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Abstract

We consider the problem of dynamic admission control in a Markovian loss system with two classes of customers with different service rates and revenues, where arrivals occur in batches. We establish a monotonicity property of optimal value functions, which reduces the number of possibly optimal actions. We also show that under certain conditions there exists a preferred class. These results are valid under a Markov arrival process.

1 Introduction

The importance of dynamic admission control strategies has received increasing recognition in recent years in many applications. The improvements in automation has made it possible to implement sophisticated rules, for example in computer-controlled manufacturing facilities and communication systems with automated digital switches. Thus, dynamic admission policies are used more widely as revenue management tools especially in telecommunications, and in particular, in telephone service and support applications.

Dynamic rules offer the possibility of significantly improved performance through increased flexibility of resourse allocation among different customer classes, as compared to static rules. A static admission policy determines the priorities among different customer classes only according to their service requirements and revenues, and so these rules are completely independent of the system state. On the other hand, a dynamic admission policy decides on admitting an arriving customer based on the system state as well as the customer's class. A dynamic admission policy usually specifies a resource idling rule, since a customer can be denied service, even in the presence of available resources, to provide a better service level for future customers of a more profitable class.

There has been an increasing interest on multiclass loss networks recently due to the growth in telecommunications systems. Admission control is one of the main research areas on loss networks, see Chapter 4 of Ross (1995) for a comprehensive review. Most of the research on the area concentrates on certain type of policies (e.g., coordinate convex policies, trunk reservation policies) rather than analyzing the optimal policy; except the following studies: In early 1970's, Lippman & Ross (1971) analyze the optimal admission rules for a system with one server and no waiting room which receives offers from customers according to a joint service time and reward probability distribution, whereas Miller (1971) for a system with c parallel identical servers, no waiting room and k different customer classes. Ku & Jordan (1997) consider two stations in tandem each with no waiting room and parallel servers. Carrizosa, Conde & Munoz-Marquez (1998) present a static control policy for acceptance/rejection of k classes in an $M/G/c/c$ queue. In Örmeci, Burnetas & Wal (1999*b*), the same system considered here is analyzed with single arrivals. Indeed, all these studies, except for Lippman & Ross (1971), assume that the system has single arrivals. However, batch arrivals are quite common especially in telecommunications. A loss system with batch arrivals is considered by Puhalskii & Reiman (1998), where they restrict the domain of admission policies to the set of trunk reservation policies.

In this paper, we consider a system, which consists of c servers with no waiting room and two classes of customers: each served class-i customer brings a reward of $r_i > 0$ upon his (her) departure. Arrivals occur according to a Poisson process with rate λ and a batch of j_1 class-1 and j_2 class-2 customers arrives with probability $p_{j_1 j_2}$. A class-i customer requires an exponential service time with rate μ_i. We assume that we can accept some of the customers and reject the rest from each batch. This is substantially different than the case in which it is allowed to accept or reject the whole batch only. For the latter case, we present an example which violates any expected monotonicity properties of an optimal policy.

We prove a monotonicity of the value functions, which leads to a reduction in the number of possible actions that can be optimal. In addition, we develop a set of sufficient conditions which ensure that a customer class is "preferred", in the sense that its customers are always admitted whenever there are free servers, regardless of the system congestion level. The meaning of "preferred" is different for systems which receive batches consisting of only one class of customers and systems receiving mixed batches, although its definition remains the same. With uniform batches, basicly two actions are compared at an arrival epoch: keeping a server empty or filling him with a customer of the batch class. Thus, in this case, 'preferring class i' means that 'a server working on a class-i customer is more beneficial than having him idle', so that being preferred does not involve a comparison between the two classes, which, in turn, allows to have more than one preferred class. However, when the system receives batches consisting of both class 1 and 2, at most one of the classes can be preferred: Assume that only one of the servers is empty and a mixed batch arrived, then only one customer can be accepted, and so both classes cannot be preferred at the same time. Thus, with mixed batches, 'preferring class i' means that 'a server occupied with a class-i customer brings more benefit than both an empty server, and a server working on a class-k customer, $k \neq i$'. Therefore, systems with uniform and mixed batches require different sets of conditions for each class to be preferred. Our results still hold under Markov arrival process (MAP), which is shown to approximate any independent arrival process arbitrarily closely by Asmussen & Koole (1993).

This paper is organized as follows: In the next section, we present the corresponding Markov Decision Process (MDP) model of the system described above. The third section proves a monotonicity of optimal value functions. The fourth section presents the conditions under which a preferred class exists and how to determine this class for systems with uniform and mixed batches. In the fifth section, we give a brief remark on systems with $\mu_1 = \mu_2$. The sixth section presents a counterexample to any possible monotonicity of the value functions when the whole batch has to be admitted or rejected. Finally, we discuss generalizations and possible future research in the last section.

2 Markov Decision Model

2.1 Discrete time model of the system

In this section, we build a discrete time Markov decision process (MDP) for the system described above with the objective of maximizing total expected discounted returns over a finite time horizon with β as the discount rate. We can consider discounting as exponential failures, i.e., the system closes down in an exponentially distributed time with rate β (for the equivalence of the process with discounting and the process without discounting but with an exponential deadline, see e.g., Walrand (1988)). We also assume without loss of generality that $\mu_1 \leq \mu_2$. Arrivals occur according to a Poisson process with rate λ, and at each arrival epoch a batch of j_i class-i customers seek admittance to the system, where $\sum_{j_1=0}^{c} \sum_{j_2=0}^{c-j_1} p_{j_1 j_2} = 1$ and $p_{00} = 0$. We denote $p_{j_1 j_2}$ by p_j occasionally. Then, maximum possible rate out of any state is $\lambda + c\mu_2 + \beta$. Since the time between each transition is always exponentially distributed and the maximum rate of transitions is finite, we can use uniformization (introduced by Lippman (1975)) to build a discrete time equivalent of the original system. Thus, we let $A = \lambda + c\mu_2 + \beta$. We observe the state of the system at each instant of a potential transition, so in every exponentially distributed time with rate A. Then, if the system is in state x, a potential transition will occur with rate A, and the actual transition will be an arrival with probability λ/A, a class-i service completion with probability $x_i\mu_i/A$, a "fictitious" service completion, which does not change the state of the system, with probability $(c\mu_2 - x_1\mu_1 - x_2\mu_2)/A$, and finally the system will close down with probability β/A. Now, we can use normalization as well, so that we can assume, using the appropriate time scale, $A = 1$. Then the system will be observed in exponentially distributed intervals with mean 1, and, as described before, there will be an arrival with probability λ and a potential service completion with probability $c\mu_2$.

The assumption $\mu_1 \leq \mu_2$ implies that class-1 customers are "slow" customers. We use this assumption quite often to couple the service times of class-1 and class-2 customers. If we want to couple service times of a certain class-1 customer, say d_1, and a class-2 customer, say d_2, we let ξ be a uniformly distributed random variable in $(0, 1)$, and we generate the service times of d_1 and d_2 using the same ξ, so customer d_2 leaves earlier than customer d_1 leaves with probability 1. In terms of discrete time, this translates to the following: Both customers leave the system with probability μ_1, and a class-2 customer departs from the system with probability $\mu_2 - \mu_1$ leaving the coupled class-1 customer in the system. Thus, coupling never allows a coupled class-1 customer to leave the system while the coupled class-2 customer is still there.

We define the state of the system including the last event occurred: Let $x = (x_1, x_2)$ be the state of the system when there are x_i class-i customers in the system and a potential service completion is observed, and $(x, j) = (x_1, x_2; j_1, j_2)$ be the state of the system if a batch of j_i customers has arrived at the system to find x_i class-i customers. Distinguishing the last event occurred in the state of the system is quite artificial, but it reflects the consequences of actions more clearly, as we see from the optimality equations given in the next subsection. Note that we always have $x_1 + x_2 \leq c$ and the actions are defined only for the states corresponding to an arrival.

2.2 Markov decision model for finite horizon

We denote the maximal expected β-discounted net benefit of the system which starts in state x and $(x;j)$ when n observation points remain in the horizon by $u^n(x)$ and $v^n(x;j)$, respectively. Let $\mathcal{S}$ be the set on which u^n's are defined, i.e., $\mathcal{S} = \{x : x_1 + x_2 \leq c\}$. $y^n(x;j) = (y_1^n(x;j), y_2^n(x;j))$ is defined as the optimal state to be in at the beginning of period $n+1$, when the system is in state $(x;j)$ at period n. We define $S(x;j)$ as the action space for state $(x;j)$:

$$S(x;j) = \Big\{y \in \mathcal{S} : x_i \leq y_i \leq x_i + j_i,\ i = 1,2\Big\},$$

so that $S(x;j)$ is the set of all feasible states which can be reached from state x when a batch of j arrives at the system. Note that $S(x;j) = \{x\}$ for $(x;j)$ with $x_1 + x_2 = c$, regardless of the value of j.

Now we can present the optimality equations. Let e_i be the vector which has a 1 at the ith coordinate, and 0 elsewhere. Then, for $x_1 + x_2 \leq c$:

$$\begin{aligned} v^n(x;j) &= \max\Big\{u^n(y) : y \in S(x;j)\Big\} \qquad (1) \\ u^{n+1}(x) &= x_1\mu_1 r_1 + x_2\mu_2 r_2 + \lambda\sum_j p_j v^n(x;j) \\ &\quad + x_1\mu_1 u^n(x - e_1) + x_2\mu_2 u^n(x - e_2) \\ &\quad + (c\mu_2 - x_1\mu_1 - x_2\mu_2)u^n(x), \end{aligned}$$

where we assume $u^n(-1, x_2) = u^n(0, x_2)$ and $u^n(x_1, -1) = u^n(x_1, 0)$. If the last event occurred is an arrival of a batch consisting of j_i class-i customers, which happens with probability $\lambda p_{j_1 j_2}$, then $y_i - x_i$ of j_i class-i customers are accepted so that the system moves to the state y. If a class-i customer finishes his service, with probability $x_i\mu_i$, the system state changes to $x - e_i$ with a reward of r_i. The "fictitious" service completions, which occur with probability $c\mu_2 - x_1\mu_1 - x_2\mu_2$, affect neither the state nor the total reward of the system. Finally, if the system closes down, with probability β, the system receives no more reward.

2.3 Infinite horizon models

We prove all our results for the objective of maximizing total expected β-discounted reward for a finite number of transitions, n, including the "fictitious" transitions due to the "fictitious" service completions. Thus, "finite" horizon problems are pseudo finite problems. They provide the powerful tool of induction to prove our results for all n, which allows us to consider the infinite horizon problems: All the results proven for finite n are true for the limit $n \to \infty$, so the corresponding conclusions are valid when total expected β-discounted reward over an infinite horizon is maximized. Moreover, since the state space and the action space in each state are finite and the results hold for all β, including $\beta = 0$, we have the same conclusions for maximizing the long-run average reward. Here, we note that for the results regarding to the preferred class, we specify the initial value function u^0 in such a way that the rewards of customers, who are still in the system at $n = 0$, are collected even if their services have not been finished. Of course, this makes no difference in the optimal policy for infinite horizon problems.

We define $v(x;j)$ $(u(x))$ as the maximal expected β-discounted reward for the system starting in state $(x;j)$ (x) over an infinite horizon. Thus, for $\beta > 0$, we have:

$$\begin{aligned} v(x;j) &= \lim_{n\to\infty} v^n(x;j) \\ u(x) &= \lim_{n\to\infty} u^n(x) \end{aligned}$$

$y(x;j) = (y_1(x;j), y_2(x;j))$ is the corresponding action in state $(x;j)$ so that it is optimal to have $y_i(x;j)$ class-i customers when a batch of j arrives at a system with x_i class-i customers. For $\beta = 0$, $u(x) \to \infty$, so we need to consider the relative value functions and the gain in the MDP usual formulation.

2.4 Effect of an additional customer

In our analysis below, the effect of an additional class-i customer in the system will be important, so we define $D^n(ik)(x)$ as the difference in the total expected discounted rewards between system A and system B if system A starts in state x 'plus' one class-i customer and system B starts in x plus a class-k customer, where $k = 0$ means that system 2 is in state x, i.e., there is no additional customer. We, occasionally, drop the arguments x and n later on, when there is no danger of confusion in the reference. The four $D^n(ik)$ functions of interest are $D^n(10)$, $D^n(20)$, $D^n(12)$ and $D^n(21)$. It is easy to see that $D^n(10)(x) = u^n(x+e_1) - u^n(x)$, $D^n(20)(x) = u^n(x+e_2) - u^n(x)$ and $D^n(12)(x) = -D^n(21)(x) = u^n(x+e_1) - u^n(x+e_2)$. We can interpret the difference $D^n(i0)(x)$ as the net benefit of the system due to an additional class-i customer in state x when there are n more transitions, whereas $D^n(12)(x)$ is the net benefit of the system when a class-1 customer already in the system is changed to a class-2 customer in state $x + e_1$.

If the arrivals were single, these functions, $D^n(ik)(x)$, could be used very effectively to determine the optimal action for state x. However, with batch arrivals, everything is more complicated since different combinations of class-1 and class-2 customers can be accepted so that the reference state x in $D^n(ik)(x)$ is no longer fixed at each decision epoch. Still, these functions prove to be useful in determining preferred class(es) as we see later in section 4.

2.5 A remark on rewards

In this model, we have considered only the rewards collected in the end of service. Rejection costs, say b_i, which are incurred at the time of the arrival of a rejected customer can be incorporated in the model by redefining the reward r_i as $r_i + \frac{\mu_i+\beta}{\mu_i} b_i$ due to the discounting. For a more general system with both rejection costs, b_i, and rewards, r_i, one can refer to the thesis Örmeci (1998), where all the equivalent results of this paper are stated with a more complicated notation, although the methods of proofs with or without rejection costs are the same.

The present value of the reward brought by a class-i customer is $\frac{r_i \mu_i}{\mu_i+\beta}$ due to the discounting. We refer to this quantity as the immediate reward of a class-i customer and denote it by R_i. Thus, r_i is the value of the reward in the end of service, whereas R_i is its value in the beginning of the service. Another quantity of interest is the average reward of a class-i customer, $r_i \mu_i$.

3 Monotonicity of Optimal Value Functions

In this section, we prove a monotonicity property of u^n's, an intuitive and simple result. However, with batch arrivals, the proof of this result is very complicated. Also, unfortunately, its implication on the optimal policy is not straighforward nor as strong as we would like. We first state the monotonicity of u^n's, and show how this reduces the number of possible optimal actions. Later on, we present several definitions and a rather technical proof of Lemma 1.

Lemma 1 *For all $x + e_1 + e_2 \in S$, we have:*

$$u^n(x) - u^n(x + e_2) - u^n(x + e_1) + u^n(x + e_1 + e_2) \leq 0 \quad \forall n \geq 1, \tag{2}$$

whenever the inequality is true for $n = 0$.

We have interpreted the difference $D^n(i0)$ as the net profit of the system due to an additional class-i customer, so inequality (2) shows that the net profit of the system due to an additional class-i customer is decreasing in the number of class-k customers, $k \neq i$. If the arrivals were single, as opposed to batch, this monotonicity of the value functions would immediately translate to a threshold policy (see Örmeci et al. (1999*b*)). However, under batch arrivals, existence of an optimal threshold policy is too strong to be deduced from the above monotonicity. Indeed, concavity of u^n's in x_i for fixed x_k, $k \neq i$, would ensure an optimal threshold policy for batch arrivals, but we have not been able to show this due to the boundary effects and state dependent service rates. Concavity of the value functions is difficult to establish even with single arrivals: Örmeci et al. (1999*b*) are able to show it only under very restrictive conditions. Nevertheless, inequality (2) decreases the number of possible actions, which can be optimal:

Theorem 1 *Let $y^* = y^n(x; j)$. Then, there are numbers $\{l_i^n(0), ..., l_i^n(c-1)\}_{\{i=1,2\}}$ such that: For $i = 1, 2$ and $k \neq i$, either $y_i^* \geq l_k^n(y_k^*)$ or $y_k^* = \min\{c - y_i^*, x_k + j_k\}$.*

Proof. We first define $l_i^n(x_i)$: For a given x_1, we define $l_1^n(x_1)$ as:

$$l_1^n(x_1) = \min\{l : u^n(x_1 + 1, l) < u^n(x_1, l)\}.$$

Similarly,

$$l_2^n(x_2) = \min\{l : u^n(l, x_2 + 1) < u^n(l, x_2)\}.$$

If there is no such l for x_i, we set $l_i^n(x_i) = c - x_i$.

Let $y^* = y^n(x; j)$. Assume, by contradiction, that there exists a state $(x; j)$ with optimal actions y^* such that $y_i^* < l_k^n(y_k^*)$ and $y_k^* \neq \min\{c - y_i^*, x_k + j_k\}$. We first observe that $y_k^* \leq \min\{c - y_i^*, x_k + j_k\}$ to be feasible, so we assume that $y_k^* < \min\{c - y_i^*, x_k + j_k\}$. Hence we can accept more class-k customers since a strictly positive number, i.e., $x_k + j_k - y_k^*$, of class-k customers are rejected while there are strictly positive number, i.e., $c - y_i^* - y_k^*$, of free servers. Let $z = (z_1, z_2)$ with $z_k = y_k^*$ and $z_i = l_k^n(y_k^*)$. Then, because $y_i^* \leq z_i - 1$:

$$0 \leq u^n(z - e_i + e_k) - u^n(z - e_i) \leq u^n(y^* + e_k) - u^n(y^*)$$

where the first inequality is by definition of l_k^n, the second inequality is due to Lemma 1 and to our assumption $y_i^* < l_k^n(y_k^*)$. But this implies that the state $y^* + e_k$ brings more benefit than y^*, which is a contradiction. □

Now, we concentrate on the proof of Lemma 1: We have to consider all the possible actions in four different states, x, $x+e_1$, $x+e_2$ and $x+e_1+e_2$. Thus, it is essential to specify the sets of states reachable from each of these states and the relations within these sets, when a batch of j arrives at the system. Let $x^1 = x + e_1$, $x^2 = x + e_2$ and $\bar{x} = x + e_1 + e_2$. Then, we define the following sets for a given $(x; j)$ with $x_1 + x_2 + 2 \le c$:

$$\begin{aligned}
S^0 &= S(x;j) \cap S(x + e_1 + e_2; j) \\
S^x_\bullet &= \{x\} \\
S^x_{x^1} &= \Big\{(y_1, x_2) : x_1 < y_1 \le \min\{x_1 + j_1, c - x_2\}\Big\} \\
S^x_{x^2} &= \Big\{(x_1, y_2) : x_2 < y_2 \le \min\{x_2 + j_2, c - x_1\}\Big\} \\
S^x &= S^x_\bullet \cup S^x_{x^1} \cup S^x_{x^2} \\
S^{\bar{x}}_\bullet &= \Big\{(x_1 + 1 + j_1, x_2 + 1 + j_2)\Big\} \cap S \\
S^{\bar{x}}_{x^1} &= \Big\{(x_1 + 1 + j_1, y_2) : x_2 + 1 \le y_2 < x_2 + 1 + j_2\Big\} \cap S \\
S^{\bar{x}}_{x^2} &= \Big\{(y_1, x_2 + 1 + j_2) : x_1 + 1 \le y_1 < x_1 + 1 + j_1\Big\} \cap S \\
S^{\bar{x}} &= S^{\bar{x}}_\bullet \cup S^{\bar{x}}_{x^1} \cup S^{\bar{x}}_{x^2}.
\end{aligned}$$

In words, S^0 is the set of states reachable from both x and $x + e_1 + e_2$, and so from $x + e_1$ and $x + e_2$ as well. S^x is the set of states reachable from x, but not from $\bar{x}$, whereas $S^{\bar{x}}$ is the set of states reachable from $\bar{x}$ and not from x. $S^x_\bullet$ is a singleton and it is reachable only from x, whereas $S^{\bar{x}}_\bullet$ is either a singleton or empty set, and reachable only from $\bar{x}$. Sets $S^x_{x^i}$ are reachable from x and x^i, but neither from $\bar{x}$ nor from x^k, $k \ne i$; similarly sets $S^{\bar{x}}_{x^i}$ are reachable from $\bar{x}$ and x^i and not from x and x^k, $k \ne i$. Also, note that sets $S^{\bar{x}}_{x^i}$ are empty if $x_1 + x_2 + 2 + j_i > c$. Lemma 2 summarizes all useful, and also obvious, relations among these sets, which is presented without a proof since all the relations are very easy to verify.

Lemma 2

(1) $S^0 = S(x + e_1; j) \cap S(x + e_2; j)$

(2) $S^x = S(x; j) \backslash S(x + e_1 + e_2; j)$

(3) $S^{\bar{x}} = S(x + e_1 + e_2; j) \backslash S(x; j)$

(4) $S^x_{x^1} \subseteq S(x + e_1; j) \cap S(x; j)$

(5) $S^x_{x^1} \cap S(x + e_2; j) = S^x_{x^1} \cap S(x + e_1 + e_2; j) = \emptyset$

(6) $S^x_{x^2} \subseteq S(x + e_2; j) \cap S(x; j)$

(7) $S^x_{x^2} \cap S(x + e_1; j) = S^x_{x^2} \cap S(x + e_1 + e_2; j) = \emptyset$

(8) $S^{\bar{x}}_{x^1} \subseteq S(x + e_1; j) \cap S(x + e_1 + e_2; j)$

(9) $S^{\bar{x}}_{x^1} \cap S(x + e_2; j) = S^{\bar{x}}_{x^1} \cap S(x; j) = \emptyset$

(10) $S^{\bar{x}}_{x^2} \subseteq S(x + e_2; j) \cap S(x + e_1 + e_2; j)$

(11) $S^{\bar{x}}_{x^2} \cap S(x + e_1; j) = S^{\bar{x}}_{x^2} \cap S(x; j) = \emptyset$

Now we can present the proof of Lemma 1:

Proof. (**Lemma 1**) Assume that u^0 satisfies inequality (2). Also assume that the statement is true for n. We first show that v^n's also satisfy the inequality. We define δ^n such that:

$$\delta^n = v^n(x;j) - v^n(x+e_2;j) - v^n(x+e_1;j) + v^n(x+e_1+e_2;j)$$

So we show that $\delta^n \leq 0$ for all possible actions. Let $y^* = (y_1^*, y_2^*) = y^n(x;j)$ and $y^{*'} = (y_1^{*'}, y_2^{*'}) = y^n(x+e_1+e_2;j)$ so that y^* and $y^{*'}$ are the optimal states to go from states $(x;j)$ and $(x+e_1+e_2;j)$, respectively. Now we differentiate the cases due to possible actions:

Case I: $y^* \in S^0$ and $y^{*'} \in S^0$

In this case, we have $u(y^*) = u(y^{*'})$ by the optimality of y^* and $y^{*'}$. Then we can assume without loss of generality that $y_i^* = y_i^{*'}$ for $i = 1, 2$. Since $S^0 = S(x+e_1;j) \cap S(x+e_2;j)$ by Lemma 2 part (1) and v^n's are optimal:

$$\delta^n \leq u^n(y^*) - u^n(y^*) - u^n(y^*) + u^n(y^*) = 0$$

Case II: $y^* \in S^0$ and $y^{*'} \in S^{\bar{x}}$

Case II.1: $y^{*'} \in S^{\bar{x}}_{x1}$

Since $S^0 = S(x+e_1;j) \cap S(x+e_2;j)$ by Lemma 2 part (1), y^* is a feasible action in state $(x+e_2;j)$, and because $S^{\bar{x}}_{x1} \subseteq S(x+e_1;j)$ by Lemma 2 part (8), $y^{*'}$ is reachable from state $(x+e_1;j)$. Then:

$$\delta^n \leq u^n(y^*) - u^n(y^*) - u^n(y^{*'}) + u^n(y^{*'}) = 0$$

Case II.2: $y^{*'} \in S^{\bar{x}}_{x2}$

This is very similar to Case II.1. We only need to observe that $y^{*'}$ is feasible for state $(x+e_2;j)$ since $S^{\bar{x}}_{x2} \subseteq S(x+e_2;j)$ by Lemma 2 part (10), and y^* is reachable from $(x+e_1;j)$ since $S^0 \subseteq S(x+e_1;j)$ by Lemma 2 part (1).

Case II.3: $y^{*'} \in S^2_{\bullet}$

Since $y^{*'} = (x_1+j_1+1, x_2+j_2+1)$, it cannot be reached from either $(x+e_2;j)$ or $(x+e_1;j)$. Therefore we cannot use the same technique as above. We first observe that the state (y_1^*, j_2) is reachable state $(x+e_2;j)$ and (j_1, y_2^*) is feasible for $(x+e_1;j)$. We set the convention as $\sum_{y=y_1}^{y_2} \alpha(y) = 0$ whenever $y_2 < y_1$ and call this *null* summation. Then

$$\begin{aligned}
\delta^n &\leq u^n(y_1^*, y_2^*) - u^n(y_1^{*'}, y_2^*) - u^n(y_1^*, y_2^{*'}) + u^n(y^{*'}) \\
&= u^n(y_1^*, y_2^*) - u^n(y_1^{*'}, y_2^*) - u^n(y_1^*, y_2^{*'}) + u^n(y^{*'}) \\
&\quad + \sum_{y_1=y_1^*+1}^{y_1^{*'}-1} \left[u^n(y_1, y_2^*) - u^n(y_1, y_2^*) - u^n(y_1, y_2^{*'}) + u^n(y_1, y_2^{*'})\right]
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{y_2=y_2^*+1}^{y_2^{*'}-1} \left[u^n(y_1^*, y_2) - u^n(y_1^*, y_2) - u^n(y_1^{*'}, y_2) + u^n(y_1^{*'}, y_2) \right] \\
& +2 \left(\sum_{y_1=y_1^*+1}^{y_1^{*'}-1} \sum_{y_2=y_2^*+1}^{y_2^{*'}-1} [u(y) - u(y)] \right) \qquad (3) \\
= & \sum_{y_1=y_1^*}^{y_1^{*'}-1} \sum_{y_2=y_2^*}^{y_2^{*'}-1} [u(y) - u(y_1+e_1) - u(y_1+e_2) + u(y_1+e_1+e_2)] \\
\leq & \ 0
\end{aligned}
$$

where the first inequality is due to the optimality of v^n's and feasibility of actions (y_1^*, j_2) and (j_1, y_2^*), in the first equality we add and subtract the same terms and in the second we organize all the terms, finally the last inequality follows from the induction hypothesis. Notice that the last summation cannot be null, because $y_i^* < y_i^{*'}$ for $i = 1, 2$ due to our assumptions $y^* \in S^0$ and $y^{*'} = (x_1 + j_1 + 1, x_2 + j_2 + 1)$.

Case III: $y^* \in S^x$ and $y^{*'} \in S^0$

We need to consider three cases; $y^* \in S^x_{x1}$, $y^* \in S^x_{x2}$ and $y^* = x$, each of which is similar to the corresponding subcase of Case II, and so the details are omitted.

Case IV: $y^* \in S^x$ and $y^{*'} \in S^{\bar{x}}$

For this case, we have 9 subcases, most of which are proven similar to each other. We consider two cases in detail and mention the similarities of the others:

Let $y^* \in S^x_{x1}$ and $y^{*'} \in S^2_1$. Then y^* and $y^{*'}$ are both reachable from $(x + e_1; j)$ and not reachable from $(x + e_2; j)$. Thus we need to add and subtract terms as in equation (3). Now observe that $(y_1^*, y_2^{*'})$ is feasible for $(x + e_2; j)$ and $(y_1^{*'}, y_2^{*'})$ is feasible for $x + e_1$. Then:

$$
\begin{aligned}
\delta^n \leq & \ u^n(y^{*'}) - u^n(y_1^{*'}, y_2^*) - u^n(y_1^*, y_2^{*'}) + u^n(y^{*'}) \\
= & \sum_{y_1=y_1^*}^{y_1^{*'}-1} \sum_{y_2=y_2^*}^{y_2^{*'}-1} [u(y) - u(y_1+e_1) - u(y_1+e_2) + u(y_1+e_1+e_2)] \\
\leq & \ 0
\end{aligned}
$$

by the induction hypothesis.

We employ the same method in the following cases: $y^* \in S^x_{x1}$ and $y^{*'} \in S^{\bar{x}}_\bullet$, $y^* \in S^x_{x2}$ and $y^{*'} \in S^{\bar{x}}_{x2}$, $y^* \in S^x_\bullet$ and $y^{*'} \in S^{\bar{x}}_{x1}$, $y^* \in S^x_\bullet$ and $y^{*'} \in S^{\bar{x}}_{x2}$, $y^* \in S^x_\bullet$ and $y^{*'} \in S^{\bar{x}}_\bullet$. Notice that in all these cases either y^* or $y^{*'}$ is reachable from neither $(x + e_1; j)$ nor $(x + e_2; j)$ or both states can be reached from only one of them. Thus we cannot cancel out the terms, instead we have add and subtract the terms as in (3).

If $y^* \in S^x_{x1}$ and $y^{*'} \in S^{\bar{x}}_{x2}$, then y^* is reachable from $(x + e_1; j)$ and $y^{*'}$ is reachable from $(x + e_2; j)$. Then by optimality of v^n's:

$$
\delta^n \leq u^n(y_1^*, y_2^*) - u^n(y_1^*, y_2^*) - u^n(y_1^{*'}, y_2^{*'}) + u^n(y_1^{*'}, y_2^{*'}) = 0
$$

Similarly, when $y^* \in S_2^1$ and $y^{*'} \in S_1^2$, y^* is reachable from $(x+e_2; j)$ and $y^{*'}$ is reachable from $(x+e_1; j)$.

Thus for all possible cases, $\delta^n \leq 0$. Now we can consider u^{n+1}:

$$\begin{array}{ll}
 & u^{n+1}(x) - u^{n+1}(x+e_2) - u^{n+1}(x+e_1) + u^{n+1}(x+e_1+e_2) \\
= \lambda \sum_{j_1,j_2} p_{j_1,j_2} & [v^n(x;j) - v^n(x+e_2;j) - v^n(x+e_1;j) + v^n(x+e_1+e_2;j)] \\
+x_1\mu_1 & [u^n(x_1-1,x_2) - u^n(x_1-1,x_2+1) - u^n(x) + u^n(x+e_2)] \\
+\mu_1 & [u^n(x) - u^n(x+e_2) - u^n(x) + u^n(x+e_2)] \\
+x_2\mu_2 & [u^n(x_1,x_2-1) - u^n(x) - u^n(x_1+1,x_2-1) + u^n(x+e_1)] \\
+\mu_2 & [u^n(x) - u^n(x) - u^n(x+e_1) + u^n(x+e_1)] \\
+\alpha & [u^n(x) - u^n(x+e_2) - u^n(x+e_1) + u^n(x+e_1+e_2)] \\
\leq & 0
\end{array}$$

where $\alpha = (c - x_1 - x_2 - 2)\mu_2 + (x_1+1)(\mu_2 - \mu_1)$. The terms in the summation are less than or equal to 0 since v^n's are shown to satisfy the inequality, the μ_1 and μ_2 terms are 0 and all other terms are non-positive by the induction hypothesis. Thus, the value functions, u^n, satisfy inequality (2) for all n whenever u^0 does. □

4 Existence of a Preferred Class

We define a preferred class as the class whose customers are always admitted to the system whenever there are available servers. As discussed earlier, this definition leads to different characterizations of preferred class(es) with uniform and mixed batches, which are analyzed seperately in this section.

In determining the preferred class, two different criteria can be considered: one is the relation between the average rewards, $r_i\mu_i$, of the two classes, which is similar to the well-known $c\mu$ rule in the stochastic scheduling literature. Recall that the $c\mu$ rule gives priority to the class with the highest average cost rate, $c_i\mu_i$, where c_i is the holding cost and μ_i is the service rate of class-i customers, so this class is preferred over the others. With the objective of maximizing revenue, this rule translates to the $r\mu$ rule, since the quantity equivalent to $c_i\mu_i$ is the average profit rate of class-i customers, $r_i\mu_i$. With single arrivals, this rule determines the preferred class under certain conditions, but not for all possible parameter values, see Örmeci et al. (1999*b*) for a counterexample. When the batches are uniform, these certain conditions for the $c\mu$ rule to hold are almost the same with single arrivals. However, with mixed batches, the $c\mu$ rule is never guaranteed. In fact, whenever $r_1\mu_1 \geq r_2\mu_2$, class 1 is preferred with both uniform and mixed batches, but for class-2 customers, this cannot be claimed: Consider a firm which has to choose one from two jobs, one of which brings $1,000 profit each month for 12 months and the other with a profit of $1,200 per month for only 3 months. The possibility that the firm will have no job after 3 months works in favor of the longer duration job. Hence the low profit job may be preferred over the high profit one. In other words, the system favors steady returns which can be attained by longer service times although the average return is somewhat lower. The second criterion to determine the preferred class is the relation between the immediate rewards, R_i. Indeed, whenever $R_2 \geq R_1$, class 2 is the preferred class for both uniform and mixed batches, but it is easy to see that this rule does not hold for class 1, since if class-1 customers are slow enough, the optimal policy may reject them even if their immediate rewards are high. We present the conditions for class 1 to be preferred in terms of average rewards, $r_i\mu_i$'s, and for class 2 in terms of immediate rewards, R_i's.

In the proofs of this section, we use, mostly, induction and sample path analysis together. The following u^0, which is briefly mentioned in Section 2, satisfies all the statements, allowing us to apply the induction:

$$u^0(x) = x_1 R_1 + x_2 R_2 \qquad \forall x \in \mathcal{S}. \tag{4}$$

This function corresponds to the assumption that the later rewards of customers, who are still in the system at $n = 0$, are collected at $n = 0$.

Before analyzing the systems with mixed and uniform batches separately, we prove the following result, which applies to both systems:

Lemma 3 *For all $x \in \mathcal{S}$ and for all $n \geq 0$:*

(1) $D^n(i0)(x) \leq R_i$ *for* $i = 1, 2$.

(2) $D^n(12)(x) = -D^n(21)(x) \leq R_1 - R_2$.

Proof. We prove the statements by a sample path analysis.

(1) Assume that system A is in state $x + e_i$ and system B in x in period n. We let system A follow the optimal policy, π, and system B imitate all the decisions of system A. We couple the two systems via the service and interarrival times, i.e., except for the additional customer in system A, all the departure and arrival times are the same in both systems. We note that system B can always imitate system A since it always has at least as many free servers as system A does. Then, the difference in the expected returns of systems A and B is only due to the additional customer in system A:

$$D^n(i0)(x) = u^n(x + e_i) - u^n(x) \leq u^n(x + e_i) - u^n_\pi(x) = R_i.$$

where $u^n_\pi(x)$ is the expected discounted return of system B and R_i is the immediate reward of the additional class-i customer in the system, which will be collected eventually due to the definition of u^0.

(2) Assume that system A starts in state $x + e_1$ and system B starts in $x + e_2$, where we now couple the additional class-1 customer, say customer d_1, in system A with the additional class-2 customer, say customer d_2, as well as all other service and interarrival times, so that, as discussed earlier, if d_1 leaves the system, d_2 also leaves. Then, we can let system A follow the optimal policy and system B imitate all the decisions of system A. Now, again, the difference in the expected discounted returns of system A and B is only due to the addititonal customers in the beginning:

$$D^n(12)(x) = u^n(x + e_1) - u^n(x + e_2) \leq u^n(x + e_1) - u^n_\pi(x + e_2) = R_1 - R_2,$$

with $u^n_\pi(x + e_2)$ the expected discounted return of system B. □

4.1 Mixed batches

In this section, we assume that there exist $j_1 > 0$ and $j_2 > 0$ with $p_{j_1 j_2} > 0$, so we have batches consisting of both classes. Then, whenever a mixed batch arrives at the system, we need to compare at least three actions, having an empty server or having a server work on class-1 or class-2 customer, i.e., we need to compare the values of $u^n(x)$, $u^n(x+e_1)$ and $u^n(x+e_2)$, respectively, when the system state is $(x;j)$ with $j_i > 0$. If $u^n(x+e_i)$ is greater than $u^n(x)$ and $u^n(x+e_k)$, $i \neq k$ for all x, then class i is preferred. Hence, the effect of changing a class-i customer to class k, $D^n(ik)(x)$, is important as well as the effect of an additional class-i customer, $D^n(i0)(x)$, in determining the preferred class. In fact, as we see later in this section, if $D^n(ik)(x) \geq 0$, then $D^n(i0)(x) \geq 0$ so that the non-negativity of $D^n(ik)(x)$ assures class i to be preferred.

We first present the sufficient conditions for class 2 to be preferred:

Theorem 2 *If $R_2 \geq R_1$, then class 2 is the preferred class.*

Proof. Assume $R_2 \geq R_1$. Then, Lemma 3 implies that

$$D^n(21)(x) \geq R_2 - R_1 \geq 0 \qquad \forall x \in \mathcal{S} \quad \forall n.$$

Hence, we only need to show that $D^n(20)(x) \geq 0$ for all x and for all n. For $D^0(20)(x) \geq 0$ for u^0 given by (4). Thus, assume that $D^n(20)(x) \geq 0$ for all x and for n, and consider $n+1$: Let system A be in state $x+e_2$ and system B in state x in period $n+1$. We let system B take the optimal actions $y^* = y^{n+1}(x;j)$, and system A imitate these actions whenever possible, i.e., whenever system B accepts at least one class-2 customer(s) so that $y_2^* > x_2$. Then, both systems end up in the same state. If $y_2^* = x_2$ and $y_1^* > x_1$, then we let system A move to state $y^* + e_2 - e_1$, and if $y_2^* = x_2$ and $y_1^* = x_1$, i.e., $y^* = x$, then system A also remains in its current state $x + e_2$. If the extra customer in system A leaves, which happens with probability μ_2, the two systems couple with a reward of r_2. If there is any other service completion, then the difference between two system remains the same due to the extra class-2 customer in system A:

$$\begin{aligned} D^{n+1}(20)(x) &= u^{n+1}(x+e_2) - u^{n+1}(x) \\ &\geq \lambda \min\{0, D^n(21)(y^*-e_1), D^n(20)(y^*)\} + \mu_2 r_2 \\ &\quad + (c-1)\mu_2 \min_{z\in\mathcal{S}}\{D^n(20)(z)\} \\ &> 0 \end{aligned}$$

where the first inequality is due to coupling and the second follows by the induction hypothesis $D^n(20) \geq 0$ and the fact that $D^n(21) \geq 0$ under the above assumption. Hence, $D^n(20)(x) \geq 0$ and $D^n(21)(x) \geq 0$ for all x and for all n, implying that class-2 customers are preferred. □

Thus, whenever $R_2 > R_1$, class 2 is the preferred class. From this result, we can easily observe the following corollary:

Corollary 1 *Whenever $r_2 \geq r_1$ so that class-2 customers bring higher rewards, and require shorter service times, class-2 customers are preferred.*

We derive a similar condition for class 1 to be preferred. However, this requires some more work:

Lemma 4 *If* $r_1\mu_1 \geq \frac{\lambda+\mu_1+\beta}{\lambda+\mu_2+\beta} r_2\mu_2$, *then for all* $x \in \mathcal{S}$ *and for all* n:

(1) $D^n(10)(x) \geq \max\left\{0, \dfrac{r_2\mu_2 - r_1\mu_1}{\mu_2 - \mu_1}\right\}$.

(2) $D^n(12)(x) \geq 0$.

Proof. We use induction on the number of transitions, n. Both statements are satisfied for u^0 defined by (4). Assume that both are true for n. Notice that $D^n(10)(x) \geq 0$ for all $x \in \mathcal{S}$ by part (1) of the induction hypothesis. Now we have to consider two pairs of systems, one for $D^{n+1}(10)(x)$ and the other for $D^{n+1}(12)(x)$.

(1) Consider the first pair: Assume that system A is in state $x + e_1$ and system B is in x in period $n+1$. We let system B follow the optimal policy, and set $y^* = y^n(x; j)$. System A takes the same action with system B, whenever it is possible, i.e., $y_1^* > x_1$; so that the two systems couple with no difference in reward. If $y_1^* = x_1$ and $y_1^* + y_2^* < c$, then system A goes to state $y^* + e_1$; and if $y_1^* = x_1$ and $y_1^* + y_2^* = c$ so that $y_2^* > x_2$, then system A goes to state $y^* + e_1 - e_2$. With the departure of the additional class-1 customer in system A, the systems again enter the same state but with a return of r_1, whereas all other service completions keep the difference between the two systems the same, which is due to the extra class-1 customer, so that the difference between two systems is at least 0 by the induction hypothesis. Then:

$$\begin{aligned} D^{n+1}(10)(x) &\geq \lambda \min\{0, D^n(10)(y^*), D^n(12)(y^* - e_2)\} + \mu_1 r_1 \\ &\quad + (c\mu_2 - \mu_1) \min_{y \in \mathcal{S}}\{D^n(10)(y)\} \\ &\geq \mu_1 r_1 + (c\mu_2 - \mu_1) \min_{y \in \mathcal{S}}\{D^n(10)(y)\} \end{aligned} \tag{5}$$

where the first inequality is due to coupling and the second due to the induction hypotheses $D^n(10)(x) \geq 0$ and $D^n(12)(x) \geq 0$. If $r_1\mu_1 \leq r_2\mu_2$, then $D^n(10)(x) \geq \frac{r_2\mu_2 - r_1\mu_1}{\mu_2 - \mu_1}$, and so we have to show that $D^{n+1}(10)(x) \geq \frac{r_2\mu_2 - r_1\mu_1}{\mu_2 - \mu_1}$ for all $x \in \mathcal{S}$:

$$\begin{aligned} D^{n+1}(10)(x) &\geq \mu_1 r_1 + (c\mu_2 - \mu_1)\frac{r_2\mu_2 - r_1\mu_1}{\mu_2 - \mu_1} \\ &= -r_1\mu_1 \frac{1 - \lambda - \mu_2 - \beta}{\mu_2 - \mu_1} + r_2\mu_2 \frac{1 - \lambda - \mu_1 - \beta}{\mu_2 - \mu_1} \\ &= \frac{r_2\mu_2 - r_1\mu_1}{\mu_2 - \mu_1} - r_2\mu_2 \frac{\lambda + \mu_1 + \beta}{\mu_2 - \mu_1} + r_1\mu_1 \frac{\lambda + \mu_2 + \beta}{\mu_2 - \mu_1} \\ &\geq \frac{r_2\mu_2 - r_1\mu_1}{\mu_2 - \mu_1} \end{aligned}$$

where the first equality is due to uniformization, and the last inequality follows by the assumption of the theorem. If $r_1\mu_1 > r_2\mu_2$, then we need to show that $D^{n+1}(10)(x) \geq 0$ for all $x \in \mathcal{S}$. By (5) and the induction hypothesis $D^n(10)(x) \geq 0$ for all $x \in \mathcal{S}$, we have:

$$D^{n+1}(10)(x) \geq r_1\mu_1 > 0.$$

Thus, the first statement is true for all $x \in S$ and for all $n \geq 0$.

(2) Now consider the second pair of systems: Let system A$'$ be in state $x + e_1$ and system B$'$ in $x + e_2$ in period $n + 1$. System B$'$ takes the optimal actions, where we set $y^{*'} = y^n(x + e_2; j)$, and we let system A$'$ go to the state $y^{*'} + e_1 - e_2$ in this period, which is always feasible. We, as in Lemma 3, couple the additional class-2 customer, say customer d_2, in system B$'$ with the additional class-1 customer, say customer d_1, as well as all other service and interarrival times. Then, if d_1 leaves the system, which happens with probability μ_1, d_2 also leaves. The departure of d_1 leads the system to couple with a reward of $r_1 - r_2$, the departure of d_2 alone, which happens with probability $\mu_2 - \mu_1$, takes the systems to two different states, $x + e_1$ and x with a reward of $-r_2$ and whenever there is any other transition, both systems continue to have their additional customers so that the difference between the two systems is only due to changing a class-1 customer to class 2:

$$\begin{aligned} D^{n+1}(12)(x) &\geq \lambda D^n(12)\left(y^{*'} - e_2\right) + \mu_1(r_1 - r_2) + (\mu_2 - \mu_1)(-r_2 + D^n(10)(x)) \\ &\quad +(c-1)\mu_2 \min_{y \in S}\{D^n(12)(y)\} \\ &\geq (\lambda + (c-1)\mu_2) \min_{y \in S}\{D^n(12)(y)\} + (\mu_2 - \mu_1) \max\left\{0, \frac{r_2\mu_2 - r_1\mu_1}{\mu_2 - \mu_1}\right\} \\ &\quad +r_1\mu_1 - r_2\mu_2 \\ &\geq \max\{r_1\mu_1 - r_2\mu_2, 0\} \geq 0 \end{aligned}$$

where the first inequality is due to the coupling, the second follows by the induction hypothesis for $D^n(10)(x)$, and the last one is due to the the induction hypothesis for $D^n(12)(x)$. This proves the second part of the lemma. □

Now we can conclude that under the following conditions class-1 customers are preferred:

Proposition 1 *If* $r_1\mu_1 \geq \frac{\lambda+\mu_1+\beta}{\lambda+\mu_2+\beta} r_2\mu_2$*, then class 1 is the preferred class.*

As discussed earlier, class 1 is the preferred class by Lemma 1 whenever the average rewards of both classes are the same.

4.2 Uniform batches

In this section, we assume $\sum_{j_1=0}^{c} p_{j_1 0} + \sum_{j_2=0}^{c} p_{0 j_2} = 1$ with $p_{00} = 0$ so that the batches can have customers of only one class. In fact, this case is very similar to the system with single arrivals, since both systems consider the drawback between having an empty server and a server occupied by a class-i customer in each state, with the only difference that batch arrivals require more states to be considered at each arrival epoch. However, 'being preferred' is a global property so that this difference does not lead to a significant change in determining which class(es) is preferred. In both systems, class i is the preferred class, if $D^n(i0)(x) \geq 0$ for all $x \in S$. The following theorem summarizes the sufficient conditions for each class to be preferred, which are the same for single arrivals with $\lambda_1 = \lambda \sum_{j_1=0}^{c} p_{j_1 0}$ and $\lambda_2 = \lambda \sum_{j_2=0}^{c} p_{0 j_2}$, see Örmeci et al. (1999*b*):

Theorem 3 *Let* $\lambda_1 = \lambda \sum_{j_1=0}^{c} p_{j_1 0}$ *and* $\lambda_2 = \lambda \sum_{j_2=0}^{c} p_{0 j_2}$.

(1) If $R_2 > \frac{\lambda_1}{\lambda_1+\mu_2+\beta}R_1$*, then for all* $x \in S$ *and for all* n*,* $D^n(20)(x) \geq 0$*, hence class 2 is a preferred class.*

(2) If $r_1\mu_1 \geq \frac{\lambda_2}{\lambda_2+\mu_2+\beta}r_2\mu_2$*, then for all* $x \in S$ *and for all* n*,* $D^n(10)(x) \geq 0$*, hence class 1 is a preferred class.*

The proof of this theorem is very similar to those of Theorem 2 and Lemma 4, and of Theorem 2 and Lemma 3 in Örmeci et al. (1999*b*), so it is omitted. The followings can be easily observed:

(1) *If* $\frac{\lambda_1(\mu_2+\beta)}{(\lambda_1+\mu_2+\beta)(\mu_1+\beta)} \leq 1$*, then the preferred class is determined by the* $r\mu$ *rule.*

(2) *Both classes are preferred, if parameter values satisfy :*

$$\frac{\lambda_1(\mu_2+\beta)}{(\lambda_1+\mu_2+\beta)(\mu_1+\beta)} \leq \frac{r_2\mu_2}{r_1\mu_1} \leq \frac{\lambda_2+\mu_2+\beta}{\lambda_2}.$$

(3)*Existence of a preferred class cannot be guaranteed if :*

$$\frac{\lambda_2+\mu_2+\beta}{\lambda_2} < \frac{r_2\mu_2}{r_1\mu_1} < \frac{\lambda_1(\mu_2+\beta)}{(\lambda_1+\mu_2+\beta)(\mu_1+\beta)}.$$

However, we still expect to have a preferred class for all parameter values when the batches are uniform:

Conjecture 1 *There always exists a preferred class for the system with uniform batches.*

5 A remark on systems with $\mu_1 = \mu_2 = \mu$

For these systems, the exponential service rates for both classes are the same, thus we only need to consider the rewards of each class. We can assume without loss of generality that $r_1 \geq r_2$. Then, it can be easily proved, via a sample path argument, that class 1 is preferred. Moreover, concavity of the optimal value functions is also easily assessed by Lemma 1 and the observation that $D^n(12)(x) = \frac{\mu}{\mu+\beta}(r_1 - r_2)$ for all $x \in S$:

Lemma 5 *For all* $x + e_1 + e_2 \in S$*,* u^n *is concave in* x_i *for fixed* x_k*,* $k \neq i$*, i.e.,:*

$$u^n(x) - 2u^n(x+e_i) + u^n(x+2e_i) \leq 0 \quad \forall n \geq 1, \tag{6}$$

whenever the inequality is true for $n = 0$*.*

Proof. Let $i = 1$, the proof is similar for $i = 2$. We can rewrite inequality (6) as follows:

$$\begin{aligned} & u^n(x) - 2u^n(x+e_1) + u^n(x+2e_1) \\ = \; & u^n(x) - u^n(x+e_2) - \frac{\mu}{\mu+\beta}(r_1 - r_2) - u^n(x+e_1) \\ & + u^n(x+e_1+e_2) + \frac{\mu}{\mu+\beta}(r_1 - r_2) \\ = \; & u^n(x) - u^n(x+e_2) - u^n(x+e_1) + u^n(x+e_1+e_2) \leq 0 \end{aligned}$$

by Lemma 1. □

As a result, the optimal policy is a trunk reservation policy with a preferred class: All customers of class 1 are accepted to the system whenever there is an idle server, and class-2 customers are admitted to the system if the number of free servers is greater than a certain number, l, where l is determined easily by using the relation for $D^n(12)(x)$ and Theorem 1. Moreover, this can be easily generalized to finite number of classes, K, with $r_i \geq r_{i+1}$. Then class 1 is still preferred, and there exists thresholds, l_k, on the number of free servers with $l_k \leq l_{k+1}$, $k = 2, ..., K$, which determine the optimal trunk reservation policy completely.

6 A counterexample

In this section, we assume that either all customers of an incoming batch are accepted or they are all rejected. Thus, we have two actions at each arrival epoch so that $a^n(x; j) = 1$ if the batch j is accepted, $a^n(x; j) = 0$ if it is rejected. Obviously, if $x + e_1 j_1 + e_2 j_2 \notin S$, the whole batch is rejected. The corresponding MDP is then as follows:

$$\begin{aligned} v^n(x; j) &= \max\{u^n(x + e_1 j_1 + e_2 j_2), u^n(x)\} \\ u^{n+1}(x) &= x_1\mu_1 r_1 + x_2\mu_2 r_2 + \lambda \sum_{j_1, j_2} p_{j_1 j_2} v^n(x; j) \\ &\quad + x_1\mu_1 u^n(x - e_1) + x_2\mu_2 u^n(x - e_2) \\ &\quad + (c\mu_2 - x_1\mu_1 - x_2\mu_2) u^n(x), \end{aligned}$$

where we set $u^n(y) = -\infty$ if $y \notin S$ to make sure that a batch j is rejected in state x whenever $x + e_1 j_1 + e_2 j_2 \notin S$.

Now we present an example which violates all possible monotonicity conditions: The system has 5 servers. The parameter values are as follows, before normalization: $\lambda = 29$, $p_{01} = 18/29$, $p_{10} = 1/29$, $p_{50} = 10/29$, $\mu_1 = \mu_2 = 2$, $r_1 = 10$, $r_2 = 3$ and $\beta = 0$. Note that, when $\beta = 0$, there is no difference in collecting the rewards in the beginning or in the end of service, and in this case we assume that they are collected in the beginning of service. Moreover, $\mu_1 = \mu_2$, so we do not need to differentiate the classes of customers in the system. Hence, we denote the system state by x, where x is the number of customers in the system. The optimal admission policy of this particular example is in Table 1, with $a_{j_1 j_2} = a(x; j)$.

x	a_{10}	a_{01}	a_{50}
0	0	0	1
1	1	0	0
2	1	0	0
3	1	1	0
4	1	1	0

Table 1: Optimal admission policy for the counter example

For this example single class-1 customers are rejected when there are no customers in the system and accepted in all other states and single class-2 customers are rejected when there are less than 3 customers in the system and accepted otherwise. This example shows that the optimal policy cannot be monotone with respect to the number of customers in the system.

7 Generalizations and Future Research

Our results still hold when the arrival process is a Markov arrival process (MAP) instead of a Poisson process, see Örmeci (1998) for details. MAPs are defined by Asmussen & Koole (1993) who have also shown that any independent arrival process with multiple classes of customers can be approximated arbitrarily closely by an MAP. The MAPs bring two main benefits: One is to be able to model the departure process of most queueing systems with exponentially distributed sojourn time, which can then be used as input to the loss system we consider. Secondly, the MAPs can model many generalizations of the exponential distribution, e.g., phase-type renewal process and Markov Modulated Poisson Process (see Hordijk & Koole (1993)).

We can also consider the system under a general arrival process, which can be modeled as an embedded MDP at arrival times. This will somewhat generalize the MAPs.

In this paper, we assume that customers of each class bring fixed rewards. Örmeci, Burnetas & Emmons (1999*a*) have considered random rewards for each class with single arrivals, where optimal policy is shown to be a threshold policy. Under random rewards, one class cannot be specified as preferred, since the reward of each customer, even if they are from the same class, varies. However, it is shown that there exist preferred customers under certain conditions, where preferred customers of each class are specified as the customers who bring at least a certain amount of reward.

The admission of customers into the system can also be controlled via pricing. Thus instead of rejecting the customers, we can propose a price, which may or may not depend on the state of the system, for which we are willing to serve the incoming customers. This kind of control has been considered in the context of social optimization for different queueing systems, see e.g., Naor (1969), Lippman & Stidham (1977) and Xu & Shantikumar (1993). The only study on loss systems with pricing, we are aware of, is Miller & Buckman (1987) who consider a static transfer pricing problem for one class in an $M/M/c/c$ system which serves as a model of a service department.

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