpha$, and a *Markov map* (or Markov fibred system) if in addition, $Ta \in \sigma(\alpha) \mod m \ \forall \, a \in \alpha$.

Write $\alpha = \{a_s : s \in S\}$ and endow $S^{\mathbb{N}}$ with its canonical (Polish) product topology. Let

$$\Sigma = \{s = (s_1, s_2, \dots) \in S^{\mathbb{N}} : \ m(\bigcap_{k=1}^{n} T^{-k} a_{s_k}) > 0 \ \ \forall \, n \geq 1\},$$

then Σ is a closed, shift invariant subset of $S^{\mathbb{N}}$, and there is a measurable map $\phi : \Sigma \to X$ defined by $\{\phi(s_1, s_2, \dots)\} := \bigcap_{k=1}^{\infty} T^{-(k-1)} a_{s_k}$.

The closed support of the probability $m' = m \circ \phi^{-1}$ is Σ, and ϕ is a conjugacy of $(X, \mathcal{B}, m, T)$ with $(\Sigma, \mathcal{B}(\Sigma), m', \text{shift})$. Thus we may, and sometimes do, assume that $X = \Sigma$, T is the shift, and $\alpha = \{[s] : s \in S\}$.

For $n \geq 1$, there are m-nonsingular inverse branches of T denoted $v_a : T^n a \to a$ and with Radon Nikodym derivatives denoted

$$v_a' := \frac{dm \circ v_a}{dm}.$$

Let $(X, \mathcal{B}, m, R)$ be a nonsingular transformation of a standard probability space. The *Frobenius-Perron* operators $P_{R^n} = P_{R^n, m} : L^1(m) \to L^1(m)$ are defined by

$$\int_X P_{R^n} f \cdot g \, dm = \int_X f \cdot g \circ R^n dm$$

and for the locally invertible $(X, \mathcal{B}, m, T, \alpha)$ (as above) have the form

$$P_{T^n} f = \sum_{a \in \alpha_0^{n-1}} 1_{T^n a} v_a' \cdot f \circ v_a.$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

A locally invertible map $(X, \mathcal{B}, m, T, \alpha)$ has:

the *Renyi property* if $\exists\, C > 1$ such that $\forall\, n \geq 1$, $a \in \alpha_0^{n-1}$, $m(a) > 0$:

$$\left| \frac{v_a'(x)}{v_a'(y)} \right| \leq C \text{ for } m \times m\text{-a.e. } (x, y) \in T^n a \times T^n a.$$

It is well known (a proof is recalled in [A-D-U]) that any topologically mixing probability preserving Markov map with the Renyi property is *exact* in the sense that $\bigcap_{n \geq 1} T^{-n} \mathcal{B} = \{\emptyset, X\} \mod m$.

Examples include:

• topological Markov shifts equipped with Gibbs measures ([Bo],[Bo-Ru]) and

• uniformly expanding, piecewise onto C^2 interval maps $T : [0,1] \to [0,1]$ satisfying Adler's condition $\sup_{x \in [0,1]} \frac{|T''(x)|}{T'(x)^2} < \infty$ ([Ad]);

or, more generally,

• Gibbs-Markov maps as in [A-D1].

Now let $\phi : X \to \mathbb{R}^d$ be measurable and consider the skew product $T_\phi : X \times \mathbb{R}^d \to X \times \mathbb{R}^d$ defined by $T_\phi(x, y) := (Tx, y + \phi(x))$ with respect to the (invariant) product measure $m \times m_{\mathbb{R}^d}$ where $m_{\mathbb{R}^d}$ denotes Lebesgue measure.

We say that ϕ is *aperiodic* if $\gamma(\phi) = z\bar{h}h \circ T$ has no nontrivial solution in $\gamma \in \hat{\mathbb{R}}^d$, $z \in S^1$ and $h : X \to S^1$ measurable. It is not hard to show that if T_ϕ is ergodic, and T is weakly mixing, then T_ϕ is weakly mixing iff ϕ is aperiodic.

We're interested in the exactness of T_ϕ.

We establish two (partial) results in this direction.

Theorem 1.

Suppose that $(X, \mathcal{B}, m, T, \alpha)$ is a probability preserving Markov map with the Renyi property. Let $N \geq 1$ and $\phi : X \to \mathbb{R}^d$ be α_0^{N-1}-measurable (i.e. $\phi(x) = \phi(\alpha_0^{N-1}(x))$ where $x \in \alpha_0^{N-1}(x) \in \alpha_0^{N-1}$).

If T_ϕ is topologically mixing, then T_ϕ is exact.

For the other result, we assume that $(X, \mathcal{B}, m, T, \alpha)$ is an exact probability preserving locally invertible map with the property that for some Banach space $(L, \| \cdot \|_L)$ of functions with $\| \cdot \|_2 \leq \| \cdot \|_L$, such that $P_T : L \to L$ and $\exists\, M > 0$, $\theta \in (0, 1)$ such that

$$\left\| P_{T^n} f - \int_X f\, dm \right\|_L \leq M\theta^n \|f\|_L \ \forall\, f \in L.$$

This property can be obtained as a consequence of the quasi compactness of Doeblin-Fortet operators, see [D-F], [IT-M]).

Given $\phi : X \to \mathbb{R}^d$ measurable, we define the *characteristic function operators* $P_t(f) = P_T(e^{i\langle t, \phi \rangle} f)$ $(t \in \mathbb{R}^d)$.

We assume also that $P_t : L \to L$ $(t \in \mathbb{R}^d)$ and that $t \mapsto P_t$ is continuous $(\mathbb{R}^d \to \text{Hom}(L, L))$.

It is shown in [Nag] (see also theorem 4.1 of [A-D1]) that

(i) there are constants $\epsilon > 0$, $K > 0$ and $\theta \in (0, 1)$; and continuous functions $\lambda : B(0, \epsilon) \to B_{\mathbb{C}}(0, 1)$, $g : B(0, \epsilon) \to L$ such that

$$\left\| P_t^n h - \lambda(t)^n g(t) \int_X h\, dm \right\|_L \leq K\theta^n \|h\|_L \quad \forall\, |t| < \epsilon, \ n \geq 1, \ h \in L;$$

and

(*ii*) in case ϕ is aperiodic, then $\forall\ 0 < \delta < M < \infty,\ \exists\ K > 0,\ 0 < \rho < 1$ such that

$$\|P_\gamma^n h\|_L \le K\rho^n \quad \forall\ h \in L,\ n \ge 1,\ \delta \le |\gamma| \le M.$$

Examples include:
- (see [A-D1]), $(X, \mathcal{B}, m, T, \alpha)$ a Gibbs-Markov maps and $\phi : X \to \mathbb{R}^d$ uniformly Hölder continuous on partition sets. Here L is a space of Hölder continuous functions $f : X \to \mathbb{C}$.
- (see [Rou], [Ry]), $X = [0, 1]$, m Lebesgue measure, α a partition of X mod m into open intervals, and $T : a \to Ta$ an invertible, m-nonsingular homeomorphism for each $a \in \alpha$ with $\inf |T'| > 1$ and $\frac{1}{T'}$ of bounded variation on X; and $\phi : X \to \mathbb{R}^d$ either: of bounded variation on X; or constant on each $a \in \alpha$.

Set $\phi_n = \phi + \phi \circ T + \dots + \phi \circ T^{n-1}$.

Theorem 2.
Suppose that

($\diamond$) $\qquad \forall\ \lambda > 1\ \exists\ n_k \to \infty$ *such that* $\dfrac{\phi_{n_k}}{\lambda^{n_k}} \to 0$ *a.e. as* $k \to \infty$

and that ϕ is aperiodic;
then T_ϕ is exact.

Remarks.
1) Theorem 2 generalises the corresponding theorem on page 443 in [G].
2) The condition ($\diamond$) is satisfied if m-dist (ϕ) is in the domain of attraction of a stable law.
3) The condition ($\diamond$) is not satisfied iff $\exists\ \lambda > 1$ and $\epsilon > 0$ such that $m([\|\phi_n\| > \lambda^n]) \ge \epsilon\ \forall\ n \ge 1$ and there are independent processes like this.

§1 FROBENIUS-PERRON OPERATORS, EXACTNESS AND RELATIVE EXACTNESS

Let $(X, \mathcal{B}, m, R)$ be a nonsingular transformation of a standard probability space. The *tail σ-algebra* of $(X, \mathcal{B}, m, R)$ is $\mathcal{T}(R) := \bigcap_{n=1}^\infty R^{-n}\mathcal{B}$ and the nonsingular transformation R is called *exact* if $= \{\emptyset, X\}$ mod m.

Theorem 1.1 [D-L].

$$\|P_{R^n} f\|_1 \to \|E(f|\mathcal{T}(R))\|_1\ as\ n \to \infty\ \forall\ f \in L^1(m).$$

In particular (see [L]), R is exact iff $\|P_{R^n} f\|_1 \to 0\ \forall\ f \in L^1(m),\ \int_X f dm = 0$.

Proof.
First note that $|P_T f| \le P_T |f|$ whence $\|P_{R^n} f\|_1 \downarrow$ and $\exists\ \lim_{n\to\infty} \|P_{R^n} f\|_1$. Next, $\forall\ n \ge 1\ \exists\ g_n \in L^\infty(\mathcal{B})$ with $\int_X (P_{R^n} f)g_n dm = \|P_{R^n} f\|_1$, whence

$$\|P_{R^n} f\|_1 = \int_X f g_n \circ R^n dm.$$

By weak $*$ compactness, $\exists\ n_k \to \infty$ and $g \in L^\infty(\mathcal{B})$ such that $g_{n_k} \circ R^{n_k} \rightharpoonup g$ weak $*$ in $L^\infty(\mathcal{B})$.

It follows that $g \in L^\infty(\mathcal{T}(R))$, $\|g\|_\infty \leq 1$ and $\lim_{n\to\infty} \|P_{R^n} f\|_1 = \int_X fg dm$. Thus

$$\lim_{n\to\infty} \|P_{R^n} f\|_1 \leq \sup\left\{\int_X fh dm : h \in L^\infty(\mathcal{T}(R)), \|h\|_\infty \leq 1\right\} = \|E(f|\mathcal{T}(R))\|_1.$$

To show the converse inequality, note that $\exists \; g \in L^\infty(\mathcal{T}(R))$, $\|g\|_\infty = 1$ such that

$$\|E(f|\mathcal{T}(R))\|_1 = \int_X E(f|\mathcal{T}(R))g dm = \int_X fg dm$$

whence $\forall \; n \geq 1$, $\exists \; g_n \in L^\infty(\mathcal{B})$, $g = g_n \circ R^n$ and

$$\|E(f|\mathcal{T}(R))\|_1 = \int_X fg dm = \int_X fg_n \circ R^n dm = \int_X (P_{R^n} f) g_n dm \leq \|P_{R^n} f\|_1.$$

$\square$

Let $(X, \mathcal{B}, m, R)$ and $(Y, \mathcal{C}, \mu, S)$ be nonsingular transformations of standard probability spaces. A *factor map* is a function $\pi : X \to Y$ satisfying $\pi^{-1}\mathcal{C} \subset \mathcal{B}$, $\pi \circ T = S \circ \pi$, $m \circ \pi^{-1} = \mu$.

The *fibre expectation* of the factor map $\pi : X \to Y$ is an operator

$$f \mapsto E(f|\pi), \; L^1(X, \mathcal{B}, m) \to L^1(Y, \mathcal{C}, \mu)$$

defined by $\int_Y E(f|\pi)g d\mu = \int_X fg \circ \pi dm$.

The factor map $\pi : X \to Y$ is called *relatively exact* if

$$f \in L^1(\mathcal{B}), \; E(f|\pi) = 0 \text{ a.e.} \implies \|P_{R^n} f\|_1 \to 0.$$

The corollary below appears in [G]. For the convenience of the reader, we supply a (possibly different) proof.

Proposition 1.2. *Suppose that $\pi : X \to Y$ is relatively exact, then $\mathcal{T}(R) = \pi^{-1}\mathcal{T}(S) \mod m$.*

Proof.

Evidently, $\pi^{-1}\mathcal{T}(S) \subseteq \mathcal{T}(R)$. We show that $\pi^{-1}\mathcal{T}(S) \supseteq \mathcal{T}(R)$.

By relative exactness and theorem 1.1, if $f \in L^1(\mathcal{B})$ and $E(f|\pi) = 0$ a.e., then $\int_X fg dm = 0 \; \forall \; g \in L^\infty(\mathcal{T}(R))$.

Thus if $f \in L^2(\mathcal{B}) \ominus L^2(\pi^{-1}\mathcal{C})$, then $E(f|\pi) = 0$ a.e. and so

$$\int_X fg dm = 0 \; \forall \; g \in L^\infty(\mathcal{T}(R)), \implies f \perp L^2(\mathcal{T}(R)).$$

Thus $L^2(\mathcal{B}) \ominus L^2(\pi^{-1}\mathcal{C}) \subset L^2(\mathcal{B}) \ominus L^2(\mathcal{T}(R))$ whence $L^2(\mathcal{T}(R)) \subset L^2(\pi^{-1}\mathcal{C})$ and $\mathcal{T}(R) \subset \pi^{-1}\mathcal{C} \mod m$.

To see that in fact $\mathcal{T}(R) \subseteq \pi^{-1}\mathcal{T}(S) \mod m$, fix $N \geq 1$, then

$$\mathcal{T}(R) = \bigcap_{n\geq 1} R^{-n}\mathcal{B} = \bigcap_{n\geq N+1} R^{-n}\mathcal{B}$$

$$= R^{-N}\mathcal{T}(R) \subset R^{-N}\pi^{-1}\mathcal{C} = \pi^{-1}S^{-N}\mathcal{C}.$$

Taking the intersection over N shows the claim. $\square$

Corollary 1.3 ([G], proposition 1).

If S is exact and $\pi : X \to Y$ is relatively exact, then T is exact.

§2 Proof of theorem 1

For a nonsingular transformation $(X, \mathcal{B}, m, R)$, define the *tail relation* of R:

$$\mathfrak{T}(R) := \{(x, y) \in X \times X : \exists\, n \geq 0, \; R^n x = R^n y\}.$$

Evidently $\mathfrak{T}(R)$ is an equivalence relation and if $(X, \mathcal{B}, m)$ is standard, then $\mathfrak{T}(R) \in \mathcal{B}(X \times X)$.

If R is locally invertible, then $\mathfrak{T}(R)$ has countable equivalence classes and is nonsingular in the sense that $m(\mathfrak{T}(R)(A)) = 0 \; \forall \; A \in \mathcal{B}, \; m(A) = 0$ where $\mathfrak{T}(R)(A) := \{y \in X : \exists\, x \in A \; (x, y) \in \mathfrak{T}(R)\}$.

A set $A \in \mathcal{B}(X)$ is *invariant* under the equivalence relation $\mathfrak{T} \in \mathcal{B}(X \times X)$ if $\mathfrak{T}(A) = A$ and the equivalence relation $\mathfrak{T}$ is called *ergodic* if $\mathfrak{T}$-invariant sets have either zero, or full measure.

The collection of invariant sets under $\mathfrak{T}(R)$ is the tail σ-algebra $\mathcal{T}(R)$ (whence the name "tail relation").

In order to prove theorem 1, it suffices to show that $\mathfrak{T}(T_\phi)$ is ergodic.

The tail relation of T_ϕ is given by

$$\mathfrak{T}(T_\phi)$$
$$= \{((x, s), (y, t)) \in (X \times G)^2 : \exists\, n \geq 0, \; T^n x = T^n y, \; s - t = \phi_n(y) - \phi_n(x)\}$$
$$= \{((x, s), (y, t)) \in (X \times G)^2 : (x, y) \in \mathfrak{T}(T), \; \tilde{\phi}(x, y) = s - t\}$$

where $\tilde{\phi} : \mathfrak{T}(T) \to \mathbb{R}^d$ is defined by $\tilde{\phi}(x, y) := \sum_{n=0}^{\infty}(\phi(T^n y) - \phi(T^n x))$.

We prove that $\mathfrak{T}(T_\phi)$ is ergodic by the method of Schmidt (explained in [S]), by showing that $\forall\, t \in \mathbb{R}^d$, U a neighbourhood of t and $A \in \mathcal{B}\, m(A) > 0$, $\exists\, B \in \mathcal{B}\, B \subset A$ and $\tau : B \to B$ nonsingular such that $(x, \tau(x)) \in \mathfrak{T}(T)$ and $\tilde{\phi}(x, \tau(x)) \in U \; \forall\, x \in B$.

This boils down to showing that

$$\forall\, A \in \mathcal{B}_+ \; g_0 \in \mathbb{R}^d \; \eta > 0, \; \exists\, B \in \mathcal{B}_+ \; B \subset A, \; n \geq 1$$
$$\text{and } \tau : B \to \tau B \subset A \text{ nonsingular such that}$$
$$(\ddagger) \qquad T^n \circ \tau \equiv T^n \text{ and } \|\phi_n \circ \tau - \phi_n - g_0\| < \eta \text{ on } B.$$

The proof of $(\ddagger)$ will be written as a sequence of minor claims, ¶0, ¶1, ...

¶0 We first claim that there is no loss in generality in assuming that $N = 1$ (i.e. that $\phi : X \to \mathbb{R}^d$ is α-measurable). This is because $(X, \mathcal{B}, m, T, \alpha_0^{N-1})$ is also a probability preserving Markov map with the Renyi property and inducing the same (shift) topology on X as $(X, \mathcal{B}, m, T, \alpha)$.

¶1 $\forall\, s, t \in S$, $\exists\, \kappa = \kappa_{s,t} \geq 1$ and $a = a_{s,t} = [a_1, \ldots a_\kappa]$, $b = b_{s,t} = [b_1, \ldots b_\kappa] \in \alpha_0^{\kappa-1}$, $a_1 = b_1 = s \; a_\kappa = b_\kappa = t$ such that $\|\phi_\kappa(b) - \phi_\kappa(a) - g_0\| < \eta$.

This follows from topological mixing of T_ϕ.

By the Renyi property, $\exists\, M > 1$ such that

$$M^{-1} m(u) m(v) \leq m(u \cap T^{-k} v) \leq M m(u) m(v) \; \forall\, u \in \alpha_0^{k-1}, \; v \in \alpha_0^{\ell-1}, \; [v_1] \subset T[u_k].$$

Given $u = [u_1, \ldots, u_n] \in \alpha_0^{n-1}$ with $u_n = t$, define $\tau = \tau_u : u \cap T^{-n}a \to u \cap T^{-n}b$ by

$$\tau(u_1, \ldots, u_n, a_1, \ldots a_\kappa, y) := \tau(u_1, \ldots, u_n, b_1, \ldots b_\kappa, y).$$

¶2 $\tau = \tau_u : u \cap T^{-n}a \to u \cap T^{-n}b$ is invertible nonsingular and $\frac{dm \circ \tau}{dm} = M^{\pm 4}\frac{m(b)}{m(a)}$.

PROOF

$$\int_{u \cap T^{-n}a \cap c} \frac{dm \circ \tau}{dm} dm = m(u \cap T^{-n}b \cap c)$$

$$= M^{\pm 2}\frac{m(b)}{m(a)}m(u)m(b)m(c)$$

$$= M^{\pm 4}\frac{m(b)}{m(a)}m(u \cap T^{-n}a \cap c).$$

□

¶3 PROOF OF ‡

Fix $0 < \epsilon < M^{-1}\min\{m(a_{s,t}), m(b_{s,t})\}$, then

$$m(u \cap T^{-n}a_{s,t}), \ m(u \cap T^{-n}b_{s,t}) \geq \epsilon m(u) \ \forall \ u \in \alpha_0^{n-1}, \ [s] \subset T[u_n].$$

Let $\delta > 0$ be so small that $\delta < \frac{m(b)(\epsilon-\delta)}{M^4 m(a)}$.

$\exists \ n \geq 1$ and $u \in \alpha_0^{n-1}$ such that $m(A \cap u) \geq (1 - \delta)m(u)$ and $[s] \subset T[u_n]$.

Consider $\tau_u : u \cap T^{-n}a \to u \cap T^{-n}b$ as in ¶2. Evidently $T^{n+\kappa} \circ \tau \equiv T^{n+\kappa}$ and $\|\phi_{n+\kappa} \circ \tau - \phi_{n+\kappa} - g_0\| < \eta$ on $u \cap T^{-n}a$.

To complete the proof we claim that $\exists \ B \in \mathcal{B}_+ \ B \subset A \cap u \cap T^{-n}a$ such that $\tau B \subset A$.

To see this, note that

$$m(u \cap T^{-n}a \cap A) \geq m(u \cap T^{-n}a) - m(u \setminus A) \geq (\epsilon - \delta)m(u),$$

whence using ¶2,

$$m(\tau(u \cap T^{-n}a \cap A)) \geq \frac{m(b)}{M^4 m(a)}m(u \cap T^{-n}a \cap A) \geq \frac{m(b)(\epsilon - \delta)}{M^4 m(a)}m(u).$$

Since $\tau(u \cap T^{-n}a \cap A) \subset u$, the condition on $\delta > 0$ ensures that $m(\tau(u \cap T^{-n}a \cap A) \cap A) > 0$ whence $m(B) > 0$ where $B := \tau^{-1}\left(\tau(u \cap T^{-n}a \cap A) \cap A\right) \subset A.$ □

§3 PROOF OF THEOREM 2

We prove theorem 2 via corollary 1.3. To do this, we must consider T_ϕ as a nonsingular transformation with respect to some probability $P \sim m \times m_{\mathbb{R}^d}$.

Let $p : \mathbb{R}^d \to \mathbb{R}_+$ be continuous with $\int_{\mathbb{R}^d} p(y)dy = 1$ and define a probability P on $X \times \mathbb{R}^d$ by $dP(x,y) := p(y)dm(x)dy$; then $(X \times \mathbb{R}^d, \mathcal{B}(X \times \mathbb{R}^d), P, T_\phi)$ is a nonsingular transformation with Frobenius-Perron operators given by

$$P_{T_\phi^n, P} f(x, y) = \frac{1}{p(y)} P_{T_\phi^n}(f \cdot 1 \otimes p)(x, y)$$

where $P_{T_\phi^n} := P_{T_\phi^n, m \times m_{\mathbb{R}^d}}$.

Consider the map $\pi : X \times \mathbb{R}^d \to X$ defined by $\pi(x, y) = x$. This is a factor map as it satisfies $\pi^{-1} \mathcal{B}(X) \subset \mathcal{B}(X \times \mathbb{R}^d)$, $\pi \circ T_\phi = T \circ \pi$, $P \circ \pi^{-1} = m$.

The fibre expectation of π is given by

$$E(f|\pi)(x) = \int_{\mathbb{R}^d} f(x, y) p(y) dy \quad (f \in L^1(X \times \mathbb{R}^d, \mathcal{B}(X \times \mathbb{R}^d), P)).$$

By corollary 1.3 and exactness of T, it suffices to show that π is relatively exact. To do this, we show that

$$\int_{\mathbb{R}^d} f(x, y) p(y) dy = 0 \text{ a.e.} \implies$$

$$\int_{X \times \mathbb{R}^d} |P_{T_\phi^n, P} f| dP = \int_{X \times \mathbb{R}^d} |P_{T_\phi^n}(f \cdot 1 \otimes p)| d(m \times m_{\mathbb{R}^d}) \to 0$$

as $n \to \infty$; equivalently (taking $F(x, y) := f(x, y) p(y)$),

$$(\star) \qquad \int_{\mathbb{R}^d} F(x, y) dy = 0 \text{ a.e.} \implies \int_{X \times \mathbb{R}^d} |P_{T_\phi^n} F| d(m \times m_{\mathbb{R}^d}) \to 0$$

as $n \to \infty$.

To prove $(\star)$, we first claim that
¶1 for $\lambda > 1$, $h \in L^1(m)$ and $f \in L^1(\mathbb{R}^d)$,

$$\|P_{T_\phi^{n_k}}(h \otimes f)\|_1 \leq C \lambda^{\frac{n_k d}{2}} \|P_{T_\phi^{n_k}}(h \otimes f)\|_2 + o(1)$$

as $k \to \infty$ where $C = 2^{\frac{d}{2}} m(B(0, 1))$ and $\frac{\phi_{n_k}}{\lambda^{n_k}} \to 0$ a.e..
PROOF As can be checked,

$$P_{T_\phi^n}(h \otimes f)(x, y) = P_{T^n}(h(\cdot) f(y - \phi_n(\cdot)))(x) \quad (h \in L^1(m), \ f \in L^1(\mathbb{R}^d)).$$

Denoting $E(H) := \int_X H dm$ for $H \in L^1(m)$, we have

$$(2) \quad \|P_{T^{n_k}}(h \otimes f)\|_1 = \int_{\mathbb{R}^d} |E(P_{T^{n_k}}(h(\cdot) f(y - \phi_{n_k}(\cdot))))| dy \leq \int_{|y| \leq 2\lambda^{n_k}} + \int_{|y| > 2\lambda^{n_k}} .$$

By the Cauchy-Schwartz inequality,

$$(3) \quad \int_{|y| \leq 2\lambda^{n_k}} \leq \sqrt{m_{\mathbb{R}^d}(B(0, 2\lambda^{n_k}))} \|P_{T_\phi^{n_k}}(h \otimes f)\|_2 = C \lambda^{\frac{n_k d}{2}} \|P_{T_\phi^{n_k}}(h \otimes f)\|_2$$

whereas

$$\int_{|y|>2\lambda^{n_k}} \leq \int_{|y|>2\lambda^{n_k}} |E(P_{T^{n_k}}(h(\cdot)f(y-\phi_{n_k}(\cdot))1_{[|\phi_{n_k}(\cdot)|\leq\lambda^{n_k}]})|dy$$

$$+ \int_{|y|>2\lambda^{n_k}} |E(P_{T^{n_k}}(h(\cdot)f(y-\phi_{n_k}(\cdot))1_{[|\phi_{n_k}(\cdot)|>\lambda^{n_k}]}))|dy = I + II.$$

Here as $k \to \infty$:

(4)
$$II \leq \|f\|_1 E(|h|1_{[|\phi_{n_k}(\cdot)|>\lambda^{n_k}]}) \to 0$$

since $\frac{\phi_{n_k}}{\lambda^{n_k}} \to 0$ a.e.; and

$$I \leq \int_{|y|>2\lambda^{n_k}} E(|h||f(y-\phi_{n_k})|1_{[|\phi_{n_k}(\cdot)|\leq\lambda^{n_k}]})dy$$

$$= E\left(|h|1_{[|\phi_{n_k}|\leq\lambda^{n_k}]} \int_{|y|>2\lambda^{n_k}} |f(y-\phi_{n_k})|dy\right)$$

(5)
$$\leq E(|h|) \int_{|y|>\lambda^{n_k}} |f(y)|dy \to 0,$$

Substituting (3),(4) and (5) into (2) proves ¶1. □

To complete the proof of $(\star)$, let $F \in L^1(m \times m_{\mathbb{R}^d})$ satisfy $\int_{\mathbb{R}^d} F(x,y)dy = 0$ for m-a.e. $x \in X$ and fix $\epsilon > 0$. We show that

$(\star_\epsilon)$
$$\limsup_{n\to\infty} \int_{X\times\mathbb{R}^d} |P_{T_\phi^n} F|d(m \times m_{\mathbb{R}^d}) < \epsilon.$$

Standard approximation techniques show that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, $h_1, \ldots, h_N \in L$, $g_1, \ldots, g_N \in L^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} g_k(y)dy = 0$ $(1 \leq k \leq N)$ and

$$\left\|F - \sum_{k=1}^N h_k \otimes g_k\right\|_{L^1(m\times m_{\mathbb{R}^k})} < \frac{\epsilon}{2}.$$

Next, it follows from theorems 1.6.3 and 1.6.4 in [Rud] that
$\exists f_1, \ldots, f_N \in L^1 \cap L^2$ such that
- $[\hat{f}_k \neq 0]$ is compact and bounded away from 0 $(1 \leq k \leq N)$; and
- $\|f_k - g_k\|_{L^1(m_{\mathbb{R}^d})} < \frac{\epsilon}{2N\|h_k\|_{L^1(m)}}$ $(1 \leq k \leq N)$, whence

$$\left\|\sum_{k=1}^N h_k \otimes f_k - \sum_{k=1}^N h_k \otimes g_k\right\|_{L^1(m\times m_{\mathbb{R}^d})} \leq \sum_{k=1}^N \|h_k\|_{L^1(m)} \cdot \|f_k - g_k\|_{L^1(\mathbb{R}^d)} < \frac{\epsilon}{2},$$

$$\left\|F - \sum_{k=1}^N h_k \otimes f_k\right\|_{L^1(m\times m_{\mathbb{R}^d})} < \epsilon$$

where $h \in L$ and $f \in L^1 \cap L^2$ is such that $[\hat{f} \neq 0]$ is compact and bounded away from 0.

We claim

¶2 If $h \in L$ and $f \in L^1 \cap L^2$ is such that $[\hat{f} \neq 0]$ is compact and bounded away from 0, then $\exists\ 0 < \rho < 1$ such that

$$(6) \qquad \|P_{T_\phi^n}(h \otimes f)\|_2 = O(\rho^n) \text{ as } n \to \infty.$$

PROOF

Let $[\hat{f} \neq 0] \subset B(0, M) \setminus B(0, \delta)$. By (ii) (above), $\exists\ K > 0,\ 0 < \rho < 1$ such that

$$|P_\gamma^n h(x)| \leq K\rho^n \quad \forall\ x \in X,\ n \geq 1,\ \delta \leq |\gamma| \leq M,$$

whence using the fact that the Fourier transform of $y \mapsto P_{T_\phi}^n(h \otimes f)(x, y)$ is $\gamma \mapsto \hat{f}(\gamma)P_\gamma^n h(x)$ and Plancherel's formula, we have

$$
\begin{aligned}
\|P_{T_\phi^n}(h \otimes f)\|_2^2 &= \int_X \left(\int_{\mathbb{R}^d} |P_{T_\phi^n}(h \otimes f)(x, y)|^2 dy \right) dm(x) \\
&= \int_X \left(\int_{\mathbb{R}^d} |\hat{f}(\gamma)|^2 |P_\gamma^n h(x)|^2 d\gamma \right) dm(x) \\
&= \int_{\mathbb{R}^d} |\hat{f}(\gamma)|^2 \|P_\gamma^n h\|_2^2 d\gamma \leq K^2 \rho^{2n} \int_{\mathbb{R}^d} |\hat{f}(\gamma)|^2 d\gamma
\end{aligned}
$$

proving ¶2. □

To finish the proof of theorem 2, we claim

¶3 if (6) holds for $h \in L$ and $f \in L^1 \cap L^2$, then

$$(7) \qquad \|P_{T_\phi^n}(h \otimes f)\|_1 \to 0.$$

PROOF

Fix $\lambda > 1$ such that $\lambda^{\frac{d}{2}}\rho < 1$. Suppose that $\frac{\phi_{n_k}}{\lambda^{n_k}} \to 0$ a.e.. Using (6), we have by ¶1,

$$\|P_{T_\phi^{n_k}}(h \otimes f)\|_1 \leq C\lambda^{\frac{n_k d}{2}} \|P_{T_\phi^{n_k}}(h \otimes f)\|_2 + o(1) = O(\lambda^{\frac{n_k d}{2}}\rho^{n_k}) + o(1) \to 0$$

as $k \to \infty$; establishing (7) since $\|P_{T_\phi^n}(h \otimes f)\|_1 \downarrow$. □

This completes the proof of theorem 2.

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