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E. Lerzan Örmeci, Apostolos Burnetas
and Hamilton Emmons

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Abstract

We consider the problem of dynamic admission control in a Markovian loss queueing system with two classes of customers with different service rates and random revenues. We establish the existence of an optimal monotone policy. We also show that under certain conditions there exist preferred customers from either of the classes.

1 Introduction

In this paper, we consider a loss system with c identical parallel servers, no waiting room and two classes of customers. Class-i customers arrive at the system according to a Poisson process with rate λ_i and demand an exponential service time with mean $1/\mu_i$, where $\mu_1 \leq \mu_2$. If a class-i customer is admitted to the system, a random reward of $\rho_i \geq 0$ is collected in the beginning of the service. We assume that ρ_i has a probability density function f_i with finite mean, and the rewards of successive customers are independent. Our objective is to describe the optimal policy for admitting jobs in order to maximize the total expected discounted reward with continuous discount rate β over an infinite horizon or the long-run average net profit. If we assume that rewards of customers are determined and collected in the end of service, then the system is equivalent to the one whose customers of class-i offer fixed rewards of r_i with $r_i = E[\rho_i]$. This system has been analyzed by Örmeci, Burnetas & Wal (1999).

It is intuitively clear that the optimal policy would require a minimum offer from each customer depending on his class and the current state of the system. We show that the minimum offer for class j is increasing in the number of class-k customers, $k \neq j$, in the system. This is equivalent to the following monotonicity of the optimal policy: If a class-j customer with a reward of ρ_j is rejected when there are x_i class-i customers in the system, then (s)he will still be rejected if there are the same number of class-j customers and more class-k, $k \neq j$, customers in the system. Moreover, we show that under certain conditions, there are customer from either of the classes who offer sufficiently high rewards so that they are never denied service unless all the servers are occupied. We call these customers as the "prefered" customers.

There has been an increasing interest on multiclass loss networks recently due to the growth in telecommunications systems. Admission control is one of the main research areas on loss networks, see Chapter 4 of Ross (1995) for a comprehensive review. Most of the research on the area concentrates on certain types of policies (e.g., coordinate convex policies, trunk reservation policies) rather than analyzing the optimal policy directly; except for the following studies: In early 1970's, Lippman & Ross (1971), which is also known as streetwalker's dilemma in literature, analyze optimal admission rules for a system with one server and no waiting room which receives offers from customers according to a joint service time and reward probability distribution. Miller (1971) considers a system with c parallel identical servers, no waiting room and k different customer classes. All customers demand an exponential service with the same rate, whereas they offer different fixed rewards determined by their class. Later on, in late 1990's, Ku & Jordan (1997) consider two stations in tandem each with no waiting room and parallel servers. Carrizosa, Conde & Munoz-Marquez (1998) present an optimal static control policy for acceptance/rejection of k classes in an $M/G/c/c$ queue, where each class has a different service requirement and a different reward. Örmeci et al. (1999) consider the same system described above, where the rewards that customers bring is fixed for each class. All these studies, except for Lippman & Ross (1971), assume that the rewards gained from the customers are fixed for each class, as opposed to random rewards in our sytem. However, random rewards have been considered in queueing systems; e.g., Ghoneim & Stidham (1985) analyze the optimal admission policies for a system with two queues in series and two classes of customers who bring random rewards and require different services.

This paper is organized as follows: In the next section, we present the corresponding Markov decision process model of the system described above. The third section proves the existence of an optimal monotone policy. The fourth section presents the conditions under which preferred customers exist and how to determine them. The extension of all the results to infinite time horizon problems is considered in the fifth section. Finally, we discuss generalizations and possible future research in the last section.

2 Markov Decision Model

2.1 Discrete time model of the system

In this section, we build a discrete time Markov decision process (MDP) for the system described above with the objective of maximizing total expected discounted returns over a finite time horizon with β as the discount rate. We can consider discounting as exponential failures, i.e., the system closes down in an exponentially distributed time with rate β (for the equivalence of the process with discounting and the process without discounting but with an exponential deadline, see e.g., Walrand (1988)). We also assume without loss of generality that $\mu_1 \leq \mu_2$. Then, maximum possible rate out of any state is $\lambda_1 + \lambda_2 + c\mu_2 + \beta$. Since the time between each transition is always exponentially distributed and the maximum rate of transitions is finite, we can use uniformization (introduced by Lippman (1975a)) to build a discrete time equivalent of the original system. Thus, we let $A = \lambda_1 + \lambda_2 + c\mu_2 + \beta$. We observe the state of the system at each instant of a potential transition, so in every exponentially distributed time with rate A. Then, if the system is in state x, a potential transition will occur with rate A, and the actual transition will be a class-j arrival with probability λ_j/A, a class-i service completion with probability $x_i\mu_i/A$, a "fictitious" service completion, which does not change the state of the system, with probability $(c\mu_2 - x_1\mu_1 - x_2\mu_2)/A$, and finally the system will close down with probability β/A. Now, we can use normalization as well, so that we can assume, using the appropriate time scale, $A = 1$.

Then the system will be observed in exponentially distributed intervals with mean 1, and, as described before, there will be an arrival with probability $\lambda_1 + \lambda_2$ and a potential service completion with probability $c\mu_2$.

The assumption $\mu_1 \leq \mu_2$ implies that class-1 customers are "slow" customers. We use this assumption quite often to couple the service times of class-1 and class-2 customers. If we want to couple service times of a certain class-1 customer, say d_1, and a class-2 customer, say d_2, we let ξ be a uniformly distributed random variable in $(0,1)$, and we generate the service times of d_1 and d_2 using the same ξ, so customer d_2 leaves earlier than customer d_1 leaves with probability 1. In terms of discrete time, this translates to the following: Both customers leave the system with probability μ_1, and a class-2 customer departs from the system with probability $\mu_2 - \mu_1$ leaving the coupled class-1 customer in the system. Thus, coupling never allows a coupled class-1 customer to leave the system while the coupled class-2 customer is still there.

We define the state of the system depending on the type of the last transition: If the transition is due to a class-j arrival with a reward of ρ_j, then the state is $(x;j) = (x_1, x_2; j, \rho_j)$, where x_i is the number of class-i customers in the system. Otherwise, i.e., if there is a potential service completion, the state is $x = (x_1, x_2)$. Distinguishing the last event occurred in the state of the system is quite artificial, but it reflects the consequences of actions more clearly, as we see from the optimality equations given in the next subsection. Note that we always have $x_1 + x_2 \leq c$ and the actions are defined only for the states corresponding to an arrival.

2.2 Markov decision model for finite horizon

The objective is to maximize the total expected discounted reward over a finite horizon; let $u_n(x)$ and $v_n(x;j)$ be the maximal expected β-discounted profit, starting in state x and $(x;j)$, respectively, when n transitions remain in the horizon. Computing $v_n(x;j)$ requires a comparison of two actions: accepting the incoming class-j customer which implies moving to state $x + e_j$ with a reward of ρ_j, where e_j is defined as the vector which has a 1 at the jth coordinate, and 0 elsewhere, and rejecting him (her) so that the system remains in the same state with no reward or cost. We define $a_n(x;j)$ as the optimal action in state $(x;j)$ when there are n remaining transitions, where $a_n(x;j) = 1$ if it is optimal to accept the arriving customer of class j and 0 otherwise. We let S be the set on which u_n's are defined, i.e., $S = \{x : x_1 + x_2 \leq c\}$.

The optimality equations of this model are as follows: For $x_1 + x_2 < c$:

$$\begin{aligned} v_n(x;j) &= \max\{\rho_j + u_n(x+e_j), u_n(x)\} \\ u_{n+1}(x) &= \lambda_1 E[v_n(x;1)] + \lambda_2 E[v_n(x;2)] + \\ &\quad x_1\mu_1 u_n(x-e_1) + x_2\mu_2 u_n(x-e_2) + \\ &\quad (c\mu_2 - x_1\mu_1 - x_2\mu_2) u_n(x), \end{aligned} \tag{1}$$

where we set $u_n(-1, x_2) = u_n(0, x_2)$ and $u_n(x_1, -1) = u_n(x_1, 0)$. For $x_1 + x_2 = c$, no customers can be accepted so that $a_n(x;j) = 0$, and thus $v_n(x;j) = u_n(x)$. If the last event occurred is a class-j arrival, which happens with probability λ_j, (s)he is either accepted so that the system moves to the state $x + e_j$ with a random reward of ρ_j, or rejected, which keeps the system in the same state x. If a class-i customer finishes his service, with probability $x_i\mu_i$, the system state changes to $x - e_i$. The "fictitious" service completions, which occur with probability $c\mu_2 - x_1\mu_1 - x_2\mu_2$, affect neither the state nor the total reward of the system. Finally, if the system closes down, with probability β, the system receives no more reward.

2.3 Infinite horizon models

We prove all our results for the objective of maximizing total expected β-discounted reward for a finite number of transitions, n, including the "fictitious" transitions due to the "fictitious" service completions. Thus, "finite" horizon problems are pseudo finite problems. They provide the powerful tool of induction to prove our results for all n, which allows us to consider the infinite horizon problems: All the results proven for finite n are true for the limit $n \to \infty$, so the corresponding conclusions are valid when total expected β-discounted reward over an infinite horizon is maximized.

We consider the objective of maximizing expected long-run return as well. All our results hold for all β, including $\beta = 0$ and the model is unichain since state (0,0) is reachable from all states. Then, it is straightforward to verify that the conditions introduced by Lippman (1975*b*) are satisfied in this system, see Örmeci (1998) for details. Therefore, we have the same conclusions for the value functions, which maximizes the long-run average reward.

We define $v(x;j)$ and $u(x)$ as the maximal expected β-discounted reward for the system starting in state $(x;j)$ and x, respectively, over an infinite horizon. Thus, for $\beta > 0$, we have:

$$\begin{aligned} v(x;j) &= \lim_{n\to\infty} v_n(x;j) \\ u(x) &= \lim_{n\to\infty} u_n(x) \end{aligned}$$

$a(x;j)$ is the corresponding action in state $(x;j)$ so that $a(x;j) = 1$ if it is optimal to the customer and $a(x;j) = 0$ otherwise. For $\beta = 0$, $u(x) \to \infty$, so we need to consider the relative value functions and the gain in the usual MDP formulation.

2.4 Minimum offer required from an incoming customer

We choose to serve a customer, if both rejecting and accepting him (her) is optimal. Then, from equation (1), we easily observe that:

$$a_n(x;j) = 1 \iff u_n(x) - u_n(x + e_j) \le \rho_j$$

Now let $D_n(0j)(x) = u_n(x) - u_n(x + e_j)$. Hence, the optimal policy accepts a class-j customer only if (s)he offers at least $D_n(0j)(x)$ as a reward when the system is in state x. Thus, $D_n(0j)(x)$ is the minimum offer that the optimal policy demands from an incoming class-j customer, i.e., it determines a threshold level in state x such that if the random reward ρ_j of a class-j customer who finds the system in state x exceeds this threshold level, (s)he will be admitted to the system, otherwise (s)he will be rejected.

More generally, we define $D_n(ij)(x)$ as the difference in the total expected discounted rewards between system 1 and system 2 if system 1 starts in state x 'plus' one type i job and system 2 starts in x plus a type j job, when there are n more transitions in the horizon, where $i = 0$ means that system 1 is in state x, i.e., there is no additional customer. We, occasionally, drop the arguments x and n later on, when there is no danger of confusion in the reference. The four $D_n(ij)$ functions of interest are $D_n(01)$, $D_n(02)$, $D_n(12)$ and $D_n(21)$ given by $D_n(01)(x) = u_n(x) - u_n(x+e_1)$, $D_n(02)(x) = u_n(x) - u_n(x+e_2)$ and $D_n(21)(x) = -D_n(12)(x) = u_n(x+e_2) - u_n(x+e_1)$. We can interpret the difference $D_n(0j)(x)$ in a slightly different way as well, so that it corresponds to the expected burden that an additional class-j customer brings to the system in state x when there are n more transitions. Then, the difference $D_n(21)(x)$ also finds its interpretation as the expected burden of changing a class-2 customer who is already in the system to a class-1 customer in state $x+e_2$.

3 Existence of a monotone optimal policy

In this section we show that the minimum offer, $D_n(0j)(x)$, demanded from class-j customers in state x is increasing in the number of class-k customers in the system, $k \neq j$, so that it is more difficult for a class-j customer to be admitted when there are more class-k customers in the system. Intuitively, we expect that the required minimum offer is increasing with the congestion level in the system. Thus, the result of this section is less than our expectation, since we have not shown that the minimum offer for class j is increasing in the number of class-j customers; which corresponds to the concavity of u_n in x_j for fixed x_k, $k \neq j$. However, we were not able to prove the concavity of u_n because of the boundary effects. We also note that concavity of u_n's would lead to monotonicity of $D_n(0j)(x)$.

Lemma 1 *For all x with $x_1+x_2+2 \leq c$ (or equivalently for all x such that $x+e_1+e_2 \in S$):*

$$u_n(x) - u_n(x+e_2) - u_n(x+e_1) + u_n(x+e_1+e_2) \leq 0 \quad \forall n \geq 1, \tag{2}$$

for all functions $u_0(x)$ that also satisfy (2), and are otherwise arbitrary.

Proof. Assume that u_0 satisfies (2). Note that many functions satisfy this inequality including $u_0(x) = 0$. We prove the statement by induction on the number of remaining transitions, so assume that the statement is true for n.

We first show that $v_n(.;1)$'s also satisfy this monotonicity. We define δ such that:

$$\delta = v_n(x;1) - v_n(x+e_1;1) - v_n(x+e_2;1) + v_n(x+e_1+e_2;1)$$

Whenever there is a class-1 arrival, there are four possible cases due to the actions $a_n(x;1)$ and $a_n(x+e_1+e_2;1)$:

Case I: $a_n(x;1) = a_n(x+e_1+e_2;1) = 0$
$\delta \leq u_n(x) - u_n(x+e_1) - u_n(x+e_2) + u_n(x+e_1+e_2) \leq 0$

where the first inequality follows from the case assumptions and optimality of v_n's and the second one from the induction hypothesis.

Case II: $a_n(x;1) = 1$ and $a_n(x+e_1+e_2;1) = 0$
$\delta \leq u_n(x+e_1) + \rho_1 - u_n(x+e_1) - u_n(x+e_1+e_2) - \rho_1 + u_n(x+e_1+e_2) = 0$

where the inequality follows from the case assumptions and optimality of v_n's.

Case III: $a_n(x;1) = 0$ and $a_n(x+e_1+e_2;1) = 1$

$$\begin{aligned}\delta &\leq u_n(x) - u_n(x+2e_1) - \rho_1 - u_n(x+e_2) + u_n(x+2e_1+e_2) + \rho_1 \\ &= u_n(x) - u_n(x+e_2) - u_n(x+e_1) + u_n(x+e_1+e_2) \\ &\quad + u_n(x+e_1) - u_n(x+e_1+e_2) - u_n(x+2e_1) + u_n(x+2e_1+e_2) \leq 0\end{aligned}$$

where the first inequality is true due to the case assumptions and optimality of v_n's and the second one due to the induction hypothesis.

Case IV: $a_n(x;1) = 1$ and $a_n(x+e_1+e_2;1) = 1$

$$\delta \leq u_n(x+e_1) - u_n(x+2e_1) - u_n(x+e_1+e_2) + u_n(x+2e_1+e_2) \leq 0$$

where the first inequality is true due to the case assumptions and optimality of v_n's and the second one due to the induction hypothesis.

Hence, $v_n(x;1)$'s satisfy inequality (2) for all ρ_1, and so $E[v_n(x;1)]$ also satisfies the inequality. $E[v_n(x;2)]$ can be proved to satisfy inequality (2) in a similar way. Thus, we can consider u_{n+1}'s:

$$\begin{aligned}
& u_{n+1}(x) - u_{n+1}(x+e_2) - u_{n+1}(x+e_1) + u_{n+1}(x+e_1+e_2) \\
= \lambda_1 \quad & (E[v_n(x;1)] - E[v_n(x+e_2;1)] - E[v_n(x+e_1;1)] + E[v_n(x+e_1+e_2;1)]) \\
+\lambda_2 \quad & (E[v_n(x;2)] - E[v_n(x+e_2;2)] - E[v_n(x+e_1;2)] + E[v_n(x+e_1+e_2;2)]) \\
+x_1\mu_1 \quad & [u_n(x-e_1) - u_n(x-e_1+e_2) - u_n(x) + u_n(x+e_2)] \\
+\mu_1 \quad & [u_n(x) - u_n(x+e_2) - u_n(x) + u_n(x+e_2)] \\
+x_2\mu_2 \quad & [u_n(x-e_2) - u_n(x) - u_n(x+e_1-e_2) + u_n(x+e_1)] \\
+\mu_2 \quad & [u_n(x) - u_n(x) - u_n(x+e_1) + u_n(x+e_1)] \\
+\alpha \quad & [u_n(x) - u_n(x+e_2) - u_n(x+e_1) + u_n(x+e_1+e_2)] \\
\leq \quad & 0
\end{aligned}$$

where $\alpha = c\mu_2 - (x_1+1)\mu_1 - (x_2+1)\mu_2$. The first two terms are less than or equal to 0 since $E[v_n]$'s are shown to satisfy the inequality, the third, fifth and seventh terms are also non-positive by the induction hypothesis whereas fourth and sixth terms are 0. Thus, the value functions, u_n, satisfy inequality (2) for all n whenever u_0 does. □

This lemma guarantees the following monotonicity of the optimal policy:

Theorem 1 *If it is optimal to reject a class-j customer in state $(x;j)$, then it is optimal to reject him (her) in all states $(x+le_k;j)$ with $l \geq 1$ and $k \neq j$.*

4 Existence of preferred customers

We define the preferred customers as the customers who are always admitted to the system whenever there is at least one idle server. In this section, we show that under certain conditions there are preferred customers from either of the classes. This is natural in the sense that whenever customers are willing to pay sufficiently high prices, they should receive immediate service if there is an available server.

First, we consider the system in which the rewards of class j are unbounded, i.e., we assume that for all $M > 0$, $P\{\rho_j \geq M\} > 0$. Then we easily conclude that there are always preferred customers of this class: We have interpreted the difference $D_n(0j)(x)$ as the burden of an additional class-j customer, and seen that a customer of class j bringing a reward of ρ_j is admitted to the system if $D_n(0j)(x) \leq \rho_j$. Since the set $\mathcal{S}$ is finite, $D_{max}(0j) = \max_{x \in \mathcal{S}}\{D_n(0j)(x)\} < \infty$. Then, $P\{\rho_j \geq D_{max}(0j)\} > 0$ so that there exist class-j customers who are willing to pay high enough prices to be served immediately whenever there is an available server.

Proposition 1 *If rewards of class-j customers, ρ_j's, are unbounded, then there are preferred class-j customers.*

From now on, we assume that rewards are bounded for both classes, so there exists a $\bar{\rho}_j < \infty$, such that $P\{\rho_j > \bar{\rho}_j\} = 0$ and $P\{\rho_j \leq \bar{\rho}_j\} = 1$ for $j = 1, 2$. Clearly, from practical point of view as well as the mathematics of the analysis, this is the more interesting case. Under this assumption, we need to analyze the behavior of the differences $D_n(ij)(x)$ in more detail to be able to show the existence of preferred customers. Indeed, showing that $D_n(0j)(x) < \bar{\rho}_j$ for all $x \in \mathcal{S}$ is equivalent to prove the existence of preferred customers of class j, since then class-j customers who offer high enough rewards arrive at the system with the positive probability of $P\{D_{max}(0j) \leq \rho_j < \bar{\rho}_j\}$. We first show the non-negativity of $D_n(0j)(x)$'s and $D_n(21)(x)$:

Lemma 2 *For $j = 1, 2$, for all $x \in \mathcal{S}$ and $n \geq 0$:*
(1) $D_n(0j)(x) \geq 0$.
(2) $D_n(21)(x) = -D_n(12)(x) \geq 0$.

Proof. We prove the statements by a sample path analysis.

(1) Assume that system A is in state x and system B in $x + e_j$ in period n. We let system B follow the optimal policy, π, and system A imitate all the decisions of system B. We couple the two systems via the service and interarrival times, i.e., except for the additional customer in system B, all the departure and arrival times are the same in both systems. We note that system A can always imitate system B since it always has at least as many free servers as system B does. Then, all future rewards of system A and B are the same:

$$D_n(0j)(x) = u_n(x) - u_n(x + e_j) \geq u_n^{\pi}(x) - u_n(x + e_j) = 0.$$

where $u_n^{\pi}(x)$ is the expected discounted return of system A.

(2) Assume that system A starts in state $x + e_2$ and system B starts in $x + e_1$, where we now couple the additional class-2 customer, say customer d_2, in system A with the additional class-1 customer, say customer d_1, in system B, as well as all other service and interarrival times, so that, as discussed earlier, if d_1 leaves the system, d_2 also leaves. Then, we can let system B follow the optimal policy and system A imitate all the decisions of system B. Now, again, all future rewards of both systems are equal:

$$D_n(21)(x) = u_n(x + e_2) - u_n(x + e_1) \geq u_n^{\pi}(x + e_2) - u_n(x + e_1) = 0,$$

with $u^{\pi}(x + e_2)$ the expected discounted return of system A. □

$u_n(x)$ is the expected discounted total reward of the system under the optimal policy when there are n more transitions. Thus, $u_n(x)$ refers to the future rewards: The rewards are collected in the beginning of service, hence the customers who are already in the system do not bring any benefit in the future. In other words, the customers initially in the system bring only more burden by preventing to accept more customers. Hence, it is always more preferable to be in a state where there are less or faster customers, which is, indeed, the conclusion of Lemma 2.

Now, we prove that under certain conditions, there are preferred customer of class 2:

Theorem 2 *If $\frac{\bar{\rho}_2}{\bar{\rho}_1} > \frac{\lambda_1}{\lambda_1+\mu_2+\beta}$, then for all $x \in S$ and for all n, $D_n(02)(x) \le \bar{\rho}_2$, hence there are preferred class-2 customers.*

Proof. Let $\bar{\rho}_2 > \frac{\lambda_1}{\lambda_1+\mu_2+\beta}\bar{\rho}_1$. We use induction to prove the result. The function $u_0(x) = 0$ for all $x \in S$ clearly satisfies the statement. So assume that the statement is also true for period n, and consider period $n+1$. Now we use a sample path argument: Let system A be in state x and system B in $x + e_2$ in period $n+1$. System A takes the optimal actions and system B rejects all customers in period $n+1$. Consider an arrival. If system A also rejects either of the two classes, both systems remain in their current states, preserving the extra class-2 customer. Acceptance of a class-1 customer with a reward of ρ_1 to system A leads two systems to two different states $x + e_1$ and $x + e_2$ with a difference of ρ_1 in the value functions. If a class-2 customer bringing a reward of ρ_2 is admitted to system A, then the two systems couple with a difference of ρ_2 in reward. With the departure of the additional class-2 customer in system B, the systems again enter the same state, but with no difference in reward, whereas all other service completions keep the extra class-2 customer in system A. Then:

$$\begin{aligned}
D_{n+1}(02)(x) &= u_{n+1}(x) - u_{n+1}(x+e_2) \\
&\le \lambda_1 \max\{D_n(12)(x) + \rho_1, D_n(02)(x)\} + \lambda_2 \max\{\rho_2, D_n(02)(x)\} + \mu_2 \times 0 \\
&\quad +(c-1)\mu_2 \max_{y \in S}\{D_n(02)(y)\} \\
&\le \lambda_1 \max\{\bar{\rho}_1, \bar{\rho}_2\} + \lambda_2\bar{\rho}_2 + (1 - \lambda_1 - \lambda_2 - \mu_2 - \beta)\bar{\rho}_2 \\
&\le \lambda_1 \max\{\bar{\rho}_1 - \bar{\rho}_2, 0\} + (1 - \mu_2 - \beta)\bar{\rho}_2
\end{aligned}$$

where the first inequality is due to the coupling, the second inequality follows from the definition of $\bar{\rho}_j$, the induction hypothesis, part 2 of Lemma 2 and uniformization. If $\bar{\rho}_2 \ge \bar{\rho}_1$, then the statement is proven. Otherwise, we have:

$$D_{n+1}(02)(x) \le \lambda_1(\bar{\rho}_1 - \bar{\rho}_2) + (1 - \mu_2 - \beta)\bar{\rho}_2 = \bar{\rho}_2 - (\lambda_1 + \mu_2 + \beta)\bar{\rho}_2 + \lambda_1\bar{\rho}_1 < \bar{\rho}_2$$

where the last inequality is due to the assumption of the theorem. □

We derive a similar condition for class 1 to be preferred. However, this requires some more work, since we have to consider an upper bound on $D_n(21)(x)$ simultaneously with the minimum offer for class 1, $D_n(01)(x)$.

Lemma 3 *If $\frac{\bar{\rho}_2(\mu_2+\beta)}{\bar{\rho}_1(\mu_1+\beta)} \le \frac{\lambda_2+\mu_2+\beta}{\lambda_2}$, then for all $x \in S$ and for all n:*

(1) $D_n(01)(x) \le \bar{\rho}_1$.

(2) $D_n(21)(x) \le \dfrac{\mu_2 - \mu_1}{\mu_2 + \beta}\bar{\rho}_1$.

Proof. We use induction on the number of transitions, n. Both statements are satisfied for $u_0(x) = 0$ for all $x \in \mathcal{S}$. Assume that both are true for n. Now we have to consider two pairs of systems, one for $D_{n+1}(01)(x)$ and the other for $D_{n+1}(12)(x)$.

(1) Consider the first pair: Assume that system A is in state x and system B is in $x+e_1$ in period $n+1$, and we couple the two systems in such a way that system A follows the optimal policy, whereas system B rejects all customers in period $n+1$. If upon an arrival system A also rejects either of the two classes, both systems remain in their current states, preserving the extra class-1 customer. Acceptance of a class-1 customer with a reward of ρ_1 to system A leads both systems to enter the same state with a difference of ρ_1 in reward. If a class-2 customer bringing ρ_2 is admitted to system A, then the systems move to two different states $x+e_2$ and $x+e_1$ with a difference of ρ_2. With the departure of the additional class-1 customer in system B, the systems again enter the same state but with no return, whereas all other service completions keep the difference between the two systems the same. Then:

$$
\begin{aligned}
D_{n+1}(01)(x) &\leq \lambda_1 \max\{\rho_1, D_n(01)(x)\} + \lambda_2 \max\{D_n(21)(x) + \rho_2, D_n(01)(x)\} \\
&\quad + (c\mu_2 - \mu_1) \max_{y \in \mathcal{S}}\{D_n(01)(y)\} \\
&\leq \lambda_1 \max\{\bar{\rho}_1, D_n(01)(x)\} + \lambda_2 \max\{D_n(21)(x) + \bar{\rho}_2, \bar{\rho}_1\} \\
&\quad + (1 - \lambda_1 - \lambda_2 - \mu_1 - \beta)\bar{\rho}_1 \\
&\leq \lambda_1\bar{\rho}_1 + \lambda_2 \max\left\{\frac{\mu_2 - \mu_1}{\mu_2 + \beta}\bar{\rho}_1 + \bar{\rho}_2, \bar{\rho}_1\right\} + (1 - \lambda_1 - \lambda_2 - \mu_1 - \beta)\bar{\rho}_1 \\
&\leq \lambda_2 \max\left\{\bar{\rho}_2 - \frac{\mu_1 + \beta}{\mu_2 + \beta}\bar{\rho}_1, 0\right\} + (1 - \mu_1 - \beta)\bar{\rho}_1
\end{aligned}
$$

where the first inequality is due to coupling, the second due to the definition of $\bar{\rho}_j$, the induction hypothesis for $D_n(10)(x)$ and uniformization, and the third one follows from the induction hypotheses for $D_n(10)(x)$ and $D_n(21)(x)$. If $\bar{\rho}_2 \leq \frac{\mu_1+\beta}{\mu_2+\beta}\bar{\rho}_1$, the statement is proven; otherwise:

$$
\begin{aligned}
D_{n+1}(01)(x) &\leq \lambda_2\bar{\rho}_2 - \lambda_2\frac{\mu_1 + \beta}{\mu_2 + \beta}\bar{\rho}_1 + (1 - \mu_1 - \beta)\bar{\rho}_1 \\
&\leq \bar{\rho}_1 + \lambda_2\bar{\rho}_2 - \frac{\mu_1 + \beta}{\mu_2 + \beta}\bar{\rho}_1(\lambda_2 + \mu_2 + \beta) < \bar{\rho}_1
\end{aligned}
$$

where the last inequality is due to the assumption of the theorem. Thus, the first statement is true for all $x \in \mathcal{S}$ and for all $n \geq 0$.

(2) Now consider the second pair of systems: Let system A′ be in state $x+e_2$ and system B′ in $x+e_1$ in period $n+1$. System A′ takes the optimal actions and system B′ imitates all the actions of system A′ in this period. We, as in Lemma 2, couple the additional class-2 customer, say customer d_2, in system A′ with the additional class-1 customer, say customer d_1 in system B′, as well as all other service and interarrival times. Then, if d_1 leaves the system, which happens with probability μ_1, d_2 also leaves. The departure of d_1 leads the system to couple with no reward, the departure of d_2 alone, which happens with probability $\mu_2 - \mu_1$, takes the systems to two different states, x and $x+e_1$ with no reward and whenever there is any other transition, both systems continue to have their additional customers so that the difference between the two systems is due to changing a class-1 customer to class

2:

$$\begin{aligned} D_{n+1}(21)(x) &\leq \mu_1 \times 0 + (\mu_2 - \mu_1) D_n(01)(x) + (\lambda_1 + \lambda_2 + (c-1)\mu_2) \max_{y \in \mathcal{S}} \{D_n(21)(y)\} \\ &\leq (\mu_2 - \mu_1)\bar{\rho}_1 + (1 - \mu_2 - \beta)\frac{\mu_2 - \mu_1}{\mu_2 + \beta}\bar{\rho}_1 = \frac{\mu_2 - \mu_1}{\mu_2 + \beta}\bar{\rho}_1 \end{aligned}$$

where the first inequality is due to the coupling and the second follows by uniformization and the induction hypotheses for both $D_n(01)(x)$ and $D_n(21)(x)$. This proves the second part of the lemma. □

This lemma immediately leads to the following theorem which gives the sufficient conditions for class 1 to be preferred:

Theorem 3 *If* $\frac{\bar{\rho}_2(\mu_2+\beta)}{\bar{\rho}_1(\mu_1+\beta)} < \frac{\lambda_2+\mu_2+\beta}{\lambda_2}$, *then there are preferred customers of class 1.*

Theorem 2 and 3 present sufficient conditions to have preferred customers of class 2 and 1, respectively. Theorem 2 implies that if the upper bound of both classes are the same, then there are always preferred class-2 customers. This is very intuitive, since we would prefer to serve the faster customers if both classes offer the same reward; i.e., in this case whenever a class-2 customer offers a random reward of $\bar{\rho}_2$, (s)he will certainly be accepted if there is an idle server. Theorem 3 is not that easy to interpret since the quantity $\frac{(\lambda_2+\mu_2+\beta)(\mu_1+\beta)}{\lambda_2(\mu_2+\beta)}$ is not necessarily greater than 1. However, we can still conclude that, in order to have preferred class-1 customers, we need stronger conditions on the ratio of upper bounds, $\frac{\bar{\rho}_2}{\bar{\rho}_1}$, to accomodate the slowness of class-1 customers. Notice that we have these conditions because the rewards are collected in the beginning of service, the conclusions can be quite different when we assume otherwise. For example, as mentioned earlier, if the rewards are determined and so collected in the end of the service, the system changes to the system in Örmeci et al. (1999) with fixed rewards of $E[\rho_i]$. Then, the slower customers are preferred whenever $E[\rho_1] = E[\rho_2]$, since they provide more steady income for the system. See Örmeci et al. (1999) for details.

5 Future Research

We can consider the system under a general arrival process, which can be modeled as an embedded MDP at arrival times. This allows us to model especially computer and communication systems in a better way, since recently it has been verified that the arrival processes in these systems do not satisfy the assumptions of Poisson arrivals (see e.g. Leland, Taqqu, Willinger & Wilson (1994) and Willinger, Taqqu, Leland & Wilson (1995)).

The admission of customers into the system can also be controlled via pricing. Thus, instead of rejecting the customers, we can propose a price, which may or may not depend on the state of the system, for which we are willing to serve the incoming customer. This kind of control has been considered in the context of social optimization for different queueing systems, see e.g., Naor (1969), Lippman & Stidham (1977) and Xu & Shantikumar (1993). The only study on loss systems with pricing is Miller & Buckman (1987) who consider a static transfer pricing problem for one class in an $M/M/c/c$ system which serves as a model of a service department.

This system can be considered under batch arrivals, which can model computer and communication systems better. Lippman & Ross (1971) are the first to consider the batch arrivals for a single server no waiting room system. Örmeci & Burnetas (1999) have also considered a system under batch arrivals which has c identical parallel servers and two classes of customers with fixed rewards and rejection costs. They were able to characterize the optimal policy only partially, but still the system has been shown to have certain monotonicity properties. Similar results may be obtained with random rewards.

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