Report 99-048

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ISSN 1389-2355

GROUP EXTENSIONS OF GIBBS-MARKOV MAPS

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ABSTRACT. We show that aperiodic cocycles over an exact Gibbs-Markov map define exact extensions. Equivalent conditions for exactness are found.

§1 INTRODUCTION

Let $(X, \mathcal{B}, m, T, \alpha)$ be an exact probability preserving Markov map as in [A1]. We can and do assume that X is a topological Markov shift:

$$X = \{x \in \alpha^{\mathbb{N}} : \ m(x_n \cap T^{-n}x_{n+1}) > 0 \ \forall \ n \geq 1\}$$

endowed with the Polish topology inherited from the product topology on $\alpha^{\mathbb{N}}$.

Then T is *locally invertible* with respect to α in the sense that for each $n \geq 1$, $a \in \alpha_0^{n-1}$ the map $T^n : a \to T^n a$ is nonsingular and invertible. The inverse of this map is denoted $v_a : T^n a \to a$ and given by $v_a(x_1, x_2, \ldots) = (a, x_1, x_2, \ldots)$.

The partition α enables definition of a Hölder class of metrics $\{d_r : \ 0 < r < 1\}$ on X:
For $n \geq 1$, define $a_n : X \to \alpha_0^{n-1}$ by $x \in a_n(x) \in \alpha_0^{n-1}$.
For $x, y \in X$ define $t(x, y) := \min\{n \geq 1 : \ a_n(x) \neq a_n(y)\}$ $(\leq \infty)$.
For $r \in (0, 1)$ define $d_r : X \times X \to \mathbb{R}$ by $d_r(x, y) := r^{t(x,y)}$.

It is easily seen that the identity : $(X, d_r) \to (X, d_s)$ is Hölder continuous $\forall \ r, s \in (0, 1)$.

Accordingly, we define the Hölder constants of a function $h : X \to M$ with values in a metric space (M, ρ) by

$$D_{r,X}(h) := \sup_{x,y \in X} \frac{\rho(h(x), h(y))}{r^{t(x,y)}}.$$

Let $L_r(M) := \{h : X \to M : \ \sup_{a \in \alpha} D_{r,a}(h) < \infty\}$. In case $M = \mathbb{R}$ we simply write $L_r := L_r(M)$ instead.

Recall (see e.g. [A-D1]) that $(X, \mathcal{B}, m, T, \alpha)$ has

the *Gibbs property* if $\exists \ C > 1, \ 0 < r < 1$ such that $\forall \ n \geq 1, \ a \in \alpha_0^{n-1}, \ m(a) > 0$:

1991 *Mathematics Subject Classification.* Primary: 28D05, 60B15; Secondary: 58F15, 58F19, 58F30.

Research supported by Eurandom and the Deutsche Forschungsgemeinschaft, Schwerpunkt Ergodentheorie, Analysis und effiziente Simulation dynamischer Systeme.

$\left|\frac{v_a'(x)}{v_a'(y)} - 1\right| \leq Cr^{t(x,y)}$ for $m \times m$-a.e. $(x,y) \in T^n a \times T^n a$.
It is called a *Gibbs-Markov map* if it has in addition the property

$$\inf_{a\in\alpha} m(Ta) > 0.$$

Recall that any topologically mixing probability preserving Markov map with the Gibbs property is exact (see for example [A-D-U]).

Now let G be a LCA, second countable group, let $\|\cdot\|$ be a Lipschitz norm on G (i.e. $\gamma : G \to S^1$ is $\|\cdot\|$-Lipschitz for every $g \in \widehat{G}$), let $\phi : X \to G$ be measurable. Consider the skew product $T_\phi : X \times G \to X \times G$ defined by $T_\phi(x,y) := (Tx, y + \phi(x))$ with respect to the (invariant) product measure $m \times m_G$. We define $\phi_n = \phi + \phi \circ T + ... + \phi \circ T^{n-1}$.

We're interested in the exactness of T_ϕ and prove

Theorem.

Let G be a LCA, second countable group, let $(X, \mathcal{B}, m, T)$ be an exact probability preserving Gibbs-Markov map and let $\phi : X \to G$ be uniformly Hölder continuous on states.

The following are equivalent:

1.) ϕ is aperiodic in the sense that $\gamma \circ \phi = \frac{zgT}{g}$ has no non-trivial solutions in $z \in S^1$ and $g : X \to S^1$ Hölder continuous.

2.) T_ϕ is weakly mixing.

3.) T_ϕ is exact.

4.) For some $A \in \mathcal{B}_+$ and $x \in A$, the smallest closed subgroup generated by

$$\left\{t \in G :\ \exists\ k_n \to \infty\ y_n, z_n \in T^{-k_n}\{x\}, \left\{\begin{array}{l} d_r(y_n, z_n) \to 0 \\ \phi_{k_n}(y_n) - \phi_{k_n}(z_n) \to t \end{array}\right.\right\}$$

is G.

5.) For every $x \in X$,

$$G = \left\{t \in G :\ \exists\ k_n \to \infty\ y_n, z_n \in T^{-k_n}\{x\}, \left\{\begin{array}{l} d_r(y_n, z_n) \to 0 \\ \phi_{k_n}(y_n) - \phi_{k_n}(z_n) \to t \end{array}\right.\right\}.$$

Remarks: In case α is a finite Markov partition and m a Gibbs measure as in [Bo], Guivarc'h ([G]) has obtained exactness of the group extension with respect to Hölder-continuous cocycles. This applies to $\mathbb{Z}^d$-extensions of the geodesic flow on compact surfaces of constant negative curvature (among others). Let $\phi : X \to \mathbb{Z}^d$ (or $\mathbb{R}^d$) be aperiodic, locally Lipschitz and in the domain of attraction of a stable distribution of order $0 < p < 2$. Exactness follows from section 7 in [A-D1] in case T is Gibbs-Markov. The assumptions on the cocycle and the dynamics in these results are rather strong. Weaker sufficient conditions can be found in [A-D2]: Let T be a Markov map with the Renyi property:
$\exists\ C > 1$ such that $\forall\ n \geq 1,\ a \in \alpha_0^{n-1},\ m(a) > 0$:
$\frac{v_a'(x)}{v_a'(y)} \leq C$ for $m \times m$-a.e. $(x,y) \in T^n a \times T^n a$. For these maps it suffices to assume that the cocycle is locally constant (on cylinders in $(\alpha)_0^N$ for some $N \geq 0$).

For locally invertible, exact endomorphisms T with the Renyi property it suffices to assume a spectral representation à la Nagaev ([N]) for the Frobenius-Perron operator and at most exponentially increasing $\phi + ... + \phi \circ T^{n_k}$ $(k = 1, 2, ...)$.

The proof of the theorem is given in the subsequent sections. The only non-trivial implications are 4.) $\Longrightarrow$ 3.) and 1.) $\Longrightarrow$ 5.). Our proofs certainly follows general concepts, like [L-R-W] and [F] for the first implication and [S] for the second. In particular the last section contains a ratio limit theorem of independent interest.

The *Frobenius-Perron operator* $\widehat{R} : L_1(m) \to L_1(m)$ of a nonsingular transformation $(X, \mathcal{B}, m, R)$ is defined by

$$\int_X \widehat{R}f \cdot gdm = \int_X f \cdot g \circ Tdm$$

where $f \in L_1(m)$ and $g \in L_\infty(m)$. For a Gibbs-Markov map T this operator has the form

$$\widehat{T}f = \sum_{a \in \alpha} 1_a(v_a)v_a' f(v_a),$$

and for the group extension T_ϕ

$$\widehat{T}_\phi^n f(x, g) = \widehat{T}^n[f(\cdot, g - \phi_n(\cdot))](x).$$

Fix some $r \in (0, 1)$ and let β denote the coarsest partition such that $T\alpha \subset \sigma(\beta)$. We define the Banach space L of all L_∞ functions $f : X \to \mathbb{R}$ with

$$D_{r,f} = \sup_{b \in \beta} D_{r,b}(f) < \infty.$$

We may assume that r is chosen so large that $D_\phi = D_{r,\phi} < \infty$. It is shown in [A-D1] that $\widehat{T}^n$ $(n \geq 1)$ has a representation

$$\widehat{T}f(x) = \int fdm + O(\rho^n \|f\|_L).$$

Proof of 4.) $\Longrightarrow$ 3.).

It is sufficient to show relative exactness (see [G], [A-D2], i.e. that

$$\int_X \int_G \left| \hat{T}_\phi^n [\Psi \otimes \Gamma](x, g) \right| dgm(dx) \to 0$$

for all $\Psi \in L_1(m)$ and $\Gamma \in L_1(G)$ satisfying $\int_G \Gamma dg = 0$. Moreover, we may and will reduce to those Ψ which are Lipschitz continuous and those Γ which are Lipschitz continuous and have compact support.

Let $\Psi \in L_1(m)$ and $\Gamma \in L_1(G)$. Then

$$\begin{aligned}
U_{n+1}(\Psi \otimes \Gamma) &:= \int_X \int_G \left| \hat{T}_\phi^{n+1}(\Psi \otimes \Gamma)(x, g) \right| dgm(dx) \\
&\leq \int_X \int_G \sum_{T(z)=x} \left| \hat{T}_\phi^n [\Psi \otimes \Gamma](z, g - \phi(z)) \right| p_n(x, z) dgm(dx) \\
&= \int_G \int_X \hat{T}\left[\left| \hat{T}_\phi^n [\Psi \otimes \Gamma](\cdot, g - \phi(\cdot)) \right|\right](x) m(dx) dg \\
&= \int_G \int_X \left| \hat{T}_\phi^n [\Psi \otimes \Gamma](x, g - \phi(x)) \right| dgm(dx) \\
&= \int_X \int_G \left| \hat{T}_\phi^n [\Psi \otimes \Gamma](x, g) \right| dgm(dx) = U_n(\Psi \otimes \Gamma).
\end{aligned}$$

Therefore

$$U_n(\Psi \otimes \Gamma) \downarrow C(\Psi \otimes \Gamma) \geq 0, \tag{1}$$

and it is left to show that $\int_G \Gamma dg = 0 \Longrightarrow C(\Psi \otimes \Gamma) = 0$.

Definition: A sequence of signed measures $\{\mu_n : n \geq 1\}$ on G is called *completely mixing* if for every $\Gamma \in L_1(G)$ with integral $\int_G \Gamma(g)dg = 0$ we have

$$\|\mu_n \star \Gamma\|_{L_1(G)} \to 0.$$

We define the operators $M_t : L_1(G) \to L_1(G)$ by $M_t F(g) = F(g+t)$. Let $\Psi \in L_1(X)$ and let the measures $\{\mu_{n,x} : n \geq 1\}$ on G be defined by

$$\mu_{n,x} = \sum_{T^n(z)=x} \Psi(z) p_n(x,z) \delta_{\phi_n(z))}.$$

We'll show that the measures $\{\mu_{n,x} : n \geq 1\}$ are completely mixing in measure.

Note that $\|\mu_{n,x} \star F\|_{L_1(G)} \leq \widehat{T}^n |\Psi|(x) \, \|F\|_{L_1(G)}$. Therefore $t \mapsto \|\mu_{n,x} \star M_t F\|_{L_1(G)}$ is Lipschitz continuous with Lipschitz constant $\widehat{T}^n |\Psi|(x) \|F - M_t F\|_{L_1(G)}$.

Proposition 1: *For every $\Gamma \in L_1(G)$ the random sequence*

$$\|\mu_{n,\cdot} \star \Gamma\|_{L_1(G)}$$

converges in $L_1(m)$ to $C(\Psi \otimes \Gamma)$. In addition,

$$C(\Psi \otimes \Gamma) \leq \|\Psi\|_{L_1(m)} \|\Gamma\|_{L_1(G)}.$$

Proof. Since $\hat{T}^n_\phi F(x,g) = \hat{T}^n F(\cdot, g - \phi_n(\cdot))(x)$, it suffices to show the theorem for a subclass which generates a dense subspace in $L_1(X \times G)$. Here we take the class of all functions $\Psi \otimes \Gamma$ where Ψ belongs to the space L and Γ is an integrable and Lipschitz continuous function on G.

It also follows from the above that

$$\begin{aligned}
&\mu_{n+1,x} \star \Gamma(g) \\
&= \int_G \Gamma(g-h) \mu_{n+1,x}(dh) = \sum_{T^{n+1}(z)=x} \Psi(z) p_{n+1}(x,z) \Gamma(g - \phi_{n+1}(z)) \\
&= \sum_{T(z)=x} p(x,z) \hat{T}^n_\phi [\Psi \otimes \Gamma](z, g - \phi(z))
\end{aligned}$$

whence as before,

$$\begin{aligned}
&\|\mu_{n+1,x} \star \Gamma\|_{L_1(G)} \\
&\leq \int_G \sum_{T(z)=x} p(x,z) \left| \hat{T}^n_\phi [\Psi \otimes \Gamma](z, g - \phi(z)) \right| dg \\
&= \sum_{T(z)=x} p(z,x) \int_G \left| \hat{T}^n_\phi [\Psi \otimes \Gamma](z, g) \right| dg \\
&= \hat{T} \left[\|\mu_{n,\cdot} \star \Gamma\|_{L_1(G)} \right](x).
\end{aligned}$$

By induction it is easily seen that for n fixed and $k \geq 1$

$$\|\mu_{n+k,x} \star \Gamma\|_{L_1(G)} \leq \hat{T}^k \left[\|\mu_{n,\cdot} \star \Gamma\|_{L_1(G)}\right](x).$$

Since the function

$$x \to \|\mu_{n,x} \star \Gamma\|_{L_1(G)}$$

is of class L it follows that for $k \to \infty$

$$\hat{T}^k \left[\|\mu_{n,\cdot} \star \Gamma\|_{L_1(G)}\right] \to \int_X \|\mu_{n,x} \star \Gamma\|_{L_1(G)} m(dx) \downarrow C(\Psi \otimes \Gamma),$$

whence

$$\limsup_{n\to\infty} \|\mu_{n,x} \star \Gamma\|_{L_1(G)} \leq C(\Psi \otimes \Gamma). \tag{2}$$

By (1) and (2), given $\epsilon > 0$, we can choose n_0 so large that for $n \geq n_0$

$$\int_{\{x:\|\mu_{n,x}\star\Gamma\|_{L_1(G)}-C(\Psi\otimes\Gamma)>0\}} \left[\|\mu_{n,x} \star \Gamma\|_{L_1(G)} - C(\Psi \otimes \Gamma)\right] m(dx) \leq \epsilon^2.$$

Using (1) once again,

$$\begin{aligned}
&m\{x : \|\mu_{n,x} \star \Gamma\|_{L_1(G)} \leq C(\Psi \otimes \Gamma) - \epsilon\} \\
&\leq \frac{1}{\epsilon} \int_{\{x:C(\Psi\otimes\Gamma)-\|\mu_{n,x}\star\Gamma\|_{L_1(G)}\geq\epsilon\}} \left[C(\Psi \otimes \Gamma) - \|\mu_{n,x} \star \Gamma\|_{L_1(G)}\right] m(dx) \\
&= \frac{1}{\epsilon} \left(C(\Psi \otimes \Gamma) - \int_X \|\mu_{n,x} \star \Gamma\|_{L_1(G)} m(dx)\right) \\
&\quad - \frac{1}{\epsilon} \int_{\{x:C(\Psi\otimes\Gamma)-\|\mu_{n,x}\star\Gamma\|_{L_1(G)}<\epsilon\}} \left[C(\Psi \otimes \Gamma) - \|\mu_{n,x} \star \Gamma\|_{L_1(G)}\right] m(dx) \\
&\leq \frac{1}{\epsilon} \int_{\{x:\|\mu_{n,x}\star\Gamma\|_{L_1(G)}-C(\Psi\otimes\Gamma)>0\}} \left[\|\mu_{n,x} \star \Gamma\|_{L_1(G)} - C(\Psi \otimes \Gamma)\right] m(dx) < \epsilon.
\end{aligned}$$

The proposition follows easily. The additional claim follows from

$$C(\Psi \otimes \Gamma) \leftarrow \|\mu_{n,x} \star \Gamma\|_{L_1(G)} \leq \widehat{T}^n |\Psi|(x) \|\Gamma\|_{L_1(G)} \to \|\Psi\|_{L_1(m)} \|\Gamma\|_{L_1(G)}.$$

In order to show the theorem it is left to prove the following

Lemma 2: *If* $\int_X \int_G \Psi(x)\Gamma(g) dg m(dx) = 0$, *then*

$$C(\Psi \otimes \Gamma) = 0.$$

Proof. The proof of this statement follows from a series of facts: Define the measures $\nu_{n,x} = \sum_{T^n(z)=x} p_n(x,z)\delta_z$ on X.

Claim 1: *Let $k \geq 0$ be fixed. We first claim that for any subsequence $\{n_l : l \in \mathbb{N}\} \subset \mathbb{N}$ there exists a further subsequence $\{m_j : j \geq 1\}$ such that for a.e. $x \in X$ and for every $B \in \mathcal{B}$*

$$\lim_{j\to\infty} \frac{1}{\nu_{k,x}(B)} \int_G \left| \int_B \left(\mu_{m_j,y} \star M_{\phi_k(y)}\Gamma\right)(g)\nu_{k,x}(dy)\right| dg = C(\Psi\otimes\Gamma). \tag{3}$$

In order to see this claim, let n_l be any subsequence and choose m_j so that

$$\|\mu_{m_j,x}\star\Gamma\|_{L_1(G)}, \|\mu_{m_j+k,x}\star\Gamma\|_{L_1(G)} \to C(\Psi\otimes\Gamma) \tag{4}$$

for $x \in \Omega$ where Ω is a T-invariant set of full measure. On the one hand it follows from this that for every B fixed

$$\begin{aligned} &\frac{1}{\nu_{k,x}(B)} \int_G \left| \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma\nu_{k,x}(dy)\right| dg \\ &\leq \frac{1}{\nu_{k,x}(B)} \int_B \|\mu_{m_j,y}\star\Gamma\|_{L_1(G)}\nu_{k,x}(dy) \to C(\Psi\otimes\Gamma), \end{aligned}\tag{5}$$

because the integrand is uniformly bounded and pointwise convergent by (4).

On the other hand, for $x \in \Omega$,

$$\begin{aligned} C(\Psi\otimes\Gamma) &= \lim_{j\to\infty} \|\mu_{m_j+k,x}\star\Gamma\|_{L_1(G)} \\ &= \lim_{j\to\infty} \int_G \left| \sum_{T^k(y)=x} p_k(x,y)\hat{T}_\phi^{m_j}[\Psi\otimes\Gamma](y, g-\phi_k(y))\right| dg \\ &\leq \lim_{j\to\infty} \int_G \left|\int_B \mu_{m_j,y}\star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy)\right| + \left|\int_{B^c} \mu_{m_j,y}\star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy)\right| dg \\ &\leq C(\Psi\otimes\Gamma) \end{aligned}$$

by (5), hence for $x \in \Omega$

$$\lim_{j\to\infty} \frac{1}{\nu_{k,x}(B)} \int_G \left| \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma\nu_{k,x}(dy)\right| dg = C(\Psi\otimes\Gamma),$$

proving claim 1.

Claim 2: *For any subsequence $\{n_l : l \in \mathbb{N}\} \subset \mathbb{N}$ there exists a further subsequence $\{m_j : j \geq 1\}$ such that for a.e. $x \in X$ every disjoint sets $A, B \in \mathcal{B}$*

$$\begin{aligned} \lim_{j\to\infty} \int_G \Bigg| &\frac{1}{\nu_{k,x}(A)} \int_A \mu_{m_j,y}\star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \\ &+ \frac{1}{\nu_{k,x}(B)} \int_B \mu_{m_j,y}\star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy)\Bigg| dg = 2C(\Psi\otimes\Gamma) \end{aligned}\tag{6}$$

Choose the subsequence and Ω as in (4). Then for $x \in \Omega$ by (3)

$$\begin{aligned}
&\int_G \left| \frac{1}{\nu_{k,x}(A)} \int_A \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma \nu_{k,x}(dy) + \frac{1}{\nu_{k,x}(B)} \int_B \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma \nu_{k,x}(dy) \right| dg \\
&\qquad \leq \frac{1}{\nu_{k,x}(A)} \int_G \left| \int_A \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma(g) \nu_{k,x}(dy) \right| dg \\
&\qquad + \frac{1}{\nu_{k,x}(B)} \int_G \left| \int_B \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma(g) \nu_{k,x}(dy) \right| dg \\
&\qquad \to 2C(\Psi \otimes \Gamma) \qquad (7)
\end{aligned}$$

and, since $A \cap B = \emptyset$ (and w.l.o.g. assume that $\nu_{k,x}(A) \leq \nu_{k,x}(B)$),

$$\begin{aligned}
&\frac{1}{\nu_{k,x}(A)} \int_G \left| \int_A \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma \nu_{k,x}(dy) + \frac{\nu_{k,x}(A)}{\nu_{k,x}(B)} \int_B \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma \nu_{k,x}(dy) \right| dg \\
&\qquad \geq \frac{1}{\nu_{k,x}(A)} \left(\int_G \left| \int_{A \cup B} \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma(g) \nu_{k,x}(dy) \right| dg \right. \\
&\qquad \left. - \left(1 - \frac{\nu_{k,x}(A)}{\nu_{k,x}(B)}\right) \int_G \left| \int_B \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma(g) \nu_{k,x}(dy) \right| dg \right) \\
&\qquad \to 2C(\Psi \otimes \Gamma). \qquad (8)
\end{aligned}$$

Claim 2 follows from (7) and (8).

Claim 3: *Let $A, B \in \alpha_0^{k-1}$ be images of inverse branches v_A and v_B of T^k, where k is still fixed. Let $\epsilon = d(A,B)$ and let Γ be Lipschitz continuous with compact support K. Then for every $n \geq 1$*

$$\begin{aligned}
&\int_G \left| \mu_{n,v_A(x)} \star M_{\phi_k(v_A(x))} \Gamma - \mu_{n,v_B(x)} \star M_{\phi_k(v_A(x))} \Gamma \right| dg \\
&\leq C_1 \|\Gamma\|_{L_1(G)} \epsilon + D_\Gamma C_0 D_\phi |B(K, C_0 D_\phi \epsilon)| \epsilon,
\end{aligned}$$

where $|\cdot|$ denotes Haar measure on G.

Let $x \in X$, $v = v_A(x)$ and $w = v_B(x)$. By the Lipschitz property of ϕ by the expanding property of T, we have for any inverse branch $v_a : A \cup B \to a \in (\alpha)_0^{n-1}$ of T^n that

$$\begin{aligned}
|\phi_n(v_a(v)) - \phi_n(v_a(w))| &\leq D_\phi \sum_{l=0}^{n-1} d(T^l(v_a(v)), T^l(v_a(w))) \\
&\leq C_0 D_\phi d(v,w) \leq C_0 D_\phi \epsilon.
\end{aligned}$$

Since Γ has compact support

$$\|\Gamma(g) - \Gamma(g + \phi_n(v_a(v)) - \phi_n(v_a(w)))\| \leq D_\Gamma C_0 D_\phi \epsilon 1_{B(K, C_0 D_\phi \epsilon)}(g).$$

Similarly, there exists a constant C_1 (also depending on the Lipschitz constant of Ψ) so that

$$|p_n(v, v_a(v)) \Psi(v_a(v)) - p_n(w, v_a(w)) \Psi(v_a(w))| \leq C_1 p_n(v, v_a(v)) d(v,w).$$

Therefore

$$\begin{aligned}
&\int_G \left|\mu_{n,v_A(x)} \star M_{\phi_k(v_A(x))}\Gamma(g) - \mu_{n,v_B(x)} \star M_{\phi_k(v_A(x))}\Gamma(g)\right| dg \\
&= \int_G \Bigg|\sum_a p_n(v, v_a(v))\Psi(v_a(v))\Gamma(g - \phi_k(v) - \phi_n(v_a(v))) \\
&\quad - \sum_a p_n(w, v_a(w))\Psi(v_a(w))\Gamma(g - \phi_k(v) - \phi_n(v_a(w)))\Bigg| dg \\
&\le \int_G \Bigg|\sum_a [p_n(v, v_a(v))\Psi(v_a(v)) - p_n(w, v_a(w))\Psi(v_a(w))] \\
&\qquad \Gamma(g - \phi_k(v) - \phi_n(v_a(v)))\Bigg| dg \\
&\quad + \int_G \Bigg|\sum_a p_n(w, v_a(w))\Psi(v_a(w)) \\
&\qquad [\Gamma(g - \phi_k(v) - \phi_n(v_a(v))) - \Gamma(g - \phi_k(v) - \phi_n(v_a(w)))]\Bigg| dg \\
&\le \left(C_1\|\Gamma\|_{L_1(G)} + D_\Gamma C_0 D_\phi |B(K, C_0 D_\phi \epsilon)|\right) \|\hat{T}^n 1\|_\infty \epsilon,
\end{aligned}$$

where $\sum_a$ extends over all $a \in (\alpha)_0^{n-1} : T^n a \supset A \cup B$.

Claim 4: *There exists a set Ω of measure 1 and a constant $C > 0$ with the following property:*
If $x \in \Omega$, $k \ge 1$ and $v, w \in T^{-k}(\{x\})$ then

$$\left|2C(\Psi \otimes \Gamma) - C(\Psi \otimes (I + M_{\phi_k(v)-\phi_k(w)})\Gamma)\right| < Cd(v, w). \tag{9}$$

By claims 1–3 there exists a subsequence $\{m_j : j \ge 1\} \subset \mathbb{N}$ and a subset Ω so that (3), (6) and (9) hold for any $x \in \Omega$, $k \ge 1$ and $v, w \in T^{-k}(\{x\})$. Therefore

$$\begin{aligned}
&\int_G \Bigg|\frac{1}{\nu_{k,x}(\{v\})} \int_{\{v\}} \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \\
&\quad + \frac{1}{\nu_{k,x}(\{w\})} \int_{\{w\}} \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy)\Bigg| dg \\
&= \int_G \left|\mu_{m_j,v} \star M_{\phi_k(v)}\Gamma(g) + \mu_{m_j,w} \star M_{\phi_k(w)}\Gamma(g)\right| dg \\
&\le \int_G \left|\mu_{m_j,v} \star M_{\phi_k(v)}\Gamma(g) - \mu_{m_j,w} \star M_{\phi_k(v)}\Gamma(g)\right| dg \\
&\quad + \int_G \left|\mu_{m_j,w} \star M_{\phi_k(w)}\Gamma(g) + \mu_{m_j,w} \star M_{\phi_k(v)}\Gamma(g)\right| dg \\
&\le \int_G \left|\mu_{m_j,w} \star (I + M_{\phi_k(v)-\phi_k(w)})\Gamma(g)\right| dg + Cd(v, w),
\end{aligned}$$

where $C = C_1\|\Gamma\|_{L_1(G)} + D_\Gamma C_0 D_\phi |B(K, C_0 D_\phi)$. The lower bound is shown in the same way, proving claim 4.

Claim 5: *Let $\Psi \in L$ and $\Gamma \in L_1(G)$. Then*

$$C(\Psi \otimes (\Gamma - M_t\Gamma)) = 0.$$

First observe that the set of $t \in G$ satisfying the claim is a group. In fact, the claim holds for the identity in G, and by proposition 1

$$\begin{aligned}
&C(\Psi \otimes (I - M_{t+s})\Gamma) \\
&= \lim_{n\to\infty} \int_X \int_G \left| \int_G (I - M_{t+s})\Gamma(g-h)\mu_{n,x}(dh) \right| dgm(dx) \\
&\le \lim_{n\to\infty} \int_X \int_G \left| \int_G (I - M_t)\Gamma(g-h)\mu_{n,x}(dh) \right| dgm(dx) \\
&+ \lim_{n\to\infty} \int_X \int_G \left| \int_G (I - M_t)M_s\Gamma(g-h)\mu_{n,x}(dh) \right| dgm(dx) \\
&= 0.
\end{aligned}$$

Hence it suffices to prove the claim for t in a generating set G_0. Moreover, it suffices to prove the claim for Γ Lipschitz continuous with compact support, since $C(\Psi \otimes \Gamma)$ is $L_1(G)$-norm continuous.

By assumption, and by claim 4 there is a measurable set $\Omega \in \mathcal{B}$ of full measure, a constant $C > 0$ and a subset $G_0 \subset G$ generating G such that for all $x \in \Omega$ and $v, w \in T^{-k}(x)$

$$\left|2C(\Psi \otimes \Gamma) - C(\Psi \otimes (I + M_{\phi_k(v)-\phi_k(w)})\Gamma)\right| < Cd(v,w), \tag{9}$$

$$\begin{aligned}
&\forall t \in G_0 \ \exists x_n \in \Omega, k_n \ge 1, v_n, w_n \in T^{-k_n}(x_n) \\
&\ni \phi_{k_n}(v_n) - \phi_{k_n}(w_n) \to t \ \& \ d(v_n, w_n) \to 0.
\end{aligned} \tag{10}$$

Since $t \to C(\Psi \otimes M_t\Gamma)$ is continuous, it follows from these properties that

$$2C(\Psi \otimes \Gamma) = C(\Psi \otimes (I + M_t)\Gamma) \quad (t \in G_0).$$

Because of continuity, this equation holds for any $\Gamma \in L_1(G)$. Hence, replacing Γ by $(I - M_t)\Gamma$ and repeating this argument for each $(I + M_t)^k(I - M_t)\Gamma$, $k \ge 0$, we obtain

$$C(\Psi \otimes (I - M_t)\Gamma) = 2^{-k}C(\Psi \otimes (I + M_t)^k(I - M_t)\Gamma)$$

for every $k \ge 0$ and $t \in G_0$.

It suffices to show the claim for $\Gamma \geq 0$. For even k

$$\begin{aligned}
&C(\Psi \otimes \left(\frac{I+M_t}{2}\right)^k (I-M_t)\Gamma) \\
&\leq \int_X |\Psi| dm \int_G \left| \left(\frac{I+M_t}{2}\right)^k (I-M_t)\Gamma(g) \right| dg \\
&= \|\Psi\|_{L_1(X)} \int_G 2^{-k} \left| I - M_t^{k+1} + \sum_{j=1}^{k} \left(\binom{k}{j} - \binom{k}{j-1} \right) M_t^j \right| \Gamma(g) dg \\
&\leq 2^{-k} \|\Psi\|_{L_1(X)} \int_G \left(I + M_t^{k+1} + \sum_{j=1}^{k/2} \left(\binom{k}{j} - \binom{k}{j-1} \right) M_t^j \right. \\
&\quad \left. + \sum_{j=k/2+1}^{k} \left(\binom{k}{j-1} - \binom{k}{j} \right) M_t^j \right) \Gamma(g) dg \\
&\leq 2^{-k+1} \|\Psi\|_{L_1(X)} \|\Gamma\|_{L_1(G)} \left(1 + \binom{k}{k/2} \right).
\end{aligned}$$

Claim 6:

$$\int_G \Gamma(g) dg = 0 \Longrightarrow C(\Psi \otimes \Gamma) = 0.$$

This fact is well known from standard arguments of ergodic transformations: Indeed, as it is well known,

$$\overline{\bigcup_{t \in G} (I - M_t) L_1(G)} = \{ f \in L_1(G) : \int f(g) dg = 0 \}.$$

Proof of 1.) $\Longrightarrow$ 5.)

Ratio limit theorem for symmetric cocycles.

Suppose that $\phi : X \to G$ is Hölder continuous, aperiodic and symmetric in the sense that there exists a probability preserving transformation $S : X \to X$ such that $ST = TS$ and $\phi \circ S = -\phi$, then there exists $u_n > 0$ such that

$$\frac{P_{T_\phi^n}(h \otimes f)(x, y)}{u_n} \to \int_{X \times G} h \otimes f \quad \forall\, h \in L,\ f \in C_c(G),\ x \in X,\ y \in G.$$

Proof.

First let (as in[A-D1]) $P_\gamma : L \to L \quad (\gamma \in \widehat{G})$ be defined by

$$P_\gamma h := P_T(\gamma \circ \phi \cdot h).$$

As shown in [A-D1], $\gamma \mapsto P_\gamma$ is continuous ($\widehat{G} \to \text{Hom}(L, L)$), and $\exists\ \epsilon > 0,\ 0 \le \theta < 1$ and continuous functions

$$\lambda : B_{\widehat{G}}(0, \epsilon) \to B_{\mathbb{C}}(0, 1),\ g : B_{\widehat{G}}(0, \epsilon) \to L$$

continuous, such that $\lambda(0) = 1,\ g(0) \equiv 1,\ \int_X g(\gamma) dm \equiv 1,$

$$|\lambda(\gamma)| \le 1 \text{ with equality iff } \gamma = 0,$$

$$P_\gamma h = \lambda h \implies |\lambda| \le |\lambda(\gamma)| \quad (\gamma \in B_{\widehat{G}}(0, \epsilon)),$$
$$P_\gamma h = \lambda(\gamma) h \iff h \in \mathbb{R} \cdot g(\gamma) \quad (\gamma \in B_{\widehat{G}}(0, \epsilon)),$$

and

$$g(-\gamma) = \overline{g}(\gamma),\ \lambda(-\gamma) = \overline{\lambda}(\gamma).$$

Noting that $S^{-1} \circ P_\gamma \circ S = P_{-\gamma}$, we see that

$$g(-\gamma) = g(\gamma) \circ S,\ \lambda(\gamma) \in \mathbb{R}.$$

Next, for $0 < \eta \le \epsilon$ set $u_n(\eta) := \int_{B(0,\eta)} \lambda(\gamma)^n d\gamma$. For η small enough (so that $\lambda > 0$ on $B(0, \eta)$), $u_n(\eta) > 0$. Choose one such $\eta_0 > 0$ and define $u_n := u_n(\eta_0)$. Note that $\rho^n = o(u_n)\ \forall\ \rho < 1$ since $\exists\ \eta < \eta_0$ such that $\min_{|\gamma| < \eta} |\lambda(\gamma)| = r > \rho$ whence

$$\frac{u_n}{\rho^n} \ge \frac{u_n(\eta)}{\rho^n} \ge \frac{r^n}{\rho^n} \cdot m(B(0, \eta)) \to \infty.$$

Also, for $0 < \eta < \eta'$,

$$u_n(\eta) = u_n(\eta') \pm O(\rho(\eta)^n)$$

where $\rho(\eta) := \sup_{\eta \le |\gamma| \le \epsilon} |\lambda(\gamma)| < 1$. Thus

$$u_n(\eta) \sim u_n \text{ as } n \to \infty\ \forall\ 0 < \eta \le \epsilon.$$

Now fix $h \in L$ and $f \in L^1(G)$ with $\hat{f} \in C_c(\widehat{G})$, then $\forall\ x \in X,\ y \in G$,

$$P_{T_\phi^n}(h \otimes f)(x, y) = \int_{\widehat{G}} \hat{f}(\gamma) \overline{\gamma}(y) P_\gamma^n h(x) d\gamma$$
$$= \int_X h dm \int_{B(0,\eta_0)} \hat{f}(\gamma) \lambda(\gamma)^n \Re(\overline{\gamma}(y) g(\gamma)(x)) d\gamma + O(\theta^n)$$

(by reality, for some $0 < \theta < 1$). Since $\hat{f}(\gamma) \Re(\overline{\gamma}(y) g(\gamma)(x)) \to 1$ as $\gamma \to 0$, it follows that

$$P_{T_\phi^n}(h \otimes f)(x, y) \sim u_n \int_X h dm \int_G f dm_G.$$

By the method of Breiman ([Brei]),

$$P_{T_\phi^n}(h \otimes f)(x, y) \sim u_n \int_X h dm \int_G f dm_G\ \forall\ h \in L, f \in C_c(G).$$

Corollary.

Under the same assumptions, $\forall\ x, y \in X,\ t \in G,\ \epsilon > 0,\ \exists n_0$ *such that* $\forall\ n \geq n_0\ \exists\ z \in T^{-n}\{x\}$ *such that* $d(y,z) < \epsilon$ *and* $\|t - \phi_n(z)\| < \epsilon$.

Proof.

Let $a = [a_1, \ldots, a_N] = B(y, \epsilon),\ h = 1_a \in L$ and let $f \in C(G),\ f \geq 0,\ [f > 0] \subset B(0, \epsilon)$. Then

$$\frac{P_{T_\phi^n}(h \otimes f)(x,t)}{u_n} \to \int_{X \times \mathbb{G}} h \otimes f dm \times m_G$$

and $\exists\ n_0$ such that $\forall\ n \geq n_0$,

$$0 < P_{T_\phi^n}(h \otimes f)(x,t) = \sum_{T^n z = x,\ d(y,z) < \epsilon} p_{x,n}(z) g(t - \phi_n(z))$$

and in particular $\exists\ \exists\ z \in T^{-n}\{x\}$ such that $d(y,z) < \epsilon$ and $\|t - \phi_n(z)\| < \epsilon$.

Exactness lemma.

Suppose that $\phi : X \to G$ *is Hölder continuous, aperiodic, then* $\forall\ x \in X,\ t \in G,\ \epsilon > 0,\ \exists n_0$ *such that* $\forall\ n \geq n_0\ \exists\ y, z \in T^{-n}\{x\}$ *such that* $d(y,z) < \epsilon$ *and* $\|t + \phi_n(y) - \phi_n(z)\| < \epsilon$.

Proof.

Consider the mixing Gibbs-Markov map $(X \times X, \mathcal{B}(X \times X), T \times T, m \times m, \alpha \times \alpha)$ equipped with the cocycle $\tilde{\phi} : X \times X \to G$ defined by $\tilde{\phi}(x, x') := \phi(x) - \phi(x')$.

The cocycle $\tilde{\phi} : X \times X \to G$ is also Hölder continuous, aperiodic, but also symmetric: $\tilde{\phi} \circ S = -\tilde{\phi}$ where $S(x, x') := (x', x)$ (evidently $S(T \times T) = (T \times T)S$). Thus the conclusion of the corollary holds and this is the lemma.

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