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Abstract

A load-balanced network with two queues Q_1 and Q_2 is considered. Each queue receives a Poisson stream of customers at rate λ_i, $i = 1, 2$. In addition, a Poisson stream of rate λ arrives to the system; the customers from this stream join the shorter of two queues. After being served in the ith queue, $i = 1, 2$, customers leave the system with probability $1 - p_i^*$, join the jth queue with probability $p(i, j)$, $j = 1, 2$ and choose the shortest of two queues with probability $p(i, \{1, 2\})$. We establish necessary and sufficient conditions for stability of the system.

Keywords: load-balanced network, stability, Markov chain, Lyapunov function

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1 Description and history of the model

We consider the following system with two queues, Q_1 and Q_2. Each queue Q_i receives a Poisson stream of customers at rate λ_i ($i = 1, 2$). In addition to this, a Poisson stream of rate λ arrives into the system. The customers belonging to this stream join the queue that is shorter at the time of arrival, breaking the ties at random. All three streams are independent. The service times are all independent and exponentially distributed with mean 1. After being served at the ith queue ($i = 1, 2$), a customer leaves the system with probability $1 - p_i^*$, remains at the ith queue with probability $p(i, i)$, goes to the other queue with probability $p(i, j)$, $j \neq i$, and chooses the shorter of the two with probability $p(i, \{1, 2\})$, again breaking the ties at random. The return probabilities are $p_i^* = p(i, 1) + p(i, 2) + p(i, \{1, 2\})$.

We are interested in necessary and sufficient conditions for stability of this system.

This model is called a load-balanced network and is determined by nine parameters λ_1, λ_2, λ, $p(1,1)$, $p(1,2)$, $p(1,\{1,2\})$, $p(2,1)$, $p(2,2)$, $p(2,\{1,2\})$. If $\lambda = p(i,\{1,2\}) = 0$, it becomes a standard Jackson network with two nodes. In general it may be viewed as a Jackson-type feedback version of the classical problem of joining the shortest of two queues, see e.g. [1, 3]. Models with J servers but without feedback have been analysed in [4, 5]. In the model of [4] the arrival stream is a collection of independent Poisson flows ξ_A of rate λ_A, for each non-empty set $A \subseteq \{1, 2, \ldots, J\}$. Customers from flow ξ_A choose the server from A with the shortest queue breaking the ties at random. The service times are independent and exponentially distributed with rate μ_i for server i. After being served, the customers leave the system. The necessary and sufficient condition for stability of this queueing system is $\sum_{B\subseteq A} \lambda_B < \sum_{i\in A} \mu_i$ for all A.

A similar model with feedback was introduced in [9]. In a simplified version of this model, the exogenous arrival stream and the service times are as above, but customers may re-enter the system after being served. Namely, on being processed by server i, a customer inspects the queues from set A with probability $p(i, A)$ and chooses the shortest of them. The probability of leaving the system is $1-p_i^*$, where $p_i^* = \sum_{A\subseteq\{1,\ldots,J\}} p(i, A) \leq 1$. A sufficient condition for stability is $\sum_{B\subseteq A} (\lambda_B + \sum_i p(i, B)) < \sum_{i\in A} \mu_i$ for all A. The authors of [9] proposed also a necessary condition and conjectured that for $J = 2$ it suffices for stability. The result of this paper confirms this conjecture.

Another aspect of load-balanced models was studied in [6, 7, 8, 9, 10]. Here, the so-called mean-field limit was analysed; the main goal being a super-exponential decay of equilibrium tail distributions.

The organisation of this paper is as follows. In Section 2 we state the main results Theorems 1 and 2, and in Section 3 we give the proofs. The proofs are based on the method of Lyapunov functions for Markov processes, in the form developed in the book [2].

2 Results

Theorem 1 *The network is stable iff one of the following conditions holds:*

$$\begin{cases} \lambda_1 + p(2,1) + p(1,1) < 1, \\ \lambda_2 + p(1,2) + p(2,2) < 1, \\ \lambda_1 + \lambda_2 + \lambda + p_1^* + p_2^* < 2, \end{cases} \tag{1}$$

or

$$\begin{cases} \lambda_1 + p(2,1) + p(1,1) \geq 1, \\ p(2,1)[\lambda_1 + \lambda_2 + p_1^* - 1] + (1 - p_2^*)[\lambda_1 + p(1,1) - 1] < 0, \end{cases} \tag{2}$$

or

$$\begin{cases} \lambda_2 + p(1,2) + p(2,2) \geq 1, \\ p(1,2)[\lambda_1 + \lambda_2 + p_2^* - 1] + (1 - p_1^*)[\lambda_2 + p(2,2) - 1] < 0. \end{cases} \tag{3}$$

To clarify the meaning of these conditions, observe that the left-hand sides of the inequalities (1) are the rates of the so-called *dedicated* traffic to queues Q_1, Q_2, and $\{Q_1, Q_2\}$, respectively. The dedicated traffic is formed by customers which join a given set of queues regardless of the state of the network. Thus the system (1) requires that the mean traffic to Q_i decreases whenever Q_i is longer than the other queue; and that the total mean traffic to the system decreases, when both Q_1 and Q_2 are not empty. If $p_1^* = p_2^* = 0$, this system coincides with the inequalities necessary and sufficient for stability of the model from [4]. In our case, system (1) is sufficient but not necessary for stability.

Consider now the inequalities (2). The first inequality implies that the mean traffic to Q_1 increases when Q_1 is longer than Q_2 and Q_2 is *not empty.* It also increases, of course, when Q_1 is shorter than Q_2, due to the load-balancing. How can the system be stable in this case ? If Q_1 grows mainly because of exogenous customers, then stability is impossible. But if Q_1 grows mainly because of customers coming from Q_2, then stability is possible, if we guarantee that Q_2 empties sufficiently often: then Q_2 will not send too many customers to Q_1. To be more precise, consider a Jackson network with independent Poisson flows of intensities λ_1 and $\lambda_2 + \lambda$ arriving to queues Q_1 and Q_2 respectively. Let the service times be independent exponential with mean 1 and the routing matrix be

$$\begin{bmatrix} p(1,1) & p(1,2) + p(1,\{1,2\}) \\ p(2,1) & p(2,2) + p(2,\{1,2\}) \end{bmatrix}.$$

Observe that if $\lambda_1 + p(1,1) + p(2,1) - 1 \geq 0$, then the Jackson network is stable if and only if the second inequality of (2) holds. Now assume that in our load-balanced network the mean traffic to Q_1 increases when Q_1 is longer than Q_2 and Q_2 is not empty (i.e. the first inequality of (2) holds). Then the system is stable if and only if the above-mentioned Jackson network is stable.

System (3) is symmetric to (2) and may be analysed in the same way.

The following theorem proves a conjecture made in [9].

Theorem 2 *The network is stable iff the system*

$$\left\{\begin{array}{l}\lambda_1+\rho_1 p(1,1)+\rho_2 p(2,1)<\rho_1,\\ \lambda_2+\rho_1 p(1,2)+\rho_2 p(2,2)<\rho_2,\\ \lambda_1+\lambda_2+\lambda+\rho_1 p_1^*+\rho_2 p_2^*-\rho_1-\rho_2=0\end{array}\right. \tag{4}$$

has a solution $(\rho_1,\rho_2)\in(0,1)^2$.

The meaning of system (4) is as follows. Suppose that the stationary distribution exists, and ρ_1, ρ_2 are the stationary traffic rates at queues Q_1 and Q_2. Then the left-hand sides of the first and the second inequalities in (4) are the rates of the traffic dedicated to Q_1 and Q_2. This traffic will come to the queues regardless of their lengths. On the other hand, the right-hand sides of these inequalities are the total traffic rates. The third equation in this system gives the balance of arrival and departure rates.

3 Proofs

We represent the network as a time-continuous random walk $(X(t),Y(t))$ on $\mathbf{Z}_+^2$, where $X(t)$ and $Y(t)$ are the numbers of customers at Q_1 and Q_2, respectively. It has six regions of spatial homogeneity: two angles $\{x>y>0\}$ and $\{y>x>0\}$, three rays $\{y=0,x>0\}$, $\{x=0,y>0\}$, $\{x=y>0\}$ and the point $(0,0)$. Consider an embedded Markov chain $\mathcal{L}$ with step transition probabilities $p_{\alpha,\beta}=\nu_{\alpha,\beta}/a(\alpha)$, $\alpha,\beta\in\mathbf{Z}^2$. For $\|\alpha-\beta\|=1$, the numbers $\nu_{\alpha,\beta}$ are indicated on Figure 1. We have also

$$\nu_{\alpha,\alpha}=\left\{\begin{array}{ll}p(1,1)+p(2,2)+p(2,\{1,2\}) & \text{if } \alpha\in\{x>y>0\},\\ p(1,1)+p(2,2)+p(1,\{1,2\}) & \text{if } \alpha\in\{y>x>0\},\\ p(1,1)+p(2,2) & \\ \quad+(p(1,\{1,2\})+p(2,\{1,2\}))/2 & \text{if } \alpha\in\{x=y>0\},\\ p(1,1) & \text{if } \alpha\in\{y=0,x>0\},\\ p(2,2) & \text{if } \alpha\in\{x=0,y>0\}.\end{array}\right.$$

For $\|\alpha-\beta\|>1$, $\nu_{\alpha,\beta}=0$. The numbers $a(\alpha)=\sum_\beta \nu_{\alpha,\beta}$ are normalising factors. Set

$$\begin{aligned} a &:= a(\alpha)=\sum_\beta \nu_{\alpha,\beta}=\lambda_1+\lambda_2+\lambda+2, \quad \text{if } \alpha=(x,y),\ x,y\neq 0;\\ a' &:= a(\alpha)=\sum_\beta \nu_{\alpha,\beta}=\lambda_1+\lambda_2+\lambda+1, \quad \text{if } \alpha=(x,0) \text{ or } (0,y),\ x,y>0.\end{aligned}$$

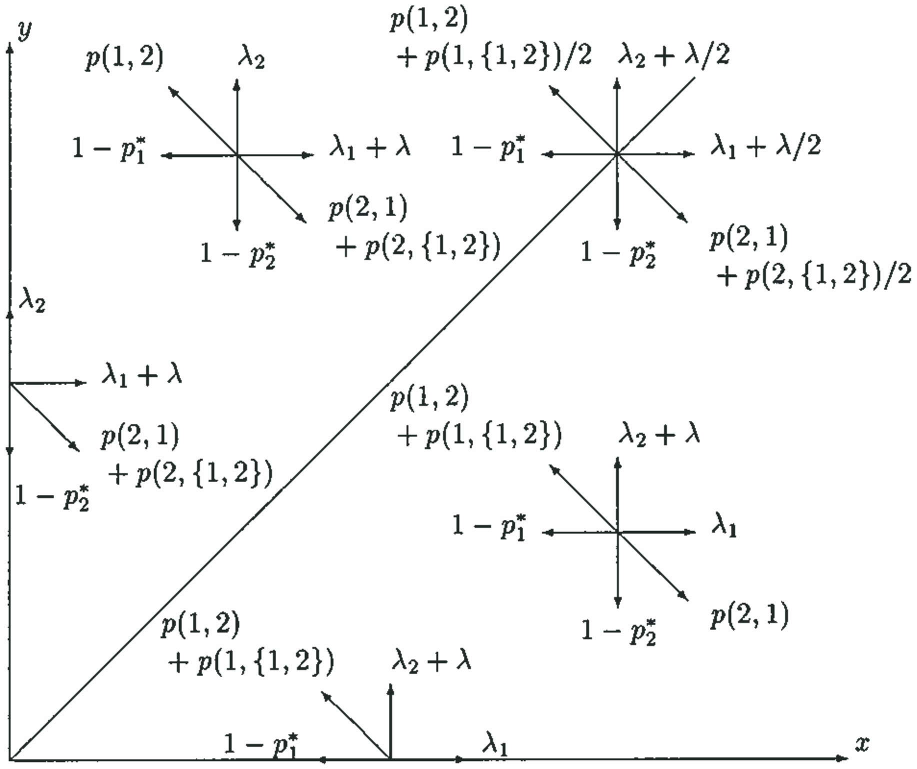

Figure 1: Random walk on $\mathbf{Z}_+^2$

It is well-known that the network is stable if and only if the Markov chain $\mathcal{L}$ is ergodic.

Let us denote the mean jump vector of $\mathcal{L}$ in angle $\{x > y > 0\}$ by (E_x^1, E_y^1), and in angle $\{y > x > 0\}$ by (E_x^2, E_y^2). We also denote by (E_x', E_y') the mean jump vector from ray $\{y = 0, x > 0\}$ and by (E_x'', E_y'') from $\{x = 0, y > 0\}$. Then

$$
\begin{aligned}
E_x^1 &= (\lambda_1 + p(2,1) + p(1,1) - 1)/a, \\
E_y^1 &= (\lambda_2 + \lambda + p_1^* + p_2^* - 1 - p(2,1) - p(1,1))/a, \\
E_x^2 &= (\lambda_1 + \lambda + p_1^* + p_2^* - 1 - p(2,2) - p(1,2))/a, \\
E_y^2 &= (\lambda_2 + p(1,2) + p(2,2) - 1)/a, \\
E_x' &= (\lambda_1 + p(1,1) - 1)/a', \\
E_y' &= (\lambda_2 + \lambda + p_1^* - p(1,1))/a', \\
E_x'' &= (\lambda_1 + \lambda + p_2^* - p(2,2))/a',
\end{aligned}
$$

$$E''_y = (\lambda_2 + p(2,2) - 1)/a'.$$

The mean jump vector from ray $\{x = y > 0\}$ is $1/2(E^1_x + E^2_x, E^1_y + E^2_y)$. Note also that

$$E^1_x + E^1_y = E^2_x + E^2_y.$$

Systems (1), (2) and (3) are equivalent to the following systems, respectively:

$$\begin{cases} E^1_x < 0, \\ E^2_x < 0, \\ E^1_x + E^1_y = E^2_x + E^2_y < 0, \end{cases} \tag{5}$$

$$\begin{cases} E^1_x \geq 0, \\ E^1_x E'_y - E'_x E^1_y < 0, \end{cases} \tag{6}$$

$$\begin{cases} E^2_y \geq 0, \\ E^2_y E''_x - E^2_x E''_y < 0. \end{cases} \tag{7}$$

Proof of Theorem 1. We will prove ergodicity of $\mathcal{L}$ if one of the systems (5), (6) or (7) holds and establish non-ergodicity otherwise.

Sufficiency. To prove ergodicity, we will use Theorem 2.2.3 (Foster's criterion) from [2]. Accordingly, Markov chain $\mathcal{L}$ is ergodic iff there exists a positive function $f(x, y)$ on $\mathbf{Z}^2_+$, a number $\varepsilon_0 > 0$ and a finite set $A \in \mathbf{Z}^2_+$, such that

$$Ef(x + \theta_x, y + \theta_y) - f(x, y) < -\varepsilon_0, \quad \text{for all } (x, y) \in \mathbf{Z}^2_+ \setminus A, \tag{8}$$

where (θ_x, θ_y) is a random vector distributed as a one-step jump of the chain $\mathcal{L}$ from the state (x, y).

Assume that (5) holds. Then the function $f(x, y) = \sqrt{x^2 + y^2}$ satisfies Foster's criterion. By Lemma 3.3.3 from [2],

$$Ef(x+\theta_x, y+\theta_y) - f(x, y) = \frac{xE\theta_x + yE\theta_y}{f(x, y)} + o(1), \quad \text{as } x^2 + y^2 \to \infty. \tag{9}$$

If $x > y > 0$, then $E\theta_x = E^1_x < 0$, $E\theta_x + E\theta_y = E^1_x + E^1_y < 0$, and we have

$$Ef(x + \theta_x, y + \theta_y) - f(x, y) \leq \frac{y(E^1_x + E^1_y)}{y\sqrt{2}} + o(1) < -\varepsilon_1 \tag{10}$$

for some $\varepsilon_1 > 0$ and all pairs (x, y) with the sum $x^2 + y^2$ sufficiently large.

Assume now that $x > 0$ and $y = 0$. Note that condition $E_x^1 < 0$ implies $E_x' < 0$, since $E_x' = E_x^1 a/a' - p(2,1)/a'$. Then

$$Ef(x+\theta_x, y+\theta_y) - f(x,y) \leq \frac{xE_x'}{x\sqrt{2}} + o(1) < -\varepsilon_2 \tag{11}$$

for some $\varepsilon_2 > 0$ and all x sufficiently large.

The case $y > x$ is symmetric to $y < x$, and (8) is verified similarly. Finally, it is easy to check (8) on the ray $\{x = y > 0\}$ using again (9) and the fact that $(E\theta_x, E\theta_y) = 1/2(E_x^1 + E_x^2, E_y^1 + E_y^2)$. Then Foster's criterion applies and the chain is ergodic.

Assume now that (6) holds. It implies the following inequalities:

$$\begin{aligned}
&E_x^1 + E_y^1 \\
&\quad = (\lambda_1 + \lambda_2 + \lambda + p_1^* + p_2^* - 2)/a \\
&\quad \leq (\lambda_1 + \lambda_2 + \lambda + p_1^* - 1)/a + (1 - p_2^*)(\lambda_1 + p(1,1) - 1)/(p(2,1)a) < 0
\end{aligned}$$

$$\begin{aligned}
E_y^1 &= (E_x^1 + E_y^1) - E_x^1 < 0, && (12)\\
E_x^2 &= E_x^1 + (\lambda + p(2,\{1,2\}) + p(1,\{1,2\}))/a > 0, && (13)\\
E_y^2 &= E_y^1 - (\lambda + p(1,\{1,2\}) + p(2,\{1,2\}))/a < 0, && (14)\\
E_x' &< (E_x^1 E_y')/(E_y^1) < 0, && (15)\\
E_y'' &= E_y^2 a/a' - p(1,2) < 0. && (16)
\end{aligned}$$

We will construct the function $f(x,y) = \sqrt{ux^2 + vy^2 + wxy}$ with an appropriate choice of $u, v > 0$, $uv > w^2/4$, satisfying (8). First choose $u, w > 0$ such that

$$\begin{aligned}
2uE_x^1 + wE_y^1 &< 0, && (17)\\
2uE_x' + wE_y' &< 0. && (18)
\end{aligned}$$

This means

$$\frac{E_y^1}{E_x^1} < -\frac{2u}{w} < \frac{E_y'}{E_x'}.$$

Since $E_x^1 E_y' - E_y^1 E_x' < 0$, this choice is possible. Next, let us take $v > w^2/(4u)$, such that

$$\begin{aligned}
&2uE_x^1 + w(E_x^1 + E_y^1) + 2vE_y^1 < 0, && (19)\\
&wE_x^2 + 2vE_y^2 < 0, && (20)\\
&2uE_x^2 + w(E_x^2 + E_y^2) + 2vE_y^2 < 0, && (21)\\
&wE_x'' + 2vE_y'' < 0. && (22)
\end{aligned}$$

This choice is possible due to the inequalities (12), (14) and (16). Then the above function $f(x,y)$ satisfies (8). By Lemma 3.3.3 from [2]

$$Ef(x+\theta_x, y+\theta_y) - f(x,y) = \frac{x(2uE\theta_x + vE\theta_y) + y(wE\theta_x + 2vE\theta_y)}{2f(x,y)} + o(1),$$

as $x^2+y^2 \to \infty$. If $x > y > 0$, then $(\theta_x, \theta_y) = (E_x^1, E_y^1)$ and by (17) and (19)

$$Ef(x+\theta_x, y+\theta_y) - f(x,y) \le \frac{y(2uE_x^1 + w(E_x^1 + E_y^1) + 2vE_y^1)}{y\sqrt{(u+v+w)}} + o(1) < -\varepsilon_1$$

for some $\varepsilon_1 > 0$ and x^2+y^2 sufficiently large. The same is true for $y > x > 0$ by (20) and (21). To check (8) in the cases $y = 0$ and $x = 0$, we use (18) and (22) respectively. Thus the chain is ergodic.

System (7) is symmetric to (6) and the proof is analogous.

Necessity. We prove non-ergodicity if none of systems (5), (6) or (7) holds. Here we apply Theorem 2.2.6 from [2]. Namely for Markov chain $\mathcal{L}$ to be non-ergodic, it is sufficient that there exist a function $f(x,y)$ on $\mathbf{Z}_+^2$ and a constant $C > 0$ such that

$$Ef(x+\theta_x, y+\theta_y) - f(x,y) \ge 0 \quad \text{for all } (x,y) \in \{(x,y) : f(x,y) > C\}, \tag{23}$$

where the sets $\{(x,y) : f(x,y) > C\}$ and $\{(x,y) : f(x,y) < C\}$ are not empty.

Let us first check non-ergodicity under assumption

$$E_x^1 + E_y^1 = E_x^2 + E_y^2 \ge 0.$$

Set $f(x,y) = x + y$. If $x, y \ne 0$, then

$$Ef(x+\theta_x, y+\theta_y) - f(x,y) = E_x^1 + E_y^1 = E_x^2 + E_y^2 \ge 0.$$

If $y = 0$, $x > 0$,

$$\begin{aligned} Ef(x+\theta_x, y+\theta_y) - f(x,y) &= E_x' + E_y' \\ &= E_x^1 a/a' - p(2,1)/a' + E_y^1 a/a' + p(2,1)/a' + (1-p_2^*)/a' > 0. \end{aligned}$$

Finally, if $x = 0$, $y \ne 0$,

$$\begin{aligned} Ef(x+\theta_x, y+\theta_y) - f(x,y) &= E_x'' + E_y'' \\ &= E_x^2 a/a' + p(1,2)/a' + (1-p_1^*)/a' + E_y^2 a/a' - p(1,2)/a' > 0. \end{aligned}$$

So, $f(x,y)$ satisfies (23) and the chain is non-ergodic.

Next, assume that

$$\begin{cases} E_x^1 \geq 0 \\ E_x^1 E_y' - E_y^1 E_x' \geq 0. \end{cases} \tag{24}$$

We shall restrict ourselves to the case $E_x^1 + E_y^1 < 0$, since the opposite case has been already considered. Here we have $E_y^1 < 0$. We may also omit the case $E_x^1 = 0$. (In fact, $E_x^1 = 0$ implies $E_x' = -p(2,1)/a' < 0$. Then by the second inequality of (24) $E_y^1 \geq 0$, which yields $E_x^1 + E_y^1 \geq 0$.)

It was proved in (13) and (14) that the assumptions $E_x^1 > 0$ and $E_y^1 < 0$ imply $E_x^2 > E_x^1 > 0$ and $E_y^2 < E_y^1 < 0$. Let us introduce a linear function

$$f(x,y) = -E_y^1 x + E_x^1 y.$$

Then for $x > y > 0$,

$$Ef(x+\theta_x, y+\theta_y) - f(x,y) = f(E_x^1, E_y^1) = 0.$$

If $x > 0$, $y = 0$, then by the second inequality of (24)

$$Ef(x+\theta_x, y+\theta_y) - f(x,y) = f(E_x', E_y') \geq 0.$$

If $y > x > 0$, since $E_x^1 + E_y^1 = E_x^2 + E_y^2 < 0$ and $E_x^2 > E_x^1$, we have

$$\begin{aligned} Ef(x+\theta_x, y+\theta_y) - f(x,y) &= f(E_x^2, E_y^2) \\ &= -E_y^1 E_x^2 + E_x^1(E_y^2 + E_x^2 - E_x^2) \\ &= (-E_y^1 - E_x^1)(E_x^2 - E_x^1) > 0. \end{aligned}$$

Moreover, if $x = 0$, $y > 0$

$$\begin{aligned} &Ef(x+\theta_x, y+\theta_y) - f(x,y) = f(E_x'', E_y'') \\ &= -E_y^1(E_x'' - E_x^2 a/a') + E_x^1(E_y'' - E_y^2 a/a') + f(E_x^2, E_y^2)a/a' \\ &\geq -(E_x^1 + E_y^1)p(1,2)/a' - E_y^1(1-p_1^*)/a' > 0. \end{aligned}$$

Thus (23) holds and non-ergodicity follows.

The case

$$\begin{cases} E_y^2 \geq 0 \\ E_y^2 E_x'' - E_y'' E_x^2 \geq 0 \end{cases}$$

is symmetric to the previous one and may be treated similarly.

Proof of Theorem 2. The proof is straightforward and is based on Theorem 1 and simple geometric considerations. It is convenient to split it into three parts. First, we assume that $\lambda_1 + p(1,1) + p(2,1) - 1 < 0$, $\lambda_2+p(1,2)+p(2,2)-1 < 0$ and prove that (4) has a solution $(\rho_1, \rho_2) \in (0,1)^2$ if and only if (1) holds. Then we assume that $\lambda_1 + p(1,1) + p(2,1) - 1 \geq 0$ $[\lambda_2 + p(1,2) + p(2,2) - 1 \geq 0]$ and prove that (4) has a solution $(\rho_1, \rho_2) \in (0,1)^2$ if and only if (2) [respectively (3)] holds.

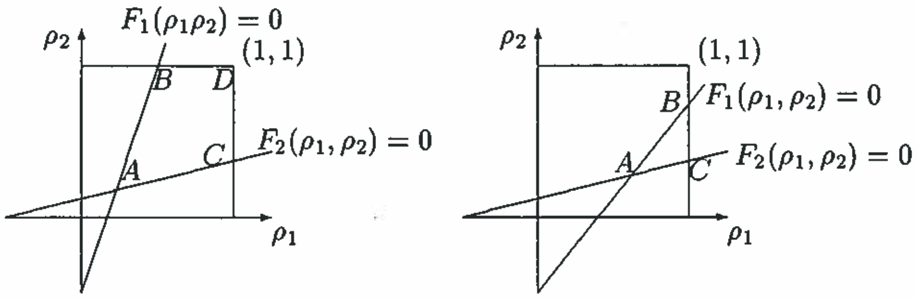

Figure 2: (a) and (b)

Assume that

$$\begin{aligned} &\lambda_1 + p(1,1) + p(2,1) - 1 < 0, \\ &\lambda_2 + p(1,1) + p(2,1) - 1 < 0. \end{aligned} \tag{25}$$

Let us draw the straight lines $F_1(\rho_1, \rho_2) = 0$ and $F_2(\rho_1, \rho_2) = 0$ on the plane of (ρ_1, ρ_2), where

$$\begin{aligned} F_1(\rho_1, \rho_2) &= \lambda_1 + \rho_1 p(1,1) + \rho_2 p(2,1) - \rho_1, \\ F_2(\rho_1, \rho_2) &= \lambda_2 + \rho_1 p(1,2) + \rho_2 p(2,2) - \rho_2. \end{aligned}$$

They intersect with the straight lines $\rho_1 = 1$ and $\rho_2 = 1$ at the points B and C respectively, where

$$B = \Big(1\,,\, \frac{\lambda_1 + p(2,1)}{1 - p(1,1)}\Big), \quad C = \Big(\frac{\lambda_2 + p(1,2)}{1 - p(2,2)}\,,\, 1\Big).$$

The second coordinate of B and the first coordinate of C are less than 1 due to the assumptions (25). Moreover, the straight lines $F_1(\rho_1, \rho_2) = 0$ and $F_2(\rho_1, \rho_2) = 0$ have a point of intersection $A = (\rho_1(A), \rho_2(A)) \in (0,1)^2$. The domain of $(0,1)^2$, where $F_1(\rho_1, \rho_2) < 0$ and $F_2(\rho_1, \rho_2) < 0$, is the quadrangle $ABCD$, where $D = (1,1)$, see Figure 2 (a). Let us introduce the function

$$F(\rho_1, \rho_2) = \lambda_1 + \lambda_2 + \lambda + \rho_1 p_1^* + \rho_2 p_2^* - \rho_1 - \rho_2.$$

Note that

$$F(\rho_1(A), \rho_2(A)) = \lambda + \rho_1(A)p(1, \{1, 2\}) + \rho_2(A)p(2, \{1, 2\}) > 0.$$

Let us suppose that (1) holds. Then

$$F(\rho_1(D), \rho_2(D)) = F(1, 1) = \lambda_1 + \lambda_2 + p_1^* + p_2^* - 2 < 0.$$

So, since $F(\rho_1(A), \rho_2(A)) > 0$ and $F(\rho_1(D), \rho_2(D)) < 0$, then in some point $(\rho_1, \rho_2) \in ABCD$ $F(\rho_1, \rho_2) = 0$ and this point is a solution of (4).

Suppose now that (4) has a solution in the unit square. Then there is a point $(\rho_1, \rho_2) \in ABCD$, where $F(\rho_1, \rho_2) = 0$. Then the straight line $F(\rho_1, \rho_2) = 0$ crosses two of four segments AB, BC, CD and AD. This pair can not be AB and BC or AC and CD because the line $F(\rho_1, \rho_2) = 0$ forms an angle $\gamma > \pi$ with the positive direction of the ρ_1-axis (this is easily seen from its definition). Then $F(\rho_1, \rho_2) = 0$ crosses AB and AC or BD and BC. It divides the plane into two parts and in both cases A and D lie in different parts. Since $F(\rho_1(A), \rho_2(A)) > 0$, then $F(\rho_1(D), \rho_2(D)) < 0$ and this is exactly the third inequality of (1).

A similar argument can be performed in case of system (2) (it is illustrated on Figure 2 (b)) and (3).

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References

[1] I.J. Adan, J. Wessels and W.H.M. Zijm, Analysis of the asymmetric shortest queue problem, Queueing Systems 8 (1991) 1–58.

[2] G. Fayolle, V.A. Malyshev and M.M. Menshikov, *Constructive Theory of Markov Chains* (Cambridge Univ. Press, 1995).

[3] L. Flatto and H. McKean, Two queues in parallel, Comm. Pure Appl. Math., 30 (1977) 255–263.

[4] R. Foley and D. McDonald, Join the shortest queue: stability and exact asymptotics, Preprint, Georgia Institute of Technology and University of Ottawa (1999).

[5] S. Foss and N. Chernova, On the stability of partially accessible queue with state-dependent routing, Queueing Systems 29 (1998) 55–73.

[6] J.B. Martin and Yu.M. Suhov, Fast Jackson networks, to appear in Ann. Appl. Prob. (1999).

[7] M. Mitzenmacher, *The Power of Two Choices in Randomized Load Balancing*, PhD Thesis, University of California, Berkeley (1996).

[8] M. Mitzenmacher and B. Voecking, The asymptotics of selecting the shortest of two, improved, Technical Report, Harvard University (1999).

[9] Yu.M. Suhov and N.D. Vvedenskaya, Fast Jackson-type networks with dynamic routing, Preprint (1999).

[10] N.D. Vvedenskaya, R.L. Dobrushin and F.I. Karpelevich, A queueing system with selection of the shortest of two queues: an asymptotical approach, Problems of Information Transmission, 32 (1996) 15–27.