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RECONSTRUCTING A 2-COLOR SCENERY IN POLYNOMIAL TIME BY OBSERVING IT ALONG A SIMPLE RANDOM WALK PATH WITH HOLDING

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Abstract

Let $\{\xi(k)\}_{k \in \mathbb{Z}}$ be a 2-color random scenery, that is a random coloration of $\mathbb{Z}$ in two colors, such that the $\xi(k)$'s are i.i.d. Bernoulli variables with parameter $\frac{1}{2}$. Let $\{S(k)\}_{k \in \mathbb{N}}$ be a symmetric random walk starting at 0 such that $P(S(1) \in \{0, 1, -1\}) = 1$ and $P(S(1) = 0) \neq 1$. Our first result shows that a.s., $\xi \circ S$ (the sequence $\xi(S(0)), \xi(S(1)), \xi(S(2)), ...$) determines ξ up to translation and reflection. Thus, this gives a positive answer to the question of Harry Kesten whether by observing the scenery ξ along the random walk path S, we can a.s. reconstruct ξ up to translation and reflection if the random walk is with holding. Our second result shows that with high probability, one can reconstruct a finite piece of the scenery ξ of length l located close to the origin if one is given only a power of l observations of ξ along the path of S.

1 Introduction

For the history of the scenery distinguishing problem and the scenery reconstruction problem we refer the reader to the survey paper [2] by Harry Kesten as well as to the introduction to [4]. Here we are merely going to give a very brief and incomplete introduction and mention who posed the two problems which we solve in this paper. Let us give a definition: A scenery will be defined to be a function from $\mathbb{Z}$ to $\{0, 1\}$. Let ξ and $\bar{\xi}$ be two sceneries. We say that ξ and $\bar{\xi}$ are equivalent iff there exists $a \in \mathbb{Z}$ and $b \in \{-1, 1\}$ such that for all $k \in \mathbb{Z}$ we have that $\xi(k) = \bar{\xi}(a + bk)$. In this case we write $\xi \approx \bar{\xi}$.

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In other words, two sceneries are equivalent iff they can be obtained from each other by shift and/or reflection around the origin. The scenery reconstruction problem can now be described as follows: Let $\{S(k)\}_{k\geq 0}$ be a recurrent random walk starting at the origin. Given a scenery ξ which is unknown to us, can we "reconstruct" ξ if we are only given the scenery ξ seen along one path-realization of $\{S(k)\}_{k\geq 0}$. Thus, does one path realization of the process $\{\xi(S(k))\}_{k\geq 0}$ uniquely determine ξ? In other words does one outcome of the random sequence $\xi(S(0)), \xi(S(1)), \xi(S(2)), ...$ determine ξ? The answer to the above question in those general terms is no. First, if ξ and $\bar{\xi}$ are equivalent, we can in general not know whether the observations come from ξ or from $\bar{\xi}$. Second, it is clear that the reconstruction will in the best case work only almost surely. As a matter of fact, if the random walk $\{S(k)\}_{k\geq 0}$ would decide to walk only to the left (which it could do with probability zero), then we would have no information about the right side of the scenery ξ and thus not be able to reconstruct the scenery ξ. So the best we can hope for is a reconstruction algorithm which works almost surely. Eventually, Lindenstrauss in [3] has been able to exhibit sceneries which one can not reconstruct. However, in [5] we were able, in the case where $\{S(k)\}_{k\geq 0}$ is a simple random walk, to prove that it is possible to reconstruct a lot of typical sceneries up to equivalence and almost surely. For this we took the scenery ξ to be itself the outcome of a random process which is independent of $\{S(k)\}_{k\geq 0}$ in such a way that the $\xi(k)'s$ are i.i.d. Bernoulli variables with parameter $\frac{1}{2}$. (As already mentioned we took $\{S(k)\}_{k\geq 0}$ to be a simple random walk starting at the origin.) So the result in [5] states that, up to equivalence, almost every scenery ξ can be reconstructed a.s., provided we are given the observation of ξ seen along a path of the simple random walk $\{S(k)\}_{k\geq 0}$. (By almost every scenery we mean almost every scenery with respect to the measure which makes the $\xi(k)'s$ i.i.d. Bernoulli with parameter $\frac{1}{2}$). Now Kesten asked whether one might still be able to reconstruct the scenery ξ up to equivalence if instead of being a simple random walk $\{S(k)\}_{k\geq 0}$ would be a simple random walk with holding. One of the two main theorems in this paper provides a positive answer to that question. Let us formulate this main theorem in a precise way:

Theorem 1 *Let $p, q \geq 0$ such that $2p + q = 1$ and such that $p > 0$. Let $\{S(k)\}_{k\geq 0}$ be a random walk on $\mathbb{Z}$ starting at the origin such that $P(S(1) = 1) = P(S(1) = -1) = p$ and $P(S(1) = 0) = q$. Let $\{\xi(k)\}_{k\in\mathbb{Z}}$ be a random process living on the same probability space as $\{S(k)\}_{k\geq 0}$ and independent of $\{S(k)\}_{k\geq 0}$ such that the $\xi(k)'s$ are i.i.d. Bernoulli variables with parameter $1/2$. (We will denote by ξ the path of the process $\{\xi(k)\}_{k\in\mathbb{Z}}$. Thus, ξ is a scenery which is the outcome of a random process.) Then, one path realization of the process $\{\xi(S(k))\}_{k\geq 0}$ a.s. determines ξ up to equivalence. In other words, there exists a measurable function $\mathcal{A} : \{0,1\}^{\mathbb{N}} \longrightarrow \{0,1\}^{\mathbb{Z}}$ such that $P(\mathcal{A}(\xi \circ S) \approx \xi) = 1$. (Here $\xi \circ S$, denotes the path of the process $\{\xi(S(k))\}_{k\geq 0}$).*

The function $\mathcal{A}$ can be viewed as an algorithm which takes as input the observations $\xi \circ S$ and produces a.s. as output a scenery which is equivalent to

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ξ. Note that the observations $\xi \circ S$ contain an infinite number of bits and thus our algorithm would take an infinite time to process all the input.

The other problem which we solve in this paper and which was asked first by ????????? is the problem of reconstructing a finite piece of scenery in polynomial time. To explain this problem more in detail we need some definitions. A piece of scenery will be defined to be a function from an integer interval into $\{0,1\}$. By integer interval we mean the intersection of a real interval with $\mathbb{Z}$. We will write $[x, y]$ for the integer interval consisting of all the integers between the points x and y, where $x < y$. Let $\psi : D \longrightarrow \{0,1\}$ and $\bar{\psi} : \bar{D} \longrightarrow \{0,1\}$ be two pieces of sceneries. We say that ψ and $\bar{\psi}$ are equivalent iff there exists $a \in \mathbb{Z}$ and $b \in \{-1,1\}$ such that $a + bD = \bar{D}$ and for all $k \in D$ we have that $\psi(k) = \bar{\psi}(a + bk)$. In this case we write $\psi \approx \bar{\psi}$. In other words, two pieces of sceneries are equivalent iff they can be obtained from each other by shift and/or reflection around the origin. Let x, y be two integers such that $x < y$ and let $\psi : [x, y] \longrightarrow \{0,1\}$ be a piece of scenery. Then we call the number $y - x$ the length of the piece of scenery ψ.

Now the question is whether one might reconstruct a piece of the scenery ξ in polynomial time. By this we mean whether we might be able to reconstruct with high probability a piece of ξ close to the origin if one would be given only a finite number of observations from $\xi(S(0)), \xi(S(1)), \xi(S(2)), ...$ (Note, that the condition that the piece of ξ we want to reconstruct be close to the origin is essential. As a matter of fact, because the scenery ξ is i.i.d. each finite piece of scenery will occur up to shift infinitely often in different places in the scenery ξ. Thus if we construct a finite piece of scenery and just say: "this is a piece of ξ" then this kind of statement wouldn't make much sense. However if we say instead "this piece of scenery is a piece of ξ which is located close to the origin" then this kind of statement has "more" content.) Let us formulate this in a more precise way. For this we need the following definitions and conventions: If f is a function and A a subset of the domain of f then we write $f|A$ for the restriction of f to the set A. As already mentioned $\xi \circ S$ denotes the sequence $\xi(S(0)), \xi(S(1)), \xi(S(2)), ...$ Also, expressions of the type βe^{n^γ} should be seen as integers. As a matter of fact simply read floor function of βe^{n^γ} when we write βe^{n^γ}.

Theorem 2 *There exists $\alpha > 0$ and $\beta_0, \beta_1, \beta_2 > 0$ such that for each $n > 0$ there exists a function $ALGORITHM^n$, where $ALGORITHM^n$ is a function from $\{0,1\}^{e^{10\alpha n}}$ to the set of all piece of sceneries, that is to $\bigcup_{k>0} \{0,1\}^k$ such that if E^n designates the event $\{$there exists an integer interval I (maybe random) such that $[-e^n, e^n] \subset I \subset [-4e^n, 4e^n]$ and such that $ALGORITHM^n(\xi \circ S|[0, e^{10\alpha n}])$ is equivalent to the piece of scenery $\xi|I.\}$ then we have that $P(E^{nc}) \leqslant \beta_0 e^{-\beta_1 n^{\beta_2}}$. (Here, E^{nc} designates the complement of the event E^n.)*

So the function $ALGORITHM^n$ can be viewed as an algorithm which takes only the first $e^{10\alpha n}$ bits of the observations $\xi \circ S$ in order to reconstruct with high probability a piece of scenery of length of order e^n which is very likely to be contained (up to equivalence) in ξ somewhere close to the origin (i.e. with

high probability the algorithm $ALGORITHM^n$ produces as output a piece of scenery which is equivalent to a piece of scenery obtained by restricting the scenery ξ to an interval I, where "I is close to the origin and of length order e^n". Since the length of the piece of scenery which gets reconstructed is of order e^n and since we need only $e^{10\alpha n}$ bits of observations $\xi \circ S$ for the reconstruction this means that we only need a number of bits which is equal to a power of the length of the piece of scenery we want to reconstruct. This is why we call the algorithm $ALGORITHM^n$, the polynomial reconstruction algorithm at level n. In order to prove theorem 2, we are going to explicitly define for each $n > 0$ an algorithm $ALGORITHM^n$ and then show that $ALGORITHM^n$ satisfies the condition that $P(E^{nc}) < \beta_0 e^{-\beta_1 n^{\beta_2}}$. Although we won't prove it, it is easy to check that the algorithm $ALGORITHM^n$ uses only a "polynomial number of elementary calculation steps in e^n". This implies that the algorithm $ALGORITHM^n$ can be implemented in the praxis. This is very different from all the previously known reconstruction algorithms which take exponentially many observations (in the length of the piece of scenery one wants to reconstruct).

Now theorem 1 is a simple corollary of theorem 2. Let us explain why.

First note that when theorem 2 holds, then, since $P(E^{nc}) < \beta_0 e^{-\beta_1 n^{\beta_2}}$ we have that $\Sigma_{n>0} P(E^{nc}) < \infty$. Thus a.s. E^n holds for all but a finite number of n's. Let ξ^n designate the piece of scenery which is the outcome of the algorithm $ALGORITHM^n$. More precisely, let ξ^n be equal to $ALGORITHM^n(\xi \circ S|[0, e^{10\alpha n}])$. With this notation we get that theorem 2 implies that for all but a finite number of n's, we have that ξ^n is equivalent to the piece of scenery obtained by restricting ξ to an interval I^n, where $[-e^n, e^n] \subset I^n \subset [-4e^n, 4e^n]$. Now, with high probability, we have that " each piece of scenery of length e^n appears at most once in $\xi|[-4e^{n+1}, 4e^{n+1}]$". More precisely, let E_0^n designate the event $E_0^n = \{$if $i_1, i_2, i_3, i_4 \in [-4e^{n+1}, 4e^{n+1}]$ are such that $|i_1 - i_2|, |i_3 - i_4| = e^n$ and such that for all $k \in 0, 1, ..., e^n$, we have that $\xi(i_1 + k(i_2 - i_1)/|i_2 - i_1|) = \xi(i_3 + k(i_4 - i_3)/|i_4 - i_3|)$, then $i_1 = i_3$ and $i_2 = i_4.\}$. It is easy to show that $\Sigma_{n>0} P(E_0^{nc}) < \infty$.(Here E_0^{nc} designates the complement of the event E_0^n. For the proof of this see [5].) Thus a.s. we have that for all but a finite number of n 's E_0^n hold. But since we also have that for all but a finite number of n 's the event E^n holds, we get that we can "assemble the pieces of sceneries ξ^n "and a.s. get as a limit a scenery $\hat{\xi}$ which will be equivalent to ξ. This means that a.s. and up the equivalence we can reconstruct the scenery ξ. The assembling rule which we have to use, can be described as follows: take the pieces of sceneries ξ^n and move them around on $\mathbb{Z}$ using shift and reflection until for all but a finite number of n's, the moved ξ^n and the moved ξ^{n+1} coincide on at least an interval of length e^n. Take then $\hat{\xi}$ to be equal to the pointwise limit as $n \longrightarrow \infty$ of the moved ξ^n's. More precisely, let G designate the subgroup of all bijections on $\mathbb{Z}$ generated by the shifts and the reflection around the origin. Determine any sequence $g_1, g_2, g_3, ...$ of elements of G such that for all but a finite number of n's, one has that the piece of scenery $\xi^{n+1} \circ g_{n+1} : g_{n+1}^{-1}(D^{n+1}) \longrightarrow \{0, 1\}$ and the piece of scenery $\xi^n \circ g_n : g_n^{-1}(D^n) \longrightarrow \{0, 1\}$ coincide on (at least) an interval of length e^n. (Here D^n designates the domain of the piece of scenery

ξ^n. Thus, $\xi^n \circ g_n$ corresponds to the "moved piece of scenery ξ^{n}".) Take then the pointwise limit of the pieces of sceneries $\xi^n \circ g_n : g_n^{-1}(D^n) \longrightarrow \{0,1\}$ to get as a limit the scenery $\hat{\xi}$. That pointwise limit is defined in the following way: For all natural number k, if k is contained in all but a finite number of intervals $g_n^{-1}(D^n)$, then define $\hat{\xi}(k)$ to be equal to the limit $\lim_{n\longrightarrow\infty}\xi^n \circ g_n(k)$ (where the limit is defined by taking those n's for which $\xi^n \circ g_n(k)$ is well defined). Let us sum up: theorem 2 implies that for all but a finite number of n's we have that a.s. E^n holds. We also have that a.s. for all n's but a finite number the events E_0^n all hold. However, when for all n's but a finite number E_0^n and E^n hold, then the above described assemblage procedure produces as output (by taking the pointwise limit) a piece of scenery which is equivalent to ξ. This proves that theorem 2 implies theorem 1 . So from now on we are going to only focus on theorem 2.

2 General ideas

In this section we are going to describe in an informal way the major ideas behind the algorithm $ALGORITHM^n$ and theorem 2.

First idea) Let $\mathcal{H}_k$ designate the σ-algebra $\sigma(S(0),S(1),S(2),...,S(k);\xi(k)|$ $k \in \mathbb{Z})$. Let $\mathcal{H}$ be the filtration $\cup_{k\geq0}\mathcal{H}_k$. The first idea is concerned with how to reconstruct a piece of ξ close to the origin of length order e^n if one is given on top of the observations $\xi \circ S$ a collection of $e^{\alpha n}$ strictly increasing stopping times which with high probability all "stop the random walk $\{S(k)\}_{k\geq0}$ in the interval $[-e^n, e^n]$". At this point we are not going to describe the idea behind this "reconstruction when one has stopping times which stop $\{S(k)\}_{k\geq0}$ in the interval $[-e^n, e^n]$" since there is going to be a whole section dedicated to it, (section 4). In section 4 we will define an algorithm $SUBALGI^n$ which reconstructs with high probability a piece of ξ close to the origin of length order e^n if it is given on top of the observations $\xi \circ S$ a collection of $e^{\alpha n}$ strictly increasing stopping times which with high probability all "stop the random walk $\{S(k)\}_{k\geq0}$ in the interval $[-e^n, e^n]$". (Actually, the algorithm also needs as input a small piece of scenery of length at least n^2 which is equivalent to a restriction of $\xi|[-e^n, e^n]$.) The algorithm $SUBALGI^n$ will be used in the next section to define the algorithm $ALGORITHM^n$, and thus $SUBALGI^n$ is a subalgorithm of $ALGORITHM^n$. We will prove that $SUBALGI^n$ "works" with high probability in section 4. However, let us at this stage already formulate the main theorem concerning $SUBALGI^n$. Let $\tau(1),\tau(2),....,\tau(e^{\alpha n})$ be any finite $\mathcal{H}$-adapted sequence of $e^{\alpha n}$ stopping times. We will write τ for the random vector $(\tau(1),\tau(2),....,\tau(e^{\alpha n}))$. Let $E_{\tau,0}^n$ designate the event $E_{\tau,0}^n = \{$ for all $k \in 1,2,...,e^{\alpha n}$ we have that $S(\tau(k)) \in [-e^n, e^n]\}$. Now we will see in section 4 that $SUBALGI^n$ needs as input three things: a) a collection of $e^{\alpha n}$ stopping times b) the observations $\xi \circ S$ but only restricted between the first and last stopping time (and only up to shift) c) a little piece of scenery of length at least n^2. Thus formally the algorithm $SUBALGI^n$ can be described as a function from $\mathbb{N}^{e^{\alpha n}} \times (\cup_{k>0}\{0,1\}^k) \times (\cup_{k>0}\{0,1\}^k)$ into $(\cup_{k>0}\{0,1\}^k)$.

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Let $E^n_{\tau,1}$ be the event that the algorithm $SUBALGI^n$ constructs as output a piece of ξ of length order e^n close to the origin , when $SUBALGI^n$ is given as input the stopping times $\tau(1), \tau(2), ..., \tau(e^{\alpha n})$ and the observations $\xi \circ S$ restricted to the time between $\tau(1)$ and $\tau(e^{\alpha n})$ and any "little" piece of scenery of length at least n^2 contained in $\xi|[-e^n, e^n]$. More precisely, $E^n_{\tau,1} = \{$ for each piece of scenery ψ of length at least n^2 such that ψ is equivalent to a piece of scenery obtained by restricting $\xi|[-e^n, e^n]$ to an interval contained in $[-e^n, e^n]$, there exists an interval I such that $[-e^n, e^n] \subset I \subset [-4e^n, 4e^n]$ and such that $SUBALGI^n(\tau; \xi \circ S|[\tau(1), \tau(e^{\alpha n})]; \psi)$ is equivalent to the piece of scenery $\xi|I\}$. Let $E^{nc}_{\tau,1}$ denote the complement of $E^n_{\tau,1}$. (We will always use the notation E^{nc}_i for the complement of an event E^n_i.) We are now ready to formulate the main theorem concerning the algorithm $SUBALGI^n$:

Theorem 3 *There exists $\beta_3, \beta_4 > 0$ such that for all $n > 0$ and all strictly increasing $\mathcal{H}$-adapted sequences of stopping times.$\tau(1), \tau(2), ..., \tau(e^{\alpha n})$ we have that $P(E^{nc}_{\tau,1} \cap E^n_{\tau,0}) \leq \beta_3 e^{-\beta_4 n}$. In other words, the probability that $SUBALGI^n$ fails in reconstructing a piece of ξ of length order e^n close to the origin despite the fact that $SUBALGI^n$ is given the right input, is exponentially small in n. By "given the right input" we mean that $SUBALGI^n$ is given a $\mathcal{H}$-adapted sequence of strictly increasing stopping times all stopping the random walk $\{S(k)\}_{k \geq 0}$ in $[-e^n, e^n]$ as well as a piece of length at least n^2 of $\xi|[-e^n, e^n]$ and the observations $\xi \circ S$ restricted to the period between the first and the last stopping time.*

The above theorem implies that in order to be able to reconstruct with high probability a piece of ξ of length order e^n close to the origin we only need to be able to construct an $\mathcal{H}$-adapted sequence of $e^{\alpha n}$ strictly increasing stopping times which are very likely to all stop the random walk $\{S(k)\}_{k \geq 0}$ in $[-e^n, e^n]$. The second idea is due to Kesten and helps constructing stopping times which all stop $\{S(k)\}_{k \geq 0}$ in the same area. To explain the second idea we need a few definitions:

Definition 4 *Let $\psi : D \longrightarrow \{0,1\}$ be a piece of scenery. (Thus for example, ψ could also be equal to a scenery or ψ could be equal to the observations $\xi \circ S|[0, e^{3\alpha n}]$.) Let $x, y \in D$. Then, we say that $\{x, y\}$ is a block of ψ iff $|x-y| \geq 2$ and for all integer z strictly between x and y, we have that $\psi(x) = \psi(y) \neq \psi(z)$. We call $|x - y|$ the length of the block $\{x, y\}$. If $x < y$ then we say that x, resp. y is the right, resp. left end of the block $\{x, y\}$. Let $\{t_1, t_2\}$ be a block in the observations $\xi \circ S$ and let $\{x, y\}$ be a block of ξ. Then we say that the block $\{t_1, t_2\}$ was generated by S on the block $\{x, y\}$ iff $\{x, y\} \supset \{S(t_1), S(t_2)\}$ and for all t strictly between t_1 and t_2 we have that $S(t)$ lies strictly between x and y. Note that each block of $\xi \circ S$ is generated on one and only one block of ξ. Let $\{x^{n+}_1, x^{n+}_2\}$ where $x^{n+}_1 < x^{n+}_2$, designate the first block of ξ after zero of length $\geq n$. More precisely $\{x^{n+}_1, x^{n+}_2\}$ is a block of ξ such that $0 \leq x^{n+}_1$; $x^{n+}_2 - x^{n+}_1 \geq n$ and such that there exists no block $\{x, y\}$ of ξ of length $\geq n$ such that $x, y \in [0, x^{n+}_1 + 1]$. Let $\{x^{n-}_1, x^{n-}_2\}$ where $x^{n-}_1 > x^{n-}_2$, be the first block of*

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ξ *before zero of length* $\geq n$. *Let* $\{x_1^n, x_2^n\}$ *designate the one of the two blocks* $\{x_1^{n+}, x_2^{n+}\}$ *and* $\{x_1^{n-}, x_2^{n-}\}$ *which* $\{S(k)\}_{k \geq 0}$ *visits first.*

Second idea) (due to Kesten) A block of length n^2 in the observations $\xi \circ S$ is very unlikely to have been generated on a block of length $\leq n^{0.4}$. As a matter of fact the probability that a given block of ξ has length n has probability $(\frac{1}{2})^n$. When $\{S(k)\}_{k \geq 0}$ crosses however a block of ξ of length n, $\{S(k)\}_{k \geq 0}$ typically produces a block of length order n^2 in the observations. Thus, the probability to see a block of length n^2 in the observations $\xi \circ S$ which has been produced on a block of length n, is not smaller than order $(\frac{1}{2})^n$. On the other hand, the probability for $\{S(k)\}_{k \geq 0}$ to produce a bock of $\xi \circ S$ of length n^2 on a block of ξ of length $\leq n^{0.4}$ is smaller than $e^{-\beta_5 n^2/n^{0.8}} = e^{-\beta_5 n^{1.2}}$, where $\beta_5 > 0$ is a constant not depending on n. Now, $e^{-\beta_5 n^{1.2}}$ is much smaller than $(\frac{1}{2})^n$, and thus when we first observe a block of length order n^2 in the observation $\xi \circ S$ that block is likely to have been generated on a block of ξ of length $\geq n^{0.4}$. Now the first thing which $ALGORITHM^n$ tries to achieve is to construct a stopping time which is likely to stop $\{S(k)\}_{k \geq 0}$ at the block $\{x_1^n, x_2^n\}$. In order to achieve this, it is not quite enough to just take the first observed block of length order n^2 in the observations $\xi \circ S$. Instead, we will take the first $n^{0.2}$ blocks of length order n^2 in $\xi \circ S$ which have the average of their lengths higher than a certain number. We will prove in the next section that this method constructs a stopping time ν_0^n which with high probability stops $\{S(k)\}_{k \geq 0}$ at the block $\{x_1^n, x_2^n\}$.

Third idea) When one has constructed the stopping time ν_0^n which stops $\{S(k)\}_{k \geq 0}$ at $\{x_1^n, x_2^n\}$ with high probability it is easy to construct $e^{\alpha n^{0.2}}$ additional stopping times which are very likely to all stop $\{S(k)\}_{k \geq 0}$ at $\{x_1^n, x_2^n\}$. For this, simply take in the observations $\xi \circ S$ the right ends of the first $e^{\alpha n^{0.2}}$ blocks of length $\geq n^2$ after the stopping time ν_0^n. These right ends are, with high probability, all times when $\{S(k)\}_{k \geq 0}$ is at $\{x_1^n, x_2^n\}$ for the following reason: with high probability there is no block of ξ of length longer than $n^{0.4}$ in a radius $e^{n^{0.3}}$ of the points x_1^n, x_2^n, (other than the block $\{x_1^n, x_2^n\}$ itself). Now we saw that it is "very unlikely" that there is a block of length $\geq n^2$ produced on a block of ξ of length $\leq n^{0.4}$. However when, $S(\nu_0^n) \in \{x_1^n, x_2^n\}$ then for $e^{n^{0.3}}$ time after ν_0^n we have that $\{S(k)\}_{k \geq 0}$ remains in a radius $e^{n^{0.3}}$ of the points x_1^n, x_2^n. Thus, for $e^{n^{0.3}}$ time after ν_0^n the only block of ξ of length $\geq n^{0.4}$ with which $\{S(k)\}_{k \geq 0}$ is in contact is $\{x_1^n, x_2^n\}$. So, with high probability all the blocks of $\xi \circ S$ of length $\geq n^2$ which we observe within time $e^{n^{0.3}}$ after ν_0^n are likely to all have been generated on the block $\{x_1^n, x_2^n\}$. Furthermore, is easy to check that within time $e^{n^{0.3}}$ after ν_0^n we have that $\{S(k)\}_{k \geq 0}$ is likely to generate more than $e^{\alpha n^{0.2}}$ blocks of length $\geq n^2$ on $\{x_1^n, x_2^n\}$. This is so because within time $e^{n^{0.3}}$, $\{S(k)\}_{k \geq 0}$ comes back order $e^{0.5 n^{0.3}}$ times to $\{x_1^n, x_2^n\}$ and every time it comes back the probability to get a block of order n^2 is approximately $1/n$.

Fourth idea) With those $e^{\alpha n^{0.2}}$ stopping times which are all very likely to stop $\{S(k)\}_{k \geq 0}$ at $\{x_1^n, x_2^n\}$ we can reconstruct ξ in a radius of $e^{n^{0.2}}$ of $\{x_1^n, x_2^n\}$.

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To achieve this goal we will use a slightly modified version of $SUBALGI^{n^{0.2}}$. This modified algorithm will get precisely defined in section 5, where we will prove the properties of this algorithm. This algorithm will be denoted by $SUBALGII^n$. Thus, $SUBALGII^n$ is the second subalgorithm which we will use for the construction of $ALGORITHM^n$. Let us mention at this stage that $SUBALGII^n$ takes as input exactly $e^{n^{0.3}}$ bits. Thus $SUBALGII^n$ is a function from $\{0,1\}^{e^{n^{0.3}}}$ into $\cup_{k \geq 0} \{0,1\}^k$.

Fifth idea) As just mentioned before the subalgorithm $SUBALGII^n$ is able to reconstruct with high probability ξ in a radius of $e^{n^{0.2}}$ of $\{x_1^n, x_2^n\}$. This is not yet enough since we want to be able to reconstruct a piece of ξ of length order e^n. The idea we are going to present next shows how one can use "partial reconstruction" to construct a lot of stopping times which are all likely to stop the random walk $\{S(k)\}_{k \geq 0}$ close to the place where we did the partial reconstruction. In general the number of stopping times we are able to construct in this way is roughly speaking of order exponential power of the length of the piece of scenery constructed in the first partial reconstruction. In our case this means that since $SUBALGII^n$ reconstructs a piece of ξ of length order $e^{n^{0.2}}$ we can construct an order exponential power of $e^{n^{0.2}}$ stopping times which are likely to all stop $\{S(k)\}_{k \geq 0}$ close to $\{x_1^n, x_2^n\}$. However, in order to be able to reconstruct a piece of ξ of length order e^n close to the origin we need only e^{an} stopping times stopping $\{S(k)\}_{k \geq 0}$ in the interval $[-e^n, e^n]$. So this procedure of using a partial reconstruction to construct more stopping times provides enough stopping times for the reconstruction of a piece of ξ of length order e^n around the origin.

3 The algorithm $ALGORITHM^n$

In this section we are going to define the algorithm $ALGORITHM^n$ and prove theorem 2. We will use for this theorem 3 about the subalgorithm $SUBALGI^n$ which will be proven in section 4. Also, the description of the subalgorithm $SUBALGII^n$ as well as the proof of the theorem which goes with it will only be done in section 5, despite the fact that they will be used here. In what follows let $\{S_1(k)\}_{k \geq 0}$ denote any random walk having its increments distributed in the same way as the increments of $\{S(k)\}_{k \geq 0}$ and such that $\{S_1(k) - S_1(0)\}_{k \geq 0}$ is independent of $\{\xi(k)\}_{k \in \mathbb{Z}}$ and such that $\bar{S}_1(0) \in \{x_1^n, x_2^n\}$.

Theorem 5 *There exists constants $\beta_6, \beta_7, \beta_8, \beta_9 > 0$ and a $\sigma(\xi(k)|k \in \mathbb{Z})$-measurable event E_ξ^n such that for all $n > 0$ we have that $P(E_\xi^{nc}) \leq \beta_6 e^{-\beta_7 n^{0.2}}$ and for each scenery $\xi \in E_\xi^n$ we have that the conditional probability when conditioned under ξ of the event {the piece of scenery $SUBALGII^n(\xi \circ S_1[0, e^{n^{0.3}}])$ is not equivalent to $\xi|\{min\{x_1^n, x_2^n\} - e^{n^{0.2}}, max\{x_1^n, x_2^n\} + e^{n^{0.2}}\}\}$ is smaller than $\beta_8 e^{-\beta_9 n}$.*

In other words, the above theorem means that with high probability ξ is such that when we take any random walk starting at $\{x_1^n, x_2^n\}$ and we use the

first $e^{n^{0.3}}$ bits of observations by that random walk of the scenery ξ as input for $SUBALGII^n$, then with very high probability the output is going to be equivalent to the piece of scenery $\xi||[min\{x_1^n, x_2^n\} - e^{n^{0.2}}, max\{x_1^n, x_2^n\} + e^{n^{0.2}}]$. This implies that with high probability, every time the random walk $\{S(k)\}_{k \geq 0}$ is back visiting $\{x_1^n, x_2^n\}$ and we take the next $e^{n^{0.3}}$ bits in the observations $\xi \circ S$ as input for $SUBALGII^n$ we get with high probability as output a piece of scenery equivalent to $\xi||[min\{x_1^n, x_2^n\} - e^{n^{0.2}}, max\{x_1^n, x_2^n\} + e^{n^{0.2}}]$. (This fact is crucial in understanding why we can use $SUBALGII^n$ as a test to try to find when $\{S(k)\}_{k \geq 0}$ is close to $\{x_1^n, x_2^n\}$.) Next we need a lemma:

Lemma 6 *Let T^n designate the first hitting time of $\{S(k)\}_{k \geq 0}$ on $\{-1, n - 1\}$. (Recall that the random walk starts at the origin.) Then there exists two constants $\beta_{10}, \beta_{xx} > 0$ such that if we define the number $i^n = \beta_{xx} n^2 ln n$ then there exists $c^n > i^n$, such that $P(T^n \geq i^n) \geq n^{-\beta_{10}}$ and for all $m < n$ we have that $E[T^m|T^m \geq i^n; S(T^m) = -1], E[T^m|T^m \geq i^n; S(T^m) = m - 1] \leqslant c^n - 1$ and for $n \leqslant m \leqslant 2n$ we have that $E[T^m|T^m \geq i^n; S(T^m) = -1], E[T^m|T^m \geq i^n; S(T^m) = m - 1] \geq c^n + 1$. Furthermore, for all m such that $n^{0.4} \leqslant m \leqslant 2n$ we have that both $E[T^m|T^m \geq i^n; S(T^m) = -1]$ and $E[T^m|T^m \geq i^n; S(T^m) = m - 1]$ are smaller by at least one unit from $E[T^{m+1}|T^{m+1} \geq i^n; S(T^{m+1}) = -1]$ and $E[T^{m+1}|T^{m+1} \geq i^n; S(T^{m+1}) = m]$.*

Proof. We are going to show that if we take the constant $\beta_{xx} > 0$ (not depending on n) big enough then the above inequalities hold. We use the notation $\mu(m, 0)$ for the defective distribution $P(S(T^m) = -1)\mathcal{L}(T^m|S(T^m) = -1)$ and $\mu(m, 1)$ for $P(S(T^m) = -1)\mathcal{L}(T^m|S(T^m) = m - 1)$. We have that $\mu(m, 0)(\{r\})$ is equal to $(p/m)\Sigma_{i=1}^{m-1}(pcos(\pi i/m) + q)^{r-1} sin^2(\pi i/m)$ and we have that $\mu(m, 0)(\{r\})$ is equal to $(p/m)\Sigma_{i=1}^{m-1}(pcos(\pi i/m) + q)^{r-1} sin(\pi i/m) sin(\pi i - \pi i/m)$, (see Feller [1]). Now, we are going to show that if we take r bigger than $\beta_{xx} n^2 ln n$ with β_{xx} big enough, then in the sum $(p/m)\Sigma_{i=1}^{m-1}(pcos(\pi i/m) + q)^{r-1} sin^2(\pi i/m)$ only the term with $i = 1$ plays an important role, and thus the distribution of $\mathcal{L}(T^m - i^n|S(T^m) = -1, T^m - i^n \geq 0)$ and $\mathcal{L}(T^m|S(T^m) = m - 1, T^m - i^n \geq 0)$ are both approximately equal to the geometric distribution with parameter $pcos(\pi/m) + q$. However the expectation of a geometric variable with parameter $pcos(\pi/m) + q$ is equal to $(pcos(\pi/m) + q)/p(1 - cos(\pi/m))$. The last expression is asymptotically equivalent to $2m^2/\pi$. By this we mean that $[(pcos(\pi/m) + q)/p(1 - cos(\pi/m))]/m^2 \longrightarrow 2/\pi$ as $m \longrightarrow \infty$. So if the distributions $\mathcal{L}(T^m - i^n|S(T^m) = -1, T^m - i^n \geq 0)$ and $\mathcal{L}(T^m|S(T^m) = m - 1, T^m - i^n \geq 0)$ would be geometric with parameter $pcos(\pi/m) + q$, then the part about the conditional expectations in our lemma would hold. Let us explain next how in the sum $(p/m)\Sigma_{i=1}^{m-1}(pcos(\pi i/m) + q)^{r-1} sin^2(\pi i/m)$ the part $(p/m)\Sigma_{i=2}^{m-1}(pcos(\pi i/m) + q)^{r-1} sin^2(\pi i/m)$ is small in comparison to the leading term in the sum when r is big enough. Note that for $i \in 2, 3, ..., m$ the value of $pcos(\pi i/m) + q$ which comes closest to $pcos(\pi/m) + q$ is $pcos(\pi 2/m) + q$. Now, $cos(\pi/m) - cos(\pi 2/m) \geq (\pi/m)sin(\pi/m)$. However, for $x \in [0, \pi/2]$ we get that $sin(x) \geq 2x/\pi$. Thus, $pcos(\pi/m) - pcos(\pi 2/m) \geq 2\pi p/m^2$. Thus, $(pcos(\pi 2/m) + q)/(pcos(\pi/m) + q) \leqslant 1 - 2\pi p/m^2$. Recall now that

$(1 - 1/k)^k \longrightarrow e^{-1}$ as k goes to infinity. So that we get, (at least for n big enough), that if r is bigger than $\beta_{xx}(lnm)m^2/\pi p$ then $(pcos(\pi 2/m) + q)^r/(pcos(\pi/m) + q)^r$ is smaller than $m^{-\beta_{xx}}$. Thus, if we take r is bigger than $\beta_{xx}(lnm)m^2/\pi p$ then $(p/m)\Sigma_{i=2}^{m-1}(pcos(\pi i/m) + q)^{r-1}sin^2(\pi i/m)$ is smaller than $m^{-\beta_{xx}} \times 1/sin^2(m/\pi)$ times $(p/m)(pcos(\pi/m) + q)^{r-1}sin^2(\pi/m)$. Since we can take β_{xx} as big as we want we get that $(p/m)\Sigma_{i=2}^{m-1}(pcos(\pi i/m) + q)^{r-1}sin^2(\pi i/m)$ is any negative power of m times smaller than the leading term $(p/m)(pcos(\pi/m)+q)^{r-1}sin^2(\pi/m)$ provided we take β_{xx} big enough. However, we have that $m > n^{0.4}$ and thus $(p/m)\Sigma_{i=2}^{m-1}(pcos(\pi i/m) + q)^{r-1}sin^2(\pi i/m)$ is any negative power of n times smaller than the leading term $(p/m)(pcos(\pi/m)+q)^{r-1}sin^2(\pi/m)$ provided we take β_{xx} big enough. It only remains to show that $P(T^n \geq i^n)$ is not smaller than a negative power of n. This is actually true for any choice of the constant $\beta_{xx} > 0$. It is enough to show that the leading term $(p/n)(pcos(\pi/n) + q)^{r-1}sin^2(\pi/n)$ for $r = \beta n^2 lnn + 1$ is not to small. Now, $pcos(\pi/n) + q \geq 1 - p\pi sin(\pi/n)/n \geq 1 - p\pi^2/n^2$. Furthermore $sin^2(\pi/n) \geq 1/n^2$. Thus $(p/n)(pcos(\pi/n) + q)^{r-1}sin^2(\pi/n)$ is bigger than $(p/n^3)\,[1 - p\pi^2/n^2]^{\beta_{xx}n^2 lnn}$ and thus bigger than $(p/n^3)n^{-\beta_{xx}2p\pi^2}$ (at least for n big enough). The last expression being a negative power of n we are done.
∎

To understand what the last lemma means note that T^m has the same distribution than "the length of a block of $\xi \circ S$ given that this block was generated on a block of ξ of length m. $E[T^m|T^m \geq i^n; S(T^m) = -1]$ would then be the conditional expectation of the length of a block of $\xi \circ S$ given that block was generated on a block of length m, that its length is longer than i^n and that, when generating the block on ξ, we have that $\{S(k)\}_{k\geq 0}$ enters the block on which it generates the block of $\xi \circ S$ on the same side than it exits it. The above lemma makes clear why we can use c^n as critical value in a test to determine whether some blocks of $\xi \circ S$ with length $\geq i^n$ have been generated on a block of ξ of length longer than n. The condition $P(T^n \geq i^n) \geq n^{-\beta_{10}}$ is there to make sure that there are enough blocks of length $\geq i^n$ generated on blocks of ξ of length $\geq n$. We are now ready to define $ALGORITHM^n(\xi \circ S|[0, e^{10\alpha n}])$:

Algorithm 7 *First step) Let $\tau_+^n(k)$, resp. $\tau_-^n(k)$ designate the right end, resp. the left end of the k-th block of $\xi \circ S$ of length $\geq i^n$. Let ν_0^n designate the smallest $\tau^n(k)$ for which $k \geq e^{n^{0.2}}$ and $(1/e^{n^{0.2}})\Sigma_{k-e^{n^{0.2}} \leq l \leq k}\tau_+^n(l) - \tau_-^n(l) \geq c^n$. Second step) Apply $SUBALGII^n$ to the observations $\xi \circ S|[\nu_0^n, \nu_0^n + e^{n^{0.3}}]$. Let ψ^n be the piece of scenery $SUBALGII^n(\xi \circ S|[\nu_0^n, \nu_0^n + e^{n^{0.3}}])$. Third step) Let SET_ν^n designate the set of all points $k \geq e^{n^{0.3}}$ such that $SUBALGII^n(\xi \circ S|[k - e^{n^{0.3}}, k])$ is equivalent to ψ^n. Let $\nu^n(k)$ designate the k-th point of the set SET_ν^n. Fourth step) if $\nu^n(e^{\alpha n}) > e^{10\alpha n}$ then let the algorithm $ALGORITHM^n$ break down. Otherwise, use $\psi^n, \nu^n = (\nu^n(1), \nu^n(2), ..., \nu^n(e^{\alpha n}))$ and $\xi \circ S|[\nu^n(1), \nu^n(e^{\alpha n})]$ as input for the subalgorithm $SUBALGI^n$ to get as output the output of $ALGORITHM^n$. Thus, $ALGORITHM^n(\xi \circ S|[0, e^{10\alpha n}])$ is defined to be equal to $SUBALGI^n(\nu^n; \xi \circ S|[\nu^n(1), \nu^n(e^{\alpha n})]; \psi^n)$ (when $\nu^n(e^{\alpha n}) \leqslant e^{10\alpha n}$).*

Note that step one is supposed to give a stopping time which with high probability stops $\{S(k)\}_{k\geq 0}$ on $\{x_1^n, x_2^n\}$, that is with high probability $S(\nu_0^n) \in \{x_1^n, x_2^n\}$. The second step is supposed to reconstruct with high probability the scenery ξ in a radius $e^{n^{0.2}}$ of $\{x_1^n, x_2^n\}$. In the third step, as we will prove later, with high probability all the stopping times $\nu^n(1), \nu^n(2), ..., \nu^n(e^{\alpha n})$ stop $\{S(k)\}_{k\geq 0}$ "close to" $\{x_1^n, x_2^n\}$. However because the proof is simpler we will only prove that with high probability $S(\nu^n(k)) \in [x_2^{n-} - e^{n^{0.2}} - e^{n^{0.3}}, x_2^{n+} + e^{n^{0.2}} + e^{n^{0.3}}]$ for all $k \in 1, 2, ..., e^{\alpha n}$. In step four, the condition that the algorithm breaks down if $\nu^n(e^{\alpha n}) \geq e^{10\alpha n}$ is there to make sure that the final output of $ALGORITHM^n$ only depends on the first $e^{10\alpha n}$ bits of the observations $\xi \circ S$. Next we are going to introduce a couple of events which we will need for the proof of theorem 2:

Let E_1^n designate the event $E_1^n = \{x_1^{n+}, x_2^{n+}, x_1^{n-}, x_2^{n-} \in [-e^n/2, e^n/2]\}$. Let E_2^n designate the event that $\{S(k)\}_{k\geq 0}$ hits on $\{-e^n/2, e^n/2\}$ before time e^{3n}. Let E_3^n designate the event that up to time $2e^{3n}$ there is no block of length longer or equal n^2 in the observations $\xi \circ S$ which has been generated by $\{S(k)\}_{k\geq 0}$ on a bock of ξ of length smaller or equal to $n^{0.4}$. Let E_4^n designate the event that in a vicinity of radius $e^{n^{0.3}}$ of $x_1^{n+}, x_2^{n+}, x_1^{n-}, x_2^{n-}$ the only blocks of ξ of length longer than $n^{0.4}$ are $\{x_1^{n+}, x_2^{n+}\}$ or $\{x_1^{n-}, x_2^{n-}\}$. Let E_5^n be the event that within time $e^{n^{0.3}}$ of the first visit by $\{S(k)\}_{k\geq 0}$ to $\{x_1^n, x_2^n\}$ we have that $\{S(k)\}_{k\geq 0}$ produces at least $e^{n^{0.2}}$ blocks of length $\geq i^n$ on the block $\{x_1^n, x_2^n\}$. Let E_6^n be the event that the average of the lengths of the first $e^{n^{0.2}}$ blocks of length $\geq i^n$ produced by $\{S(k)\}_{k\geq 0}$ on the block $\{x_1^n, x_2^n\}$ is bigger than c^n. Recall that $\tau_+^n(k)$, resp. $\tau_-^n(k)$ designate the right end, resp. the left end of the k-th block of $\xi \circ S$ of length $\geq i^n$. Let E_7^n be the event that for all k such that $e^{n^{0.2}} \leq k \leq 2e^{3n}$ we have that the average $(1/e^{n^{0.2}})\Sigma_{k-e^{n^{0.2}} \leq l \leq k} \tau_+^n(l) - \tau_-^n(l)$ is smaller than c^n whenever all the bocks $\{\tau_+^n(l), \tau_-^n(l)\}$ with $k - e^{n^{0.2}} \leq l \leq k$ are generated on blocks of ξ which have length strictly smaller than n.

Now all the events we have defined so far are here to make sure that the stopping times constructed in step one of the algorithm $ALGORITHM^n$ stops $\{S(k)\}_{k\geq 0}$ on $\{x_1^n, x_2^n\}$. We have $\cap_{i \in 1,2...,7} E_i^n \subset E_8^n$ where $E_8^n = \{S(\nu_0^n) \in \{x_1^n, x_2^n\}\}$. Next we will need a few more events:

Let E_9^n be the event that the piece of scenery $SUBALGII^n(\xi \circ S|[\nu_0^n, \nu_0^n + e^{n^{0.3}}])$ is equivalent to $\xi|[min\{x_1^n, x_2^n\} - e^{n^{0.2}}, max\{x_1^n, x_2^n\} + e^{n^{0.2}}]$. Let E_{10}^n be the event that ξ is such that the conditional probability when conditioned under ξ of the event $\{$the piece of scenery $SUBALGII^n(\xi \circ S_1|[0, e^{n^{0.3}}])$ is not equivalent to $\xi|[min\{x_1^n, x_2^n\} - e^{n^{0.2}}, max\{x_1^n, x_2^n\} + e^{n^{0.2}}]\}$ is smaller than $\beta_8 e^{-\beta_9 n^{0.2}}$ and this is true for any $\{S_1(k)\}_{k\geq 0}$ random walk having its increments distributed in the same way than the increments of $\{S(k)\}_{k\geq 0}$ and such that $\{S_1(k) - S_1(0)\}_{k\geq 0}$ is independent of $\{\xi(k)\}_{k\in\mathbb{Z}}$ and $\bar{S}_1(0) \in \{x_1^n, x_2^n\}$. Let E_{11}^n designate the event that up to time $e^{10\alpha n}$ we have that $\{S(k)\}_{k\geq 0}$ visits $\{x_1^n, x_2^n\}$ at least $2e^{\alpha n}e^{n^{0.3}}$ times. Let E_{12}^n designate the event that for at least half of the k's such that $k \in 1, 2, 3, ..., 2e^{\alpha n}$ we have that if t is the $ke^{n^{0.3}}$-th visit to $\{x_1^n, x_2^n\}$ by $\{S(k)\}_{k\geq 0}$ then $SUBALGII^n(\xi \circ S|[t, t + e^{n^{0.3}}])$ is equivalent

to $\xi|[min\{x_1^n,x_2^n\} - e^{n^{0.2}}, max\{x_1^n,x_2^n\} + e^{n^{0.2}}]$. Let E_{13}^n designate the event that for every $t \leqslant e^{10\alpha n}$ such that $SUBALGII^n(\xi \circ S|[t, t+e^{n^{0.3}}])$ is equivalent to $\xi|[min\{x_1^n,x_2^n\} - e^{n^{0.2}}, max\{x_1^n,x_2^n\} + e^{n^{0.2}}]$ we have that $S(t+e^{n^{0.3}})$ is in $[x_2^{n-} - e^{n^{0.2}} - e^{n^{0.3}}, x_2^{n+} + e^{n^{0.2}} + e^{n^{0.3}}]$. Let E_{14}^n designate the event {the piece of scenery ψ^n constructed in step 2 of the algorithm $ALGORITHM^n$ is equivalent to a piece of scenery of length at least n^2 obtained by restricting $\xi|[-e^n, e^n]$ to an interval} $\cap$ { for all $k \in 1, 2, ..., e^{\alpha n}$ we have that $S(\nu^n(k)) \in [-e^n, e^n]$} $\cap$ { $\nu^n(k) < e^{10\alpha n}$}. Now when E_{14}^n holds, we have that in step five of the algorithm $ALGORITHM^n$ the subalgorithm $SUBALGI^n$ gets "correct input". So, theorem 3 implies that $P(E^{nc} \cap E_{14}^n) \leqslant \beta_3 e^{-\beta_4 n}$ for all $n > 0$. (Recall that E^{nc} is the complement of the event that the algorithm $ALGORITHM^n$ works, i.e. the complement of $E^n =$ {there exists an integer interval I (maybe random) such that $[-e^n, e^n] \subset I \subset [-4e^n, 4e^n]$ and such that $ALGORITHM^n(\xi \circ S|[0, e^{10\alpha n}])$ is equivalent to the piece of scenery $\xi|I$.}). Now since $P(E^{nc} \cap E_{14}^n) \leqslant \beta_3 e^{-\beta_4 n}$ we have that $P(E^{nc}) \leqslant P(E_{14}^{nc}) + \beta_3 e^{-\beta_4 n}$. Thus in order to prove theorem 2 it is enough to prove that $P(E_{14}^{nc})$ is exponentially small in a positive power of n. Now it is easy to check that $E_1^n \cap E_9^n \cap E_{11}^n \cap E_{12}^n \cap E_{13}^n \subset E_{14}^n$. This in terms implies that $P(E_{14}^{nc}) \leqslant P(E_1^{nc}) + P(E_8^{nc}) + P(E_{10}^{nc}) + P(E_9^{nc} \cap E_8^n \cap E_{10}^n) + P(E_{11}^{nc}) + P(E_{12}^{nc} \cap E_{10}^n) + P(E_{13}^{nc})$. The last inequality implies that in order to prove theorem 2 it is enough to prove that each of the quantities: $P(E_1^{nc})$, $P(E_8^{nc})$, $P(E_{10}^{nc})$, $P(E_9^{nc} \cap E_8^n \cap E_{10}^n)$, $P(E_{11}^{nc})$, $P(E_{12}^{nc} \cap E_{10}^n)$ and $P(E_{13}^{nc})$ are all exponentially small in a positive power of n. This is what we are going to do next. For $P(E_1^{nc})$ the proof is simple so we leave it to the reader. Theorem 5 implies that $P(E_{10}^{nc}) \leqslant \beta_6 e^{-\beta_7 n}$. Thus, $P(E_{10}^{nc})$ is exponentially small in a positive power of n. (The proof of theorem 5 will be given in section 5.)

Let us now prove that $P(E_9^{nc} \cap E_8^n \cap E_{10}^n)$ is exponentially small in a positive power of n. By conditioning under ξ and because of the strong Markov property of $\{S(k)\}_{k \geq 0}$, we get that for $\xi \in E_{10}^n$ we have $P(E_9^{nc}|\xi, E_8^n) \leqslant \beta_8 e^{-\beta_9 n^{0.2}}$ and thus $P(E_9^{nc} \cap E_8^n|\xi) \leqslant \beta_8 e^{-\beta_9 n^{0.2}}$. Integrating over $\xi \in E_{10}^n$ yields $P(E_9^{nc} \cap E_8^n \cap E_{10}^n) \leqslant \beta_8 e^{-\beta_9 n^{0.2}}$.

Let us now prove that $P(E_{11}^{nc})$ is exponentially small in a positive power of n. Let E_{111}^n be the event that the first visit to $\{x_1^n, x_2^n\}$ by $\{S(k)\}_{k \geq 0}$ takes place before time e^{3n}. Let E_{112}^n be the event that within time $e^{9\alpha n}$ after the first visit to $\{x_1^n, x_2^n\}$ by $\{S(k)\}_{k \geq 0}$, we have that $\{S(k)\}_{k \geq 0}$ visits $\{x_1^n, x_2^n\}$ at least $e^{2\alpha n}$ times. Since (at least for n big enough and we suppose $\alpha \geq 1$) $2e^{\alpha n} \cdot e^{n^{0.3}} \leqslant e^{2\alpha n}$ and $e^{3n} + e^{9\alpha n} \leqslant e^{10\alpha n}$ we get that $E_{111}^n \cap E_{112}^n \subset E_{11}^n$. Thus, $P(E_{11}^{nc}) \leqslant P(E_{111}^{nc}) + P(E_{112}^{nc})$. Now, $E_1^n \cap E_2^n \subset E_{111}^n$. Thus, we get that $P(E_{11}^{nc}) \leqslant P(E_1^{nc}) + P(E_2^{nc}) + P(E_{112}^{nc})$. We already mentioned that $P(E_1^{nc})$ is exponentially small in a positive power of n. Furthermore we will prove in a subsequent proof that the same thing is true for $P(E_2^{nc})$. So it only remains to prove that $P(E_{112}^{nc})$ is exponentially small in a positive power of n. This is done as follows: let $X(1), X(2), ...$ denote a sequence of i.i.d. random variables which have the same distribution as the first return time of $\{S(k)\}_{k \geq 0}$ to the origin. (Recall that $\{S(k)\}_{k \geq 0}$ starts at the origin.) For the sake of this proof, let us use the following notations: $x = e^{9\alpha n}$ and $y = e^{2\alpha n}$.

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Then, we have that $P(E_{112}^{nc}) \leqslant P(X(1) + X(2) + ... + X(y) \geq x)$. For any set of positive numbers $\{a, b, s, d, e, ...\}$ we have that $(a + b + c + d + ...)^3 \geq a^3 + b^3 + c^3 + ...$ Thus, $X(1)^{1/3} + X(2)^{1/3} + ... + X(y)^{1/3} \geq (X(1) + X(2) + ... + X(y))^{1/3}$. This implies, that $P(X(1) + X(2) + ... + X(y)) \geq x)$ is smaller than $P(X(1)^{1/3} + X(2)^{1/3} + ... + X(y)^{1/3} \geq (x)^{1/3})$. By Chebichev, we get that $P(X(1)^{1/3} + X(2)^{1/3} + ... + X(y)^{1/3} \geq x^{1/3})$ is smaller or equal than $E[X(1)^{1/3}]y/((x)^{1/3})$. Thus, $P(E_{112}^{nc}) \leqslant E[X(1)^{1/3}]e^{-\alpha n}$. Since it is well known that $E[X(1)^{1/3}]$ is finite, we get that $P(E_{112}^{nc})$ is exponentially small in n and this finishes this proof.

Let us now prove that $P(E_{12}^n \cap E_{10}^n)$ is exponentially small in a positive power of n. Let $Y(k)$ denote the Bernoulli variable which is equal to one when $SUBALGII^n(\xi \circ S|[t_k, t_k + e^{n^{0.3}}])$ is equivalent to $\xi|[min\{x_1^n, x_2^n\} - e^{n^{0.2}}, max\{x_1^n, x_2^n\} + e^{n^{0.2}}]$,where t_k designates the $ke^{n^{0.3}}$-th visit to $\{x_1^n, x_2^n\}$ by $\{S(k)\}_{k \geq 0}$. Then, $E_{12}^{nc} = \{\Sigma_{k \in 1, 2, ..., 2e^{\alpha n}} Y(k)/2e^{\alpha n} \leqslant \frac{1}{2}\}$. When we condition under ξ, we have for each $\xi \in E_{10}^n$ that $\Sigma_{k \in 1, 2, ..., 2e^{\alpha n}} Y(k)$ is stochastically bounded below by a binomial variable with parameters $2e^{\alpha n}$ and $1 - \beta_8 e^{-\beta_9 n^{0.2}}$. Let us assume that for example $1 - \beta_8 e^{-\beta_9 n^{0.2}} \geq \frac{2}{3}$, (which is true for n big enough.) Then, it is well known by a large deviation principle that the probability for a binomial variable with parameters $2e^{\alpha n}$ and $1 - \beta_8 e^{-\beta_9 n^{0.2}}$ to have a value smaller than $\frac{1}{2}(2e^{\alpha n})$ is exponentially small in $e^{\alpha n}$. This is much smaller than exponentially small in n. Thus, if we condition under ξ, we get for each $\xi \in E_{10}^n$ that $P(E_{12}^{nc}|\xi)$ is smaller than an expression of the type $e^{-\beta_{121} e^{\alpha n}}$, (where $\beta_{121} > 0$ is a constant not depending on n nor on $\xi \in E_{10}^n$). Integrating over $\xi \in E_{10}^n$ yields $P(E_{12}^{nc} \cap E_{10}^n) \leqslant e^{-\beta_{121} e^{\alpha n}}$.

Let us now prove that $P(E_{13}^{nc})$ is exponentially small in a positive power of n. Let for any $x \in \mathbb{N}$, $\xi_{13}(x)$ be equal to $\xi(x + x_2^{n+} + e^{n^{0.2}} + 1)$ and let for each $x < 0$, $\xi_{13}(x)$ be equal to $\xi(x + x_2^{n-} - e^{n^{0.2}})$. Then it is easy to check that $\{\xi_{13}(x)\}_{x \in \mathbb{Z}}$ is an i.i.d. scenery with parameter $\frac{1}{2}$ which is independent of $\xi|[x_2^{n-} - e^{n^{0.2}}, x_2^{n+} + e^{n^{0.2}}]$. Now for $k > 0$ and $x \in \mathbb{Z}$ let E_{13xk}^n designate the event that $SUBALGII^n(\xi \circ S|[t_{xk}, t_{xk} + e^{n^{0.3}}])$ is not equivalent to $\xi|[min\{x_1^n, x_2^n\} - e^{n^{0.2}}, max\{x_1^n, x_2^n\} + e^{n^{0.2}}]$ where t_{xk} designates the k-th visit by $\{S(k)\}_{k \geq 0}$ to the point $x + x_2^{n+} + e^{n^{0.2}} + 1$ when $x \geq 0$ and the k-th visit by $\{S(k)\}_{k \geq 0}$ to the point $x + x_2^{n-} - e^{n^{0.2}}$ when $x < 0$. Now for x such that $|x| \geq e^{n^{0.3}}$ we have that $\xi \circ S|[t_{xk}, t_{xk} + e^{n^{0.3}}]$ is independent of $\xi|[x_2^{n-} - e^{n^{0.2}}, x_2^{n+} + e^{n^{0.2}}]$. Thus, in the case that $|x| \geq e^{n^{0.3}}$ we have that $SUBALGII^n(\xi \circ S|[t_{xk}, t_{xk} + e^{n^{0.3}}])$ is independent of $\xi|[min\{x_1^n, x_2^n\} - e^{n^{0.2}}, max\{x_1^n, x_2^n\} + e^{n^{0.2}}]$ Furthermore, all the bits of ξ outside $[x_2^{n-}, x_2^{n+}]$ are i.i.d. so that the piece of scenery $\xi|[min\{x_1^n, x_2^n\} - e^{n^{0.2}}, max\{x_1^n, x_2^n\} + e^{n^{0.2}}]$ has at least $e^{n^{0.2}}$ i.i.d. bits. Thus, we get for x such that $|x| \geq e^{n^{0.3}}$ that $P(E_{13xk}^{nc}) \leq (\frac{1}{2})^{e^{n^{0.2}}}$. Now because, the random walk $\{S(k)\}_{k \geq 0}$ starts at the origin and because up to time $e^{10\alpha n}$ it can not visit any point more than $e^{10\alpha n}$ times or any point outside $[-e^{10\alpha n}, e^{10\alpha n}]$, we get that $\cap_{e^{10\alpha n} \geq |x| \geq e^{n^{0.3}}; 0 \leqslant k \leqslant e^{10\alpha n}} E_{13xk}^n \subset E_{13}^n$. Thus, $P(E_{13}^{nc}) \leq \Sigma_{e^{10\alpha n} \geq |x| \geq e^{n^{0.3}}; 0 \leqslant k \leqslant e^{10\alpha n}} P(E_{13xk}^{nc})$. This implies that

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$P(E_{13}^{nc}) \leqslant 2e^{20\alpha n}(\frac{1}{2})^{e^{n^{0.2}}}$. The expression on the right side of the last inequality being much smaller than exponentially small in a positive power of n we are done with the proof that $P(E_{13}^{nc})$ is exponentially small in a positive power of n.

Let us now prove that $P(E_8^{nc})$ is exponentially small in a positive power of n. Note that $E_1^n \cap E_2^n \cap E_3^n \cap E_4^n \cap E_5^n \cap E_6^n \cap E_7^n \subset E_8^n$. Thus, $P(E_8^{nc}) \leqslant \Sigma_{i \in 1,2,...,7} P(E_i^{nc})$. So we only need to show that for all $i \in 1, 2, ..., 7$, $P(E_i^{nc})$ is exponentially small in a positive power of n. This is what we are going to do next. We already mentioned that we will leave it to the reader to prove that $P(E_1^{nc})$ is exponentially small in n. So let us start with:

Proof that $P(E_2^{nc})$ is exponentially small in a positive power of n. It is a well known fact that there exists constants $\beta_{11}, \beta_{12} > 0$ not depending on k or l such that for each $k, l > 0$ we have that the probability for the random walk $\{S(k)\}_{k \geq 0}$ to stay in the interval $[-k, k]$ up to time l is smaller than $\beta_{11} e^{-\beta_{12} l/k^2}$. Thus, we have that $P(E_2^{nc}) \leqslant \beta_{11} e^{-4\beta_{12} e^n}$. The expression on the right side of the last inequality is obviously much smaller than exponentially small in a positive power of n.

Proof that $P(E_3^{nc})$ is exponentially small in a positive power of n. Note that from what we said about the probability for the random walk $\{S(k)\}_{k \geq 0}$ to stay in the interval $[-k, k]$ up to time l it follows that the conditional probability for a block of $\xi \circ S$ given that it was generated on a block of ξ of length k, to be longer than l is smaller than $\beta_{11} e^{-\beta_{12} l/k^2}$. Let $E_3^n(k)$ designate the event that the k-th block of $\xi \circ S$ has length strictly shorter than n^2 whenever it has been generated on a block of ξ of length shorter than $n^{0.4}$. Then, we have that $P(E_3^{nc}(k)) \leqslant \beta_{11} e^{-\beta_{12} n^{1.2}}$. However, $\cap_{k \in 1,2,...,2e^{3n}} E_3^n(k) \subset E_3^n$. Thus, $P(E_3^{nc}) \leqslant \Sigma_{k \in 1,2,...,2e^{3n}} P(E_3^{nc}(k))$ and $P(E_3^{nc}) \leqslant 2e^{3n}(\beta_{11} e^{-\beta_{12} n^{1.2}})$. The expression on the right side of the last inequality is smaller than exponentially small in n and thus we are done with this proof.

Proof that $P(E_5^{nc})$ is exponentially small in a positive power of n. Let for this proof only, $X(k)$ designate the time of the k-th visit by $\{S(k)\}_{k \geq 0}$ to $\{x_1^n, x_2^n\}$ and $Y(k)$ designate the Bernoulli variables which is equal to 1 iff $X(k)$ is the left end of a block of $\xi \circ S$ which has been generated by $\{S(k)\}_{k \geq 0}$ on $\{x_1^n, x_2^n\}$ and which is longer or equal to i^n. Let E_{51}^n designate the event that within time $e^{n^{0.3}}$ of the first visit by $\{S(k)\}_{k \geq 0}$ to $\{x_1^n, x_2^n\}$ we have that $\{S(k)\}_{k \geq 0}$ visits $\{x_1^n, x_2^n\}$ more than $e^{(1/6)n^{0.3}}$ times. Let E_{52}^n be the event that among the first $e^{(1/6)n^{0.3}}$ visits by $\{S(k)\}_{k \geq 0}$ to $\{x_1^n, x_2^n\}$ more than $e^{n^{0.2}}$ happen to be left ends of blocks of $\xi \circ S$ which were generated by $\{S(k)\}_{k \geq 0}$ on $\{x_1^n, x_2^n\}$ and which are longer or equal to i^n. Then, we have that $E_{51}^n \cap E_{52}^n \subset E_5^n$ and thus $P(E_5^{nc}) \leqslant P(E_{51}^{nc}) + P(E_{52}^{nc})$. Now one can prove in a similar manner to the proof for $P(E_{11}^{nc})$, that $P(E_{51}^{nc})$ is exponentially small in $e^{n^{0.3}}$. So it only remains to prove that $P(E_{52}^{nc})$ is small enough. Now $P(E_{52}^{nc}) \leqslant P(\Sigma_{1 \leqslant k \leqslant e^{(1/6)n^{0.3}}} Y(k) \leqslant e^{n^{0.2}})$. However by the strong Markov property of $\{S(k)\}_{k \geq 0}$ and by lemma 6 we have that $\Sigma_{1 \leqslant k \leqslant e^{(1/6)n^{0.3}}} Y(k)$ is stochastically bounded below by the sum of $e^{(1/6)n^{0.3}}$ i.i.d. Bernoulli variables with parameter

$\frac{1}{2}n^{-\beta_{10}}$. For k such that $0 < k \leqslant e^{(1/6)n^{0.3}}/(2n^{\beta_{10}}e^{n^{0.2}})$ let us define the variable $Y_1(k)$ to be equal to $\Sigma_{k(2n^{\beta_{10}}e^{n^{0.2}}) \leqslant l \leqslant (k+1)(2n^{\beta_{10}}e^{n^{0.2}})} Y(l)$. Then because of the Poisson convergence theorem, we have that there exists a constant $\beta_{13} > 0$ not depending on n or k such that $P(Y_1(k) \leqslant e^{n^{0.2}}) \leqslant 1 - \beta_{13}$. Now, we have that $P(\Sigma_{1 \leqslant k \leqslant e^{(1/6)n^{0.3}}} Y(k) \leqslant e^{n^{0.2}}) \leqslant P(\cap_{0 < k \leqslant e^{(1/6)n^{0.3}}/(2n^{\beta_{10}}e^{n^{0.2}})}\{Y_1(k) \leqslant e^{n^{0.2}}\})$. Now the expression on the right side of the last inequality is equal to $P(Y_1(k) \leqslant e^{n^{0.2}})$ to the power $[e^{(1/6)n^{0.3}}/(2n^{\beta_{10}}e^{n^{0.2}})]$. Thus $P(E_{52}^{nc}) \leqslant (1 - \beta_{13})^{[e^{(1/6)n^{0.3}}/(2n^{\beta_{10}}e^{n^{0.2}})]}$ and $P(E_{52}^{nc}) \leqslant e^{-\beta_{13}[e^{(1/6)n^{0.3}}/(2n^{\beta_{10}}e^{n^{0.2}})]}$. The expression on the right side of the last inequality is much smaller than exponentially small in n and so we are done.

Proof that $P(E_6^{nc})$ is exponentially small in a positive power of n. Condition under the information which among the first $e^{n^{0.2}}$ blocks of length $\geq i^n$ produced by $\{S(k)\}_{k \geq 0}$ on the block $\{x_1^n, x_2^n\}$ are produced by $\{S(k)\}_{k \geq 0}$ crossing $\{x_1^n, x_2^n\}$ and which are obtained by $\{S(k)\}_{k \geq 0}$ entering $\{x_1^n, x_2^n\}$ on the same side then it leaves it. When we condition under this information, the lengths of the first $e^{n^{0.2}}$ blocks of length $\geq i^n$ produced by $\{S(k)\}_{k \geq 0}$ on the block $\{x_1^n, x_2^n\}$ become independent and their distributions are equal to $\mathcal{L}[T^m | T^m \geq i^n; S(T^m) = -1]$ or $\mathcal{L}[T^m | T^m \geq i^n; S(T^m) = m - 1]$, where m designates the length of the block $\{x_1^n, x_2^n\}$. (For the definition of the random variable T^m see lemma 6.) We assume that $m \leqslant 2n$ (this holds anyhow up to an exponentially small quantity in n.) Then it is easy to check that the variables having their law equal to $\mathcal{L}[T^m | T^m \geq i^n; S(T^m) = -1]$ or $\mathcal{L}[T^m | T^m \geq i^n; S(T^m) = m - 1]$ have their tails exponentially bounded in n^2. Using lemma 6 one can then apply a large deviation principle and find that $P(E_6^{nc})$ is exponentially small in $e^{n^{0.2}}$.

Proof that $P(E_7^{nc})$ is exponentially small in a positive power of n. Let $\tau_{7+}^n(k)$, resp. $\tau_{7-}^n(k)$ designate the right end, resp. the left end of the k-th block of $\xi \circ S$ of length $\geq i^n$ which has been generated by $\{S(k)\}_{k \geq 0}$ on a block of ξ of length strictly shorter than n. Let $E_7^n(k)$ be the event that the average $(1/e^{n^{0.2}})\Sigma_{k - e^{n^{0.2}} \leqslant l \leqslant k} \tau_+^n(l) - \tau_-^n(l)$ is smaller than c^n. Then, we have that $\cap_{e^{n^{0.2}} \leqslant k \leqslant 2e^{3n}} E_7^n(k) \subset E_7^n$. Thus, $P(E_7^{nc}) \leqslant \Sigma_{e^{n^{0.2}} \leqslant k \leqslant 2e^{3n}} P(E_7^{nc}(k))$. However, using similar arguments than the one used for the proof for $P(E_7^{nc})$ one gets that $P(E_7^{nc}(k)$ is exponentially small in $e^{n^{0.2}}$. Since in the sum $\Sigma_{e^{n^{0.2}} \leqslant k \leqslant 2e^{3n}} P(E_7^{nc}(k))$ there are only exponentially many terms in n, we have that the sum $\Sigma_{e^{n^{0.2}} \leqslant k \leqslant 2e^{3n}} P(E_7^{nc}(k))$ and thus $P(E_7^{nc}(k)$ are both much smaller than exponentially small in n.

4 The subalgorithm $SUBALGI^n$

The goal of this section is to define the subalgorithm $SUBALGI^n$ and to prove theorem 3. Now recall that theorem 3 says that whenever $SUBALGI^n$ is given right input, then with probability close to one $SUBALGI^n$ gives as output a piece of scenery for which there exists a (random) interval I such that

$[-e^n, e^n] \subset I \subset [-4e^n, 4e^n]$ and such that $\xi|I$ is equivalent to that piece of scenery. By "given the right input" we mean that $SUBALGI^n$ is given a $\mathcal{H}$-adapted sequence of strictly increasing stopping times all stopping the random walk $\{S(k)\}_{k \geq 0}$ in $[-e^n, e^n]$ as well as a piece of length at least n^2 of $\xi|[-e^n, e^n]$ and the observations $\xi \circ S$ restricted to the period between the first and the last stopping time. By close to one, we mean close to one up to a negatively exponentially small quantity in n. Next we need a definition: let $\phi : D \longrightarrow \{0, 1\}$ be a piece of scenery. Let I be an integer interval such that $I \subset D$. Let φ be a piece of scenery which is equivalent to $\phi|I$. Then we say that φ is contained in ϕ. Now let E_{00}^n be the event that the piece of scenery $\xi|[-e^{3n}, e^{3n}]$ contains every piece of scenery of length $n13/(ln2)$ at most once. More precisely, $E_{00}^n = \{$if $i_1, i_2, i_3, i_4 \in [-e^{3n}, e^{3n}]$ are such that $|i_1 - i_2|$, $|i_3 - i_4| = n13/(ln2)$ and such that for all $k \in 0, 1, ..., n13/(ln2)$, we have that $\xi(i_1 + k(i_2 - i_1)/|i_2 - i_1|) = \xi(i_3 + k(i_4 - i_3)/|i_4 - i_3|)$, then $i_1 = i_3$ and $i_2 = i_4.\}$. Now one can prove that E_{00}^n holds with high probability. More precisely, there exists $\beta_{14}, \beta_{15} > 0$ such that $P(E_{00}^{nc}) \leq \beta_{14} e^{-\beta_{15} n}$ for all $n > 0$. (The proof of this fact is very similar to the proof given in and thus we leave this proof to the reader.) Let SET be a set of pieces of scenery. Then we say that the set SET has property P_1^n iff each element of SET is a piece of scenery which is contained in $\xi|[-e^{3n}, e^{3n}]$ and if for each interval $I \subset [-3e^n, 3e^n]$ of length $n(13/(ln2)) + 1$ there is at least one piece of scenery which is a member of SET which contains the piece of scenery $\xi|I$. Now, if we would be given a piece of scenery ψ of length n^2 which is contained in $\xi|[-e^n, e^n]$ and a set SET which has property P_1^n and if on top of all that E_{00}^n would hold, then we could construct a piece of scenery ξ^n for which there would exist an interval I such that ξ^n is equivalent to $\xi|I$ and $[-e^n, e^n] \subset I \subset [-4e^n, 4e^n]$. The way to construct such a piece of scenery ξ^n can be described as follows:

Algorithm 8 *step a) Place ψ at the origin. i.e. let ψ' be any piece of scenery equivalent to ψ containing zero in its domain. step b) find any sequence of pieces of sceneries $\varphi_0, \varphi_1, \varphi_2, ..., \varphi_i, ..., \varphi_j$ such that $\varphi_0 = \psi'$ and such that for all $i \in 1, 2, ..., j$ we have that φ_i is equivalent to a piece of scenery which belongs to SET and the domain of φ_i intersects the domain of at least one piece of scenery from $\varphi_0, \varphi_1, \varphi_2, ..., \varphi_{i-1}$ in an integer interval of length at least $13/(ln2)$ and for all $i_1, i_2 \in 1, 2, ..., j$ we have that φ_{i_1} and φ_{i_2} coincide on the intersection of their respective domains. Eventually we also ask that the union of the domains of $\varphi_0, \varphi_1, \varphi_2, ..., \varphi_i, ..., \varphi_j$ cover $[-2e^n, 2e^n]$. Once you found such a sequence let ξ^n be the piece of scenery with domain $[-2e^n, 2e^n]$ which coincides with all the pieces $\varphi_0, \varphi_1, \varphi_2, ..., \varphi_i, ..., \varphi_j$ on the intersection of their domains and $[-2e^n, 2e^n]$. ξ^n designates then the output of the here described procedure.*

In other words the above algorithm works as follows: we put ψ at the origin and then from there on take pieces from SET one after another and "place them on $\mathbb{Z}$" by shifting them around and turning them around so that each piece we place on $\mathbb{Z}$ coincides with at least one previously placed piece on an interval of length at least $13/(ln2)$. We also ask that all the pieces picked and placed

on $\mathbb{Z}$ coincide pair-wise on the intersection of their domains. We try to cover at least the interval $[-2e^n, 2e^n]$. The final output will be the piece of scenery with domain $[-2e^n, 2e^n]$ which coincides with all the pieces placed on $\mathbb{Z}$ on their respective domains. Now the important fact is that the above method works, that is the above method produces a piece of scenery ξ^n such that there exist an interval I such that ξ^n is equivalent to $\xi|I$ and $[-e^n, e^n] \subset I \subset [-4e^n, 4e^n]$ as soon as the following conditions are satisfied: the piece of scenery ψ is of length at least n^2 and is equivalent to a restriction of $\xi|[-e^n, e^n]$ whilst the set SET has property P_1^n and E_{00}^n holds. To check this, note the following: first when ψ is a piece of scenery of length n^2 which is contained in $\xi|[-e^n, e^n]$ and SET has property P_1^n and E_{00}^n holds, then if two pieces of scenery φ_{i_1} and φ_{i_2} which are both contained in $\xi|[-e^{3n}, e^{3n}]$ coincide on an interval of length at least $13/(ln2)$, then the two pieces φ_{i_1} and φ_{i_2} have same relative position to each other then the two pieces of sceneries which are restrictions of $\xi|[-e^{3n}, e^{3n}]$ and which are equivalent to φ_{i_1} and φ_{i_2}. This implies that when the piece of scenery ψ of length n^2 is contained in $\xi|[-e^{3n}, e^{3n}]$ and SET has property P_1^n and E_{00}^n holds, then all the pieces of sceneries $\varphi_0, \varphi_1, \varphi_2, ..., \varphi_i, ..., \varphi_j$ have same relative position to each other then the corresponding pieces of sceneries $\bar\varphi_0, \bar\varphi_1, \bar\varphi_2, ..., \bar\varphi_i, ..., \bar\varphi_j$. (Here, $\bar\varphi_i$ denotes the only piece of scenery which is obtained by restriction from $\xi|[-e^{3n}, e^{3n}]$ and which is equivalent to φ_i.) Now this implies that if we take the piece of scenery with domain being equal to the unions of the domains of the pieces of sceneries $\varphi_0, \varphi_1, \varphi_2, ..., \varphi_i, ..., \varphi_j$ and which coincides with each φ_i on their respective domain, then that piece of scenery is equivalent to the restriction of $\xi|[-e^{3n}, e^{3n}]$ to an integer interval. (Of course assuming that the conditions which we assumed for ψ and SET hold as well as E_{00}^n.) Now the "placed "pieces of sceneries $\varphi_0, \varphi_1, \varphi_2, ..., \varphi_i, ..., \varphi_j$ are not further than $2e^n$ from the piece of scenery ψ' since ψ' corresponds to a piece of scenery which can be obtained by restriction from $\xi|[-e^n, e^n]$. This implies that the pieces $\bar\varphi_0, \bar\varphi_1, \bar\varphi_2, ..., \bar\varphi_i, ..., \bar\varphi_j$ are not further than $e^n + 2e^n$ from the origin. This already implies that the piece of scenery ξ^n is equivalent to a restriction $\xi|I$ where $I \subset [-3e^n, 3e^n]$. It remains to prove that $[-e^n, e^n] \subset I$. Now, the question is can we find enough φ_i 's such that the union of all the domains of the pieces of sceneries $\varphi_0, \varphi_1, \varphi_2, ..., \varphi_i, ..., \varphi_j$ covers the interval $[-2e^n, 2e^n]$. As a matter of fact, if they do cover $[-2e^n, 2e^n]$ then we have covered an area with radius $2e^n$ left and right from ψ'. However, ψ' is equivalent to a restriction of $\xi|[-e^n, e^n]$ denoted by $\bar\varphi_0$. Because of what we said about the same relative positions to each other, the fact that the union of the domains of $\varphi_0, \varphi_1, \varphi_2, ..., \varphi_i, ..., \varphi_j$ covers the interval $[-2e^n, 2e^n]$,(i.e. covers the domain with radius $2e^n$ left and right from φ_0) implies that the union of the domains of $\bar\varphi_0, \bar\varphi_1, \bar\varphi_2, ..., \bar\varphi_i, ..., \bar\varphi_j$ covers the domain with radius $2e^n$ left and right from $\bar\varphi_0$. Since $\bar\varphi_0$ has its domain located in the interval $[-e^n, e^n]$ this means that the union of the domains of $\bar\varphi_0, \bar\varphi_1, \bar\varphi_2, ..., \bar\varphi_i, ..., \bar\varphi_j$ covers at least $[-e^n, e^n]$ and thus in this case we would have that $[-e^n, e^n] \subset I$. So it only remains to prove that in SET there are enough pieces of sceneries to ensure that the union of all the domains of $\varphi_0, \varphi_1, \varphi_2, ..., \varphi_i, ..., \varphi_j$ covers the interval $[-2e^n, 2e^n]$. Now, P_1^n guarantees that we have "enough" pieces of

sceneries in the set SET which are equivalent to a restriction of $\xi|[-3e^n, 3e^n]$. Since $\bar{\varphi}_0$ is equivalent to a restriction of $\xi|[-e^n, e^n]$ this implies that in a radius $2e^n$ of $\bar{\varphi}_0$ we have enough restrictions of ξ which are equivalent to a piece of SET. This implies that in a radius of $2e^n$ of φ_0 we have enough pieces of sceneries $\varphi_0, \varphi_1, \varphi_2, ..., \varphi_i, ..., \varphi_j$ and thus the union of all the domains of $\varphi_0, \varphi_1, \varphi_2, ..., \varphi_i, ..., \varphi_j$ covers the interval $[-2e^n, 2e^n]$. And so we are done with proving that the above method produces a piece of scenery ξ^n such that there exist an interval I (may be random) such that ξ^n is equivalent to $\xi|I$ and $[-e^n, e^n] \subset I \subset [-4e^n, 4e^n]$ as soon as the above mentioned conditions for ψ, SET and E_{00}^n hold.

We have thus reduced the problem of constructing a piece of scenery which is equivalent to a restriction $\xi|I$ such that $[-e^n, e^n] \subset I \subset [-4e^n, 4e^n]$ to the problem of constructing a set of pieces of sceneries SET satisfying the condition P_1^n. Thus, in order to prove theorem 3, it remains to prove that there exists an algorithm for each $n > 0$ which is capable with high probability to construct a collection of pieces of sceneries SET satisfying the condition P_1^n. Now in the formulation which we choose for theorem 3, we did not assume that the stopping times $\tau(1), \tau(2), ..., \tau(e^{\alpha n})$ all stop $\{S(k)\}_{k \geq 0}$ in the interval $[-e^n, e^n]$. We rather give an upper bound for the event that "$SUBALGI^n$" does not work intersected with the event that all the stopping times $\tau(1), \tau(2), ..., \tau(e^{\alpha n})$ stop $\{S(k)\}_{k \geq 0}$ in the interval $[-e^n, e^n]$. It would be equivalent to prove theorem 3 only for those stopping times $\tau(1), \tau(2), ..., \tau(e^{\alpha n})$ which all stop $\{S(k)\}_{k \geq 0}$ in the interval $[-e^n, e^n]$. As a matter of fact, if $\tau = (\tau(1), \tau(2), ..., \tau(e^{\alpha n}))$ denotes a $\mathcal{H}$-adapted sequence of strictly increasing of stopping times which do not all stop $\{S(k)\}_{k \geq 0}$ in the interval $[-e^n, e^n]$, then we can define another $\mathcal{H}$-adapted sequence of strictly increasing stopping times.$\bar{\tau} = (\bar{\tau}(1), \bar{\tau}(2), ..., \bar{\tau}(e^{\alpha n}))$ which do all stop $\{S(k)\}_{k \geq 0}$ in the interval $[-e^n, e^n]$ and which is such that when $E_{\tau,0}^n$ holds $\tau = \bar{\tau}$. (To define such a $\bar{\tau}$ simply put $\tau = \bar{\tau}$ when all the stopping times of τ stop $\{S(k)\}_{k \geq 0}$ in the interval $[-e^n, e^n]$ and otherwise if $\tau(i)$ is the first stopping time of τ for which $S(\tau(i)) \notin [-e^n, e^n]$, define $\bar{\tau}(j) = \tau(j)$ for all $j < i$ and for all $i \leqslant j \leqslant e^{\alpha n}$ let $\bar{\tau}(j)$ be the $(j - i + 1)$-th visit by $\{S(k)\}_{k \geq 0}$ at the origin after $\tau(i)$. Now, obviously an upper bound for $P(E_{\bar{\tau}}^{nc})$ is also an upper bound for $P(E_{\tau,1}^{nc} \cap E_{\tau,0}^n)$. This proves that if we can prove theorem 3 just for those $\mathcal{H}$-adapted sequences of $e^{\alpha n}$ strictly increasing stopping times which do all stop $\{S(k)\}_{k \geq 0}$ in the interval $[-e^n, e^n]$ then theorem 3 holds. Thus from now on we will assume until the end of this section that $\tau = (\tau(1), \tau(2), ..., \tau(e^{\alpha n}))$ denotes a $\mathcal{H}$-adapted sequence of strictly increasing stopping times which do all stop $\{S(k)\}_{k \geq 0}$ in the interval $[-e^n, e^n]$. With this and the previous remark we can now formulate what remains to be done in this section:

We need to prove that for each $n > 0$ there exists an algorithm $SUBALGIII^n$ (producing as output a collection of pieces of sceneries) and constants $\beta_{16}, \beta_{17} > 0$ such that for all $n > 0$ and all $\mathcal{H}$-adapted sequence of strictly increasing stopping times.$\tau = (\tau(1), \tau(2), ..., \tau(e^{\alpha n}))$ which do all stop $\{S(k)\}_{k \geq 0}$ in the interval $[-e^n, e^n]$ we have that $P\{$for every piece of scenery ψ of length longer than n^2 and which is contained in $\xi|[-e^n, e^n]$ the collection of pieces of sceneries $SUBALGIII^n(\tau, \xi \circ S|[\tau(1), \tau(e^{\alpha n})], \psi)$ satisfies condition $P_1^n\} \geq 1 - \beta_{16} e^{-\beta_{17} n}$.

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Next we are going to give a rough description of how we are going to define the algorithm $SUBALGIII^n$. For this we will need the following definitions:

Definition 9 *Let χ denote a binary sequence of length e^{3n}. Thus, $\chi \in \{0,1\}^{e^{3n}}$. In what follows $c_1 > 0$ will designates a constant not depending on n, (for the definition see at the beginning of subsection 4.3). Let h be the "truncating" function from $\mathbb{N}$ to $\{1,2,3,4,5,6\}$ such that for all $i \geq 6$ we have $h(i) = 6$, and $h(1) = 1$, $h(2) = 2$, $h(3) = 3$, $h(4) = 4$, $h(5) = 5$. Now, let $d > 0$ and let $B^n(\chi), OBS^n(d)(\chi), F^n(d)(\chi), b^n(\chi), f^n(d)(\chi)$ denote the following things: let $B^n(\chi)$ denote the $c_1 n$ -component vector who's i-th entry for $i \leqslant c_1 n$ is equal to the value of the function h at the length of the i-th block of χ after e^{2n}, (if it exists,). In other words the $c_1 n$ components of the measurable vector $B^n(\chi)$ are made out of the truncated lengths of the first $c_1 n$ blocks of χ. Let $b^n(\chi)$ designate the right end of the $c_1 n$-th block of χ, (if it exists). Let $f^n(d)(\chi)$ designate the left end of the first block of $\chi \| [b^n(\chi) + d, e^{3n}[$. Let $OBS^n(d)(\chi)$ designate the binary word which are the observations χ between $b^n(\chi)$ and $f^n(d)(\chi)$. More precisely, $OBS^n(d)(\chi) = (\chi(b^n(\chi)), \chi(b^n(\chi) + 1), \chi(b^n(\chi) + 2), ..., \chi(f^n(d)(\chi)))$. Let $F^n(d)(\chi)$ be the $c_1 n$ -component vector made out of the truncated lengths of the first $c_1 n$ blocks of $\chi \| [f^n(d)(\chi), e^{3n}]$. (when the definitions above don't make sense because there are not enough blocks in χ then define $B^n(\chi), OBS^n(d)(\chi), F^n(d)(\chi), b^n(\chi), f^n(d)(\chi)$ in any way you want, but depending only on χ.) Let $E^n_\chi(d)$ denote the event that $\chi(b^n(\chi)+d) \neq \chi(b^n(\chi)+d+1)$. Let $0 < d_1 < d_2$ then it is easy to check that $B^n, OBS^n(d_2), F^n, b^n, f^n(d_2)$ uniquely determines $B^n, OBS^n(d_1), F^n, b^n, f^n(d_1)$. Thus,*

$$\sigma(B^n, OBS^n(d_1), F^n, f^n(d_1) - b^n) \subset \sigma(B^n, OBS^n(d_2), F^n, f^n(d_2) - b^n)$$

(on $\{0,1\}^{e^{3n}}$). Next we are going to define a couple of measures on the $\sigma-$algebra $\sigma(B^n, OBS^n(d), F^n, f^n(d) - b^n)$ for $d = 17n/ln2$. Let $\{S_x(k)\}_{k \geq 0}$ be a random walk starting at the point x which is independent of $\{\xi(k)\}_{k \in \mathbb{Z}}$, having same transition distribution as the process $\{S_x(k)\}_{k \geq 0}$. Now let $\mathcal{L}_x(.|\xi)$ be the (random) probability measure induced on $\sigma(B^n, OBS^n(d), F^n, f^n(d) - b^n)$ for $d = 17n/ln2$, by the measure for $\sigma(\chi)$ obtained by putting $\chi = \xi \circ S_x \| [0, e^{3n}]$ and conditioning under ξ. So whenever we write $P_x(...)$ or $L_x(...)$ we will see $B^n, OBS^n(d), F^n(d), f^n(d) - b^n$ as random variables living on the same space as ξ and S_x which are equal to the function $B^n, OBS^n(d), F^n(d), f^n(d) - b^n$ at $\chi = \xi \circ S_x \| [0, e^{3n}]$. For all integer $x \in [-e^n, e^n]$ let $a(x)$ be equal to the proportion of k's in $k \in 1,2,...,e^{\alpha n}/e^{3n} - 1$ such that $S(\tau(ke^{3n}) = x)$. (I.e. we only take every e^{3n}-th stopping times from $\tau = (\tau(1), \tau(2), ..., \tau(e^{\alpha n}))$ and we look at which percentage of theses stopping times stop $\{S(k)\}_{k \geq 0}$ at the point x.) Since we assumed that a.s. all stopping times in $\tau = (\tau(1), \tau(2), ..., \tau(e^{\alpha n}))$ stop $\{S(k)\}_{k \geq 0}$ in $[-e^n, e^n]$ we get that the collection of coefficients $a(x)$, with x an integer of $[-e^n, e^n]$, forms a.s. the coefficients for a convex combination. Let μ designate the random measure on $\sigma(B^n, OBS^n(17n/ln2), F^n, f^n(17n/ln2) - b^n)$ defined by $\mu = \sum_{x \in [-exp(n), exp(n)]} a(x) \mathcal{L}_x(.|\xi)$. Let $\hat{\mu}$ designate the empirical distribution of $(B^n, OBS^n(17n/ln2), F^n, f^n(17n/ln2) - b^n)$ obtained by taking the observations $\xi \circ S$ after every e^{3n}-th stopping time $\tau = (\tau(1), \tau(2), ..., \tau(e^{\alpha n}))$. More

precisely, for all i such that $1 \leqslant i \leqslant e^{\alpha n}/e^{3n} - 1$ let χ^i denote the finite sequence $\xi(S(\tau(ie^{3n})))$, $\xi(S(\tau(ie^{3n})+1))$, $\xi(S(\tau(ie^{3n})+2))$, ...$\xi(S(\tau(ie^{3n})+e^{3n}))$. Let $\hat{\mu}$ denote the empirical distribution on $\sigma(B^n, OBS^n(17n/ln2), F^n, f^n(17n/ln2) - b^n)$ based on the variables χ^i, $1 \leqslant i \leqslant e^{\alpha n}/e^{3n} - 1$. Let ε be the signed measure on $\sigma(B^n, OBS^n(17n/ln2), F^n, f^n(17n/ln2) - b^n)$ which is equal to $\varepsilon = \hat{\mu} - \mu$. If $\heartsuit$ designates a finite set containing i elements, then any probability measure on the σ-algebra of all the subsets of $\heartsuit$, i.e. on$\mathcal{P}(\heartsuit)$, will be viewed as a vector of $\mathbb{R}^i$. As a matter of fact, if for each element $\natural \in \heartsuit$ we know the value of the probability measure on $\{\natural\}$ then that probability measure is uniquely defined. Thus if ν is a measure on $\mathcal{P}(\heartsuit)$, we will represent ν as the i-dimensional vector having its j-th coordinate for $j \leq i$ equal to $\nu(\{\natural_j\})$ where $\natural_j$ is the j-th element of the set $\heartsuit$. Let $\spadesuit$ be a set with k elements. Let ν_2 be a probability measure on $\mathcal{P}(\spadesuit)$. Then the product of the two measures ν and ν_2, i.e. $\nu \otimes \nu_2$, will be seen as the $i \times k$ dimensional vector in the tensor product space $\mathbb{R}^i \otimes \mathbb{R}^j$. Following our way of representing probability measure on finite spaces as vectors of x-dimensional real vector space, we get that the vector representing $\nu \otimes \nu_2$ is equal to the tensor product of the vector representing μ multiplied with the vector representing ν. (For those not familiar with tensor products, simply see $\mu \otimes \nu$ as a $i \times k$ matrix having its (j,l)-th entry equal to the product of the two real numbers $\mu(\{\natural_j\}) \times \nu(\{\natural_l\})$ where $\natural_j$ is the j-th element of the set $\heartsuit$ and $\natural_l$ is the l-th element of the set $\spadesuit$.) Now B^n takes values in a set with $5^{c_1 n}$ elements. Thus the law $\mathcal{L}_x(B^n|\xi)$ can thus be viewed as a $5^{c_1 n}$ dimensional vector. Recall that T^n designate the first hitting time of $\{S(k)\}_{k \geq 0}$ on $\{-1, n-1\}$. Define then the defective distributions $\mu(n,0) = P(S(T^n) = -1)\mathcal{L}(T^n|S(T^n) = -1)$ and $\mu(n,1) = P(S(T^n) = n-1)\mathcal{L}(T^n|S(T^n) = n-1)$. Now a linear functional $f : \mathbb{R}^5 \otimes \mathbb{R}^5 \otimes ... \otimes \mathbb{R}^5 \longrightarrow \mathbb{R}$ which has as domain the m-times tensor product of $\mathbb{R}^5$ with itself, is called positive iff for each sequence of natural numbers $n_1, n_2, ..., n_m > 1$ and each sequence of 0 's and 1 's $l_1, l_2, ..., l_m$ we have that $f(\mu(n_1 l_1) \otimes \mu(n_2 l_2) \otimes ... \otimes \mu(n_m l_m)) \geq 0$. Let x_1 be an integer in $[-e^{3n}, e^{3n}]$ such that $\xi(x_1) \neq \xi(x_1 - 1)$. Then, we call a linear functional f a limiting functional of ξ to the left at the point x_1 iff f is a positive functional such that for all $x_2 > x_1$ with $x_2 \in [-e^{3n} - e^n, e^{3n} + e^n]$ and $not(\xi(x_2) = \xi(x_2 - 1) = \xi(x_2 + 1))$ we have that $f(\mathcal{L}_x(B^n)|\xi, S_x(b^n) = x_2)) = 0$ whenever $P(S_x(b^n) = x_3) > 0$ whilst $f(\mathcal{L}_x(B^n)|\xi, S_x(b^n) = x_1) \neq 0$. Let y_1 be an integer in $[-e^{3n}, e^{3n}]$ such that $\xi(y_1) \neq \xi(y_1 + 1)$ and such that $P(S_x(e^{2n}) = y_1) > 0$ for all $x \in [-e^n, e^n]$. Then, we call a linear functional g a limiting functional to the right of the point y_1 iff g is a positive functional such that for all $y_2 < y_1$ with $y_2 \in [-e^{3n}, e^{3n}]$ we have that $g(\mathcal{L}_x(F^n)|\xi, S_x(f^n) = y_2) = 0$ whenever $P(S_x(f^n) = y_2) > 0$ whilst $g(\mathcal{L}_x(F^n)|\xi, S_x(f^n) = y_1) \neq 0$.

We are now ready at this stage to explain informally how $SUBALGIII^n$ is able to construct pieces of sceneries contained in $\xi|[[-e^{3n}, e^{3n}]$. Now we will always assume that the right end of the $c_1 n$-th block of $\xi \circ S|[f^n(17n/ln2), \infty[$ is smaller than $\tau(ie^{3n}) + e^{3n}$. (The probability that the last statement would not hold would be negatively exponentially small in an exponential function of n, and thus would be negligeable since the other events we are dealing with are

only exponentially small in n.) Now, when calculating our empirical distribution $\hat{\mu}$ we only take every $e^{3n}-th$ stopping time from $\tau = (\tau(1), \tau(2), ..., \tau(e^{\alpha n}))$. Since the sequence of stopping times $\tau(1), \tau(2), ..., \tau(e^{\alpha n})$ is strictly increasing, the $ie^{3n} - th$ such stopping times are at least e^{3n} away from each other. Thus he different intervals $[\tau(ie^{3n}), \tau(ie^{3n}) + e^{3n}]$ don't overlap. Because of the strong Markov property of $\{S_x(k)\}_{k \geq 0}$, if χ_x^i designates the i-th χ^j such that $S(\tau(je^{3n})) = x$ then the collection $(B^n, OBS^n(17n/ln2), F^n(17n/ln2), f^n(17n/ln2) - b^n)(\chi_x^i), 1 \leq i \leq a(x)$ becomes i.i.d. (provided we condition under ξ). So if instead of taking our empirical distribution based on $\chi^i, 1 \leq i \leq e^{\alpha n}/e^{3n} - 1$ we would take an empirical distribution based on $\chi_x^i, 1 \leq i \leq a(x)$ then this empirical distribution would be an approximation of $\mathcal{L}_x(.|\xi)$. (Actually we will show in the second subsection of this section that with high probability the difference between this approximation and $\mathcal{L}_x(.|\xi)$ is negatively exponentially small in n and the coefficient in that upper bound which stand next to n can be made as small as we want by just taking the constant α to be big enough.) As already mentioned $\hat{\mu}$ is the empirical distribution base on the collection of variables $(B^n, OBS^n(17n/ln2), F^n(17n/ln2), f^n(17n/ln2) - b^n)(\chi^i), 1 \leq i \leq e^{\alpha n}/e^{3n} - 1$. So be regrouping these variables according to x in χ_x^i we get that $\hat{\mu}$ is an approximation of $\mu = \sum_{x \in [-exp(n), exp(n)]} a(x)\mathcal{L}_x(.|\xi)$. Now if we are given the observations $\xi \circ S|[\tau(1), \tau(e^{\alpha n})]$ as well as the stopping times $\tau(1), \tau(2), ..., \tau(e^{\alpha n})$ then we can compute $\hat{\mu}$. However our algorithm $SUBALGIII^n$ gets as input $\xi \circ S|[\tau(1), \tau(e^{\alpha n})]$ as well as $\tau(1), \tau(2), ..., \tau(e^{\alpha n})$. Thus the first step of the algorithm $SUBALGIII^n$ will be to compute $\hat{\mu}$. It will then analyze $\hat{\mu}$ and this will allow the algorithm $SUBALGIII^n$ to construct pieces of sceneries contained in $\xi|[-e^{3n}, e^{3n}]$. However, to explain in a rough way the general idea behind $SUBALGIII^n$ it is best to explain how one can construct pieces of sceneries contained in $\xi|[-e^{3n}, e^{3n}]$ if one would be given μ. (Note that μ is not observable from $\xi \circ S$, and thus $SUBALGIII^n$ does not know μ. As a matter of fact, since $SUBALGIII^n$ does not know ξ it doesn't know the different $\mathcal{L}_x(.|\xi)$'s for the different x's in $[-e^n, e^n]$; nor does it know the different $a(x)$'s.) Now let us explain how one can construct pieces of sceneries contained in $\xi|[-e^{3n}, e^{3n}]$ if one would be given μ. Let w be a binary word of length $d+1 \leq 17n/ln2$. (Thus w can also be seen as a piece of scenery of length $d+1$.) We saw that any distribution of the measurable object $OBS^n(d)$ can be viewed as a vector in a 2^{d+1} dimensional real space. Let $\{e_v | v \in \{0,1\}^{d+1}\}$ denote the canonical basis in that vector space. Thus, e_v is the vector which would represent the probability measure which would be a singleton at the point v. Let $\{e'_v | v \in \{0,1\}^{d+1}\}$ represent the dual basis of $\{e_v | v \in \{0,1\}^{d+1}\}$. In what follows we will denote by 1_w the linear functional e'_w. Let 1_d denote the sum $\Sigma_{v \in V} 1_v$ where $V = \{v = (v(1), v(2), ..., v(d+1)) \in \{0,1\}^{d+1} | v(d) \neq v(d+1)\}$. Let $x_1 < x_2$ be two integers such that $d = x_2 - x_1$ and such that g_1 is a left limiting functional of ξ at x_1 whilst g_2 is a right limiting functional of ξ at x_2. Let as furthermore assume that when we read the bits of ξ when starting from x_1 and going to x_2 we read the word w. In other words we assume that $\xi|[x_1, x_2]$ is equivalent up to shift with w. Then it is easy to check (by condition-

ing under $S(b^n)$ and $S(f^n)$) that $g_1 \otimes 1_w \otimes g_2(\mathcal{L}_\mu(B^n, OBS^n(d), F^n(d)|\xi) > 0$ whilst $g_1 \otimes 1_{d-1} \otimes g_2(\mathcal{L}_\mu(B^n, OBS^n(d-1), F^n(d-1)|\xi) = 0$. (Here in the last equality 1_{d-1} is here to make sure that $f^n = b^n + d - 1$.) On the other hand one can show that the converse is also true. By this we mean that if there is a binary word w of length $d \leqslant 17n/ln2$ for which there exists two positive functional g_1 and g_2 such that $g_1 \otimes 1_w \otimes g_2(\mathcal{L}_\mu(B^n, OBS^n(d), F^n(d)|\xi) > 0$ and $g_1 \otimes 1_{d-1} \otimes g_2(\mathcal{L}_\mu(B^n, OBS^n(d-1), F^n(d-1)|\xi) = 0$ then the word w (seen as a piece of scenery) is contained in $\xi|[-e^{3n}, e^{3n}]$. Thus if we would be given μ we have a very simple method to find some pieces of sceneries contained in $\xi|[-e^{3n}, e^{3n}]$. Now, when we have $\hat{\mu}$ instead of μ then in general $g_1 \otimes 1_{d-1} \otimes g_2(\mathcal{L}_{\hat{\mu}}(B^n, OBS^n(d-1), F^n(d-1)|\xi)$ is no longer equal to zero, but instead is much smaller than $g_1 \otimes 1_w \otimes g_2(\mathcal{L}_\mu(B^n, OBS^n(d), F^n(d)|\xi)$ in the case that w is a word contained in $\xi|[-e^{3n}, e^{3n}]$. So our method for reconstruction of pieces of sceneries contained in $\xi|[-e^{3n}, e^{3n}]$ can roughly be described as follows: take the binary words (=piece of sceneries) w of length d having its two last bits different from each other and such that there exists positive functionals g_1 and g_2,such that $g_1 \otimes 1_{d-1} \otimes g_2(\mathcal{L}_{\hat{\mu}}(B^n, OBS^n(d-1), F^n(d-1)|\xi)$ is much smaller than $g_1 \otimes 1_w \otimes g_2(\mathcal{L}_\mu(B^n, OBS^n(d), F^n(d)|\xi)$. Those words w will be our guesses for pieces of sceneries which are contained in $\xi|[-e^{3n}, e^{3n}]$. In the next subsection we are going to prove a theorem which shows that this method works if the total variation norm of $\varepsilon = \hat{\mu} - \mu$ is small enough and there exists enough limiting functional. In subsection 4.2 we will than prove that, with high probability, the norm of ε is small enough. In subsection 4.3 we will show that there exists enough limiting functionals and give a precise description of $SUBALGIII^n$.

4.1 Case with error

For a vector $x = (x(1), x(2), ..., x(i))$ of $\mathbb{R}^i$ we will use the two following norms: $|x| = |x(1)| + |x(2)| + ... + |x(i)|$ and $|x|_2 = \sqrt{|x(1)|^2 + |x(2)|^2 + ... + |x(i)|^2}$. Note that $|.| \geq |.|_2$. Recall that p designated the number $p = P(S(i+1) - S(i) = 1) = P(S(i+1) - S(i) = -1)$ whilst $q = P(S(i+1) - S(i) = 0)$ and that $2p + q = 1$. Next we will need a lemma.

Lemma 10 *There exists $n_0 > 0$ such that for all $n \geq n_0$ we have that : Let $2 < d \leqslant 17n/ln2$ and let $0 \leqslant x < d$. If $q \neq 0$ we have $P(S(d) = x)/P(S(d-1) = x) \leqslant 4n^2$. If $q = 0$ and $P(S(d) = x) > 0$ then $P(S(d) = x)/P(S(d-2) = x) \leqslant 4n^2$.*

Proof. We will leave the proof in the case where $q = 0$ to the reader since it is very similar to the case where $q \neq 0$. So let us assume that $q \neq 0$. Let $PATH(d, x)$ denote the set of all the paths with d step starting at time 0 at the origin, ending at the point x and which at each step go one to the right or one to the left or stay in the same position. For $path \in PATH(d, x)$ let the probability of $path$ be the probability that $\{S(k)\}_{k \geq 0}$ follows during its first d steps the path $path$. We will denote that probability by $P(path)$ and

thus $P(path) = P(S|[0,d] = path)$. If the path *path* has l_r steps to the right and l_l steps to the left and l_0 which are holdings (thus $l_r + l_l + l_0 = d$) then $P(path) = p^{l_r + l_l} q^{l_0}$. Now let $funct : PATH(d,x) \longrightarrow PATH(d-1,x)$ be a map which is defined in the following way: for each $path \in PATH(d,x)$ which contains at least one holding let $funct(path)$ be equal to the path obtained by cutting out the first holding from *path*. For $path \in PATH(d,x)$ which contains no holding , because $x < d$ exists in the path *path* at least one left step followed immediately by a step to the right or a step the right followed by a step to the left. So, in case that path $path \in PATH(d,x)$ contains no holding define $funct(path)$ to be equal to the path obtained by replacing in the first two consecutive left-right or right-left steps (take what ever comes first) by a holding. It is easy to see that for all $path \in PATH(d,x)$ we have that $P(path) \leqslant \bar{p} P(funct(path))$ where $\bar{p}$ designates the constant which is equal to the maximum between 1 and $\frac{p^2}{q}$. Furthermore it is also easy to see that in each class of elements of $PATH(d,x)$ which have same image under the function $funct$ there are at most $3d$ elements. this implies that $\Sigma_{path \in PATH(d-1,x)} P(path)$ is at least as big as $\Sigma_{path \in PATH(d,x)} P(path)$ divided by $3d\bar{p}$. Now, $P(S(d) = x) = \Sigma_{path \in PATH(d,x)} P(path)$ and $P(S(d-1) = x) = \Sigma_{path \in PATH(d-1,x)} P(path)$. Thus, we get that $P(S(d) = x)/P(S(d-1) = x) \leqslant 3d\bar{p}$. Now we assumed that $d \leqslant 17n/ln2d$. So at least for n big enough we have $3d\bar{p} \leqslant 4n^2$ and thus $P(S(d) = x)/P(S(d-1) = x) \leqslant 4n^2$. ∎

Next we are going to formulate a theorem:

Theorem 11 *Case where $q \neq 0$) Let $2 < d \leqslant 17n/ln2$ and let w designate a binary word of length $d + 1$ having its two last bits different from each other. Let us furthermore assume that there exists to positive functionals g_1 and g_2 such that the following three conditions all hold:*

a) $g_1 \otimes 1_w \otimes g_2(P_{\hat{\mu}}(E_d^n)\mathcal{L}_{\hat{\mu}}(B^n, OBS^n(d), F^n(d)|\xi, E_d^n) > 1$

b) $g_1 \otimes g_2(P_{\hat{\mu}}(E_{d-1}^n)\mathcal{L}_{\hat{\mu}}(B^n, F^n(d-1)|\xi, E_{d-1}^n) \leq 1/(9n^2)$

and

c) $|g_1 \otimes g_2|_2 \otimes |\varepsilon|_1 \leqslant \frac{1}{2}/n^2$

. Then the word w (seen as a piece of scenery) is contained in $\xi|[-e^{3n}, e^{3n}]$. Case where $q = 0$) Same thing as for $q \neq 0$ but simply replace condition b by condition b' which is:

b') $g_1 \otimes g_2(P_{\hat{\mu}}(E_{d-2}^n)\mathcal{L}_{\hat{\mu}}(B^n, OBS^n(d-2), F^n(d-2)|\xi, E_{d-2}^n) \leq 1/(9n^2)$

Proof. The case where $q = 0$ is similar to the case where $q \neq 0$ and thus we will leave the proof in the case $q = 0$ to the reader. So let us assume that $q \neq 0$. We are going to do the proof by the absurd. We assume that there exists

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no interval $I \subset [-e^{3n}, e^{3n}]$ such that $\xi|I$ is equivalent to w and show that this the three conditions a,b,c in our theorem.. We call inadmissible path a finite path which at each step goes at most one to the right or one to the left or stays at the same spot. So a map $R : J = [j_1, j_2] \longrightarrow \mathbb{Z}$ where J denotes an integer interval, is called an admissible path iff for all $j \in [j_1, j_2 - 1]$ we have that $R(j + 1) - R(j) \in \{-1, 1, 0\}$. We say that the R has $j_2 - j_1$ steps and that it starts at $R(j_1)$ and ends at $R(j_2)$. For the interval J_2 we say that R is a path in J_2 iff $R(J) \subset J_2$. The binary sequence $\xi(R(j_1)), \xi(R(j_1 + 1)), ..., \xi(R(j_2))$ is called the observations generated by R on ξ. In the case that there exists no interval $I \subset [-e^{3n}, e^{3n}]$ such that $\xi|I$ is equivalent to w , whenever we have an admissible path in $[-e^{3n}, e^{3n}]$ going from a point y to a point z in exactly d steps and generating the observations w on ξ, then $|y - z| < d$. This implies that $\{OBS^n(d)(\xi \circ S_x|[0, e^{n^3}]) = w\} \subset \{|S_x(f^n(d)) - S_x(b^n)| < d\}$. Since

$$g_1 \otimes 1_w \otimes g_2(P_x(E_d^n)\mathcal{L}_{\hat{\mu}}(B^n, OBS^n(d), F^n(d)|\xi, E_d^n) > 1$$

and since $|g_1 \otimes g_2|_2 \times |\varepsilon|_1 \leqslant \frac{1}{2}/n^2$ we get that

$$g_1 \otimes 1_w \otimes g_2(\sum_{x \in [-exp(n), exp(n)]} a(x)P_x(E_d^n)\mathcal{L}_x(B^n, OBS^n(d), F^n(d)|\xi, E_d^n))$$

is bigger than $1 - \frac{1}{2}/n^2$. Thus,

$$\sum_{x \in [-exp(n), exp(n)]} a(x)(g_1 \otimes 1_w \otimes g_2(P(E_d^n)\mathcal{L}_x(B^n, OBS^n(d), F^n(d)|\xi, E_d^n))$$

is bigger than $\frac{1}{2}$. Let us now have a closer look at

$$g_1 \otimes 1_w \otimes g_2(P(E_d^n)\mathcal{L}_x(B^n, OBS^n(d), F^n(d)|\xi, E_d^n)).$$

By law of total probability after conditioning under $S_x(f^n(d)), S_x(b^n)$ and because of the strong Markov property of $\{S_x(k)\}_{k \geq 0}$ we get

$$P(E_d^n|\xi)\mathcal{L}_x(B^n, OBS^n(d), F^n(d)|\xi, E_d^n)$$

is equal to

$$\begin{aligned} \Sigma_{y,z} P_x(S_x(b^n) &= y|\xi)\mathcal{L}_x(B^n|\xi, S_x(b^n) = y) \otimes . \\ P_x(S_x(b^n + d) &= z|S_x(b^n) = y, \xi)\mathcal{L}_x(OBS^n(d)|\xi, S_x(b^n) = y, S_x(b^n + d) = z) \otimes \\ \mathcal{L}_x(F^n(d)|\xi, S_x(f^n(d)) &= z) \end{aligned}$$

where for the summation we take y, z integers in $[-e^{3n}, e^{3n}]$ and such that there exists an admissible path with d steps going from y to z and generating the observations w on ξ. Because of our remark that $\{OBS^n(d)(\xi \circ S_x|[0, e^{n^3}]) = w\} \cap E_d^n \subset \{|S_x(f^n) - S_x(b^n)| < d\}$ we get that in the last summation we only need to consider y, z such that $|y - z| < d$. This implies that

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$g_1 \otimes 1_w \otimes g_2(P(E_d^n)\mathcal{L}_x(B^n, OBS^n(d), F^n(d)|\xi, E_d^n))$ *is equal to the sum over* y, z
as discussed before of the product with the three terms

$$g_1(P_x(S_x(b^n) = y|\xi)\mathcal{L}_x(B^n|\xi, S_x(b^n) = y))$$

,

$$1_w(P_x(S_x(b^n + d) = z|S_x(b^n) = y, \xi)\mathcal{L}_x(OBS^n(d)|\xi, S_x(b^n) = y, S_x(b^n + d) = z))$$

and

$$g_2(\mathcal{L}_x(F^n(d)|\xi, S_x(f^n(d)) = z)).$$

Now we get that the term

$$1_w(P_x(S_x(b^n + d) = z|S_x(b^n) = y, \xi)\mathcal{L}_x(OBS^n(d)|\xi, S_x(b^n) = y, S_x(b^n + d) = z))$$

is smaller than $P(S_x(b^n + d) = z)|S_x(b^n) = y)$. *By the strong Markov property
of* $\{S_x(k)\}_{k \geq 0}$ *we get that* $P(S_x(b^n + d) = z)|S_x(b^n) = y)$ *is equal to* $P(\ S(d) =
z - y)$. *Thus, by lemma 10,* $P(S_x(b^n + d - 1) = z)|S_x(b^n) = y)$ *is smaller or equal
to* $4n^2 P(S_x(b^n + d - 1) = z)|S_x(b^n) = y)$. *Furthermore it easy to see that*
$\mathcal{L}_x(F^n(d)|\xi, S_x(f^n(d)) = z)$ *is equal to* $\mathcal{L}_x(F^n(d - 1)|\xi, S_x(f^n(d - 1)) = z)$.
This then implies that

$$g_1 \otimes 1_w \otimes g_2(P_x(E_d^n)\mathcal{L}_x(B^n, OBS^n(d), F^n(d)|\xi, E_d^n))$$

is smaller than the sum over $y, z \in [-e^{3n}, e^{3n}]$ *where* $|y - z| < d$ *of the product
with the three terms*

$$g_1(P_x(S_x(b^n) = y|\xi)\mathcal{L}_x(B^n|\xi, S_x(b^n) = y))$$

,

$$4n^2 P(S_x(b^n + d - 1) = z)|S_x(b^n) = y)$$

and

$$g_2(\mathcal{L}_x(F^n(d - 1)|\xi, S_x(f^n(d - 1)) = z)).$$

Summing up over y, z *by law of total probability we get that the last expression
is smaller than* $4n^2$ *times* $g_1 \otimes g_2(P_x(E_{d-1}^n)\mathcal{L}_x(B^n, F^n(d - 1)|\xi, E_{d-1}^n))$. *Since
the coefficients* $a(x)$, *where* x *is an integer in* $[-e^n, e^n]$ *are the coefficients of a
convex combination and are thus positive, we get that*

$$g_1 \otimes 1_w \otimes g_2(\sum_{x \in [-exp(n), exp(n)]} a(x)P_x(E_d^n)\mathcal{L}_x(B^n, OBS^n(d), F^n(d)|\xi, E_d^n))$$

is smaller than $4n^2 \sum_{x \in [-e^n, e^n]} a(x)g_1 \otimes g_2(P_x(E_{d-1}^n)\mathcal{L}_x(B^n, F^n(d-1)|\xi, E_{d-1}^n))$.
Thus, $g_1 \otimes g_2(P_\mu(E_{d-1}^n)\mathcal{L}_u(B^n, F^n(d-1)|\xi, E_{d-1}^n))$ *is bigger than* $1/(8n^2)$, *which
contradicts the assumptions in the theorem.* ∎

4.2 Keeping the error small

In this subsection we are going to prove that we can get the error (that is $|\varepsilon|_1$) exponentially small in n with high probability. For this we just need to take $\alpha > 0$ big enough, and then the constant next to n in the expression for the exponential upper bound for $|\varepsilon|_1$ can be made as small as one wants. Let us formulate this in a lemma:

Lemma 12 *For each $\beta > 0$, there exists $\alpha_0 > 0$ such that if $\alpha > \alpha_0$ we have that there exists $\beta' > 0$ such that for all $n > 0$ we have $P(|\varepsilon|_1 \geq e^{-\beta n}) \leqslant e^{-\beta' n}$.*

    Proof. Let β_{18} designate the constant equal to $2\ln 5 c_1 + \ln 2(17/\ln 2) + 1$. We are going to make the assumption that the variable $f^n(17/\ln 2) - b^n$ always is smaller than e^n. (This assumption of course is a small imprecision but the probability that the variable $f^n(17/\ln 2) - b^n$ is bigger than e^n is so small that there is no harm.) Now under that assumption, we have that the variable $(B^n, OBS^n(17/\ln 2), F^n(17/\ln 2), f^n(17/\ln 2) - b^n)$ has at most $e^{\beta_{18}n}$ possible states. Let $\alpha_0 = 7\beta + 2\beta_{18} + 4$, so that $\alpha > 7\beta + 2\beta_{18} + 4$. Recall that for i such that $1 \leqslant i \leqslant e^{\alpha n}/e^{3n} - 1$ we have that χ^i designates $\xi \circ S|[\tau(ie^{3n}), \tau(ie^{3n})+e^{3n}]$. For $x \in [-e^n, e^n]$ let $\hat{\mu}_x$ designate the empirical distribution based on the variables $(B^n, OBS^n(17/\ln 2), F^n, f^n(17/\ln 2) - b^n)(\chi^i)$ for which $1 \leqslant i \leqslant e^{\alpha n}/e^{3n} - 1$ and $S(\tau(ie^{3n}) = x$. By definition, we have that $\varepsilon = \Sigma_{x \in [-e^n, e^n]} a(x)(\hat{\mu}_x - \mathcal{L}_x(.|\xi))$. Thus, $|\varepsilon| \leqslant \Sigma_{x \in [-e^n, e^n]} a(x)|\hat{\mu}_x - \mathcal{L}_x(.|\xi))|$. From which it follows that $|\varepsilon|$ is smaller than

$$[\Sigma_{x \in [-e^n, e^n]; a(x)e^{(\alpha-3)n} \geq e^{(5\beta+2\beta_{18})n}} a(x)|\hat{\mu}_x - \mathcal{L}_x(.|\xi))|] + 2e^{-2\beta n}.$$

(The last inequality follows from $\alpha > 7\beta + 2\beta_{18} + 4$.) Now let EV^n denote the event $\{|\varepsilon|_1 \leqslant e^{-\beta n}\}$. For all integer $x \in [-e^n, e^n]$, let $end(x)$ designate the number of variables $\chi^j, 1 \leqslant j \leqslant e^{\alpha n}/e^{3n} - 1$ for which $S(\tau(je^{3n}) = x$ and for $i \leqslant end(x)$ let χ_x^i designate the i-th such variable. When we condition under ξ the finite collection of variables $\chi_x^1, \chi_x^2, ..., \chi_x^{end(x)}$ becomes i.i.d.. Thus we can embed that finite sequence in an infinite sequence $\chi_x^1, \chi_x^2, ..., \chi_x^i, ...$ which is i.i.d. (when we condition under ξ). Let EV_x^n denote the event that for all $j \geq e^{(5\beta+2\beta_{18})n}$ the empirical distribution based on the variables

$$(B^n, OBS^n(17/\ln 2), F^n, f^n(17/\ln 2) - b^n)(\chi_x^i), 1 \leqslant i \leqslant j$$

is less away in the norm $|.|$ than $e^{-2\beta n}$ from $\mathcal{L}_x(.|\xi)$. Now, when for all $x \in [-e^n, e^n]$ we have that EV_x^n holds than $\Sigma_{x \in [-e^n, e^n]; a(x)e^{(\alpha-3)n} \leqslant e^{3(\beta+\beta_{18})n}} a(x)|\hat{\mu}_x - \mathcal{L}_x(.|\xi))|$ is smaller than $e^{-2\beta n}$ and thus $|\varepsilon| \leqslant e^{-2\beta n} + 2e^{-2\beta n}$. We will assume that $e^{-2\beta n} + 2e^{-2\beta n} \leqslant e^{-\beta n}$, (which is true for n big enough). Thus, $\cap_{x \in [-e^n, e^n]} EV_x^n \subset EV^n$. Let $\hat{\mu}_{x,j}$ denote the empirical distribution based on the variables

$$(B^n, OBS^n(17/\ln 2), F^n, f^n(17/\ln 2) - b^n)(\chi_x^i), 1 \leqslant i \leqslant j.$$

For any possible state z for the variable

$$(B^n, OBS^n(17/\ln 2), F^n, f^n(17/\ln 2) - b^n)$$

, let $EV_{x,z}^n$ denote the event that for all $j \geq e^{(5\beta + 2\beta_{18})n}$ we have that $(\hat{\mu}_{x,j} - \mathcal{L}_x(.|\xi)))(\{z\}) \leq e^{-2\beta n - \beta_{18} n}$. Since there are less than $e^{\beta_{18} n}$ possible states for the variable $(B^n, OBS^n(17/ln2), F^n, f^n(17/ln2) - b^n)$, we get that $\cap_z EV_{x,z}^n \subset EV_x^n$, where z must be taken in the set of all possible states of the variable

$$(B^n, OBS^n(17/ln2), F^n, f^n(17/ln2) - b^n).$$

Thus, we get $P(EV^{nc}) \leq \Sigma_{x \in [-e^n, e^n]; z} P(EV_{x,z}^{nc})$. Now there exists constants $\beta_{19}, \beta_{20} > 0$, such that if $X(1), X(2), ..., X(i), ...$ are i.i.d. Bernoulli variables then $P(|\frac{X(1)+X(2)+...+X(j)}{j} - E[X(1)]| \leq \Delta$ for all $j \geq m)$ is bigger than $1 - \beta_{19} e^{-\beta_{20} m \Delta^2}$. The constants $\beta_{19}, \beta_{20} > 0$ do not depend on m, Δ or the parameter $E[X(1)]$. By conditioning on ξ we thus get that $P(EV_{x,z}^{nc})$ is smaller than $\beta_{19} e^{-\beta_{20} m \Delta^2}$ with $m = e^{(5\beta + 2\beta_{18})n}$ and $\Delta = e^{-2\beta n - \beta_{18} n}$. thus $P(EV_{x,z}^{nc})$ is smaller than $\beta_{19} e^{-\beta_{20} e^{\beta n}}$. Now there are less than $2e^{(1+\beta_{18})n}$ ordered pairs (x, z) such that x is an integer with $x \in [-e^n, e^n]$ and z is in the set of all possible states of the variable $(B^n, OBS^n(17/ln2), F^n, f^n(n17/ln2) - b^n)$. Thus, we get that $P(EV^{nc}) \leq 2e^{(1+\beta_{18})n} \beta_{19} e^{-\beta_{20} e^{\beta n}}$. The expression on the right side of the last inequality is much smaller than $e^{-\beta' n}$ for every n for $\beta' > 0$ carefully chosen. (And that inequality is true for any constant $\beta' > 0$ not depending on n, as long as n is big enough.) Thus, we are done with our proof. ∎

4.3 Conclusion

In this subsection we are going to define $SUBALGIII^n$ in a precise way and prove that it works with high probability. The constants $c_1, c_2, c_3 > 0$ are going to be any three positive constants not depending on n and satisfying the five following inequalities: $c_3 > c_1 + 3$, $c_2 > 2/(ln153 - ln128)$, $c_1 \geq 2/\beta_{21}$, $c_4 \geq 4 + 17lnp/ln2 + 2c_1(max\{ln|\bar{x}(i)^*|_2 | i \in 2,3,4,5,6\}) + 2c_3$, $c_2 = c_1/5$ where the functionals $\bar{x}(i)^*$ will get defined on the next page. It is easy to check that the system of last three equations always has a solution. Let us next define $SUBALGIII^n$.

Algorithm 13 *First step) Compute $\hat{\mu}$. Second step) Construct the set SET^n. For this put all the words, (or pieces of sceneries) w of length d where $d \leq 17n/ln2$ for which there exists two positive functionals g_1 and g_2 such that: Case where $q \neq 0$) $g_1 \otimes 1_w \otimes g_2(P_{\hat{\mu}}(E_d^n)\mathcal{L}_{\hat{\mu}}(B^n, OBS^n(d), F^n(d)|\xi, E_d^n) > 1$ and $g_1 \otimes g_2(P_{\hat{\mu}}(E_{d-1}^n)\mathcal{L}_{\hat{\mu}}(B^n, F^n(d-1)|\xi, E_{d-1}^n) \leq 1/(9n^2)$ and c) $|g_1 \otimes g_2|_2 \leq e^{c_4 n}$. Case where $q = 0$) Same thing as for $q \neq 0$ but simply replace the second inequality by $g_1 \otimes g_2(P_{\hat{\mu}}(E_{d-2}^n)\mathcal{L}_{\hat{\mu}}(B^n, OBS^n(d-2), F^n(d-2)|\xi, E_{d-2}^n) \leq 1/(9n^2)$.*

Next we are going to prove that the above algorithm works with high probability. For this we are going to introduce a couple of events and show that if they all hold then the above algorithm works. Then we will prove that all these events hold with high probability. In general, $E_{III,i}^n$ will designate the i-th event related to the algorithm $SUBALGIII^n$, and $E_{III,i}^{nc}$ the complement

of the event $E_{III,i}^n$. E_{III}^n will designate the event that $SUBALGIII^n$ works.
More precisely,

$E_{III}^n = \{$ the collection of pieces of sceneries $SUBALGIII^n(\tau; \xi \circ S|[\tau(1), \tau(e^{\alpha n})])$
satisfies property $P_1^n\}$. In other words, E_{III}^n means that if we feed to $SUBALGIII^n$
the collection of stopping times $\tau = (\tau(1), \tau(2), ..., \tau(e^{\alpha n}))$ and the observations
$\xi \circ S$ during the time $[\tau(1), \tau(e^{\alpha n})]$ then we get as output a collection of pieces of
sceneries satisfying condition P_1^n, that is all the pieces of sceneries constructed
by $SUBALGIII^n$ are contained in $\xi|[-e^{3n}, e^{3n}]$ and all the pieces of length
$\leqslant 17n/ln2$ which are contained in $\xi|[-3e^n, 3e^n]$ are contained in a piece of
scenery of that set. (We say that a piece of scenery ψ_1 is contained in an-
other piece of scenery ψ_2 iff ψ_1 is equivalent to a restriction of ψ_2 to an integer
interval.)

Recall that T^m designates the first hitting time of the random walk $\{S(k)\}_{k \geq 0}$
(which starts at the origin) on the set $\{-1, m-1\}$. Thus, $T^m + 1$ has the
same distribution than the distribution of the length of a block in the obser-
vations $\xi \circ S$ conditioned under that block was generated on a block of ξ of
length m. Now we write $\mu(m, 1)$, resp. $\mu(m, 0)$ for the defective distribu-
tion obtained by taking $T^m + 1$ and asking that $S(T^m) = m - 1$, resp. that
$S(T^m) = -1$. We will write $\mu(m, 1)(. \geq j)$ for $\mu(m, 1)(\{x|x \geq j\})$. Next
we need the following definition: Let $\vec{x}(2) = (p, pq, pq^2, pq^3, pq^4)$ and $\vec{x}(3) =$
$(0, p^2, 2p^2q, p^4 + 3p^2q^2, \mu(3, 1)(\{. > 5\}))$ and $\vec{x}(4) = (0, 0, p^3, 3p^3q, p^5 + 6p^3q^2)$
and $\vec{x}(5) = (0, 0, 0, p^4, \mu(5, 1)(\{. > 5\})$ and $\vec{x}(6) = (0, 0, 0, 0, 1) \in \mathbb{R}^5$. Now,
we have that $\mu(2, 1) = (p, pq, pq^2, pq^3, pq^4, ...)$; $\mu(3, 0) = (p, pq, p^3 + pq^2, pq^3 +$
$3p^3q, p^5 + 6p^3q^2 + pq^4, ...)$; $\mu(3, 1) = (0, p^2, 2p^2q, p^4 + 3p^2q^2, ...)$; $\mu(4, 1) =$
$(0, 0, p^3, 3p^3q, 2p^5 + 6p^3q^2, ...)$; $\mu(5, 1) = (0, 0, 0, p^4, ...)$. Then $\mu(2, 1) \circ h^{-1} =$
$\mu(2, 0) \circ h^{-1} = \vec{x}(2) + (\mu(2, 1)(. > 6))\vec{x}(6)$. Furthermore, $\mu(3, 0) \circ h^{-1} =$
$\vec{x}(2) + \vec{x}(4) + (\mu(3, 0)(. > 6))\vec{x}(6)$ and $\mu(3, 1) \circ h^{-1} = \vec{x}(3)$ and $\mu(5, 1) \circ h^{-1} =$
$\vec{x}(5)$. Now by a symmetry principle we have that $\mu(4, 0) = \mu(2, 0) + \mu(4, 1)$.
Thus, $\mu(4, 0) \circ h^{-1} = \mu(2, 0) \circ h^{-1} + \mu(4, 1) \circ h^{-1}$ and thus, $\mu(4, 0) \circ h^{-1} =$
$\vec{x}(2) + \vec{x}(4) + ((\mu(2, 1)(. > 6)) + p^5 + \mu(4, 1)(. > 6)))\vec{x}(6)$. Now, for all $m \geq 5$,
we have that $\mu(m, 0)$ coincides on its first 6 coordinates with $\mu(4, 0)$. This is
so because all the admissible paths which start at 0 hit on $+3$ and then come
back to -1 are at least seven steps long. Thus we get that for all $m \geq 5$
we have that $\mu(m, 0) \circ h^{-1}$ is equal to $\vec{x}(2) + \vec{x}(4)$ plus a positive coefficient
times $\vec{x}(6)$. For $m > 5$ we have that $\mu(m, 1) \circ h^{-1}$ is equal to $\vec{x}(6)$ times
$\mu(m, 1)(. > 5) = 1/m$. Let $(\vec{x}(2)^*, \vec{x}(3)^*, \vec{x}(4)^*, \vec{x}(5)^*, \vec{x}(6)^*)$ designate the
dual basis of $(\vec{x}(2), \vec{x}(3), \vec{x}(4), \vec{x}(5), \vec{x}(6))$. Let $z \in [-3e^n, 3e^n]$ be such that
$\xi(z) \neq \xi(z + 1)$. Let $l_{i,z}$ designate the length of the i-th block of $\xi|[z, \infty[$.
Then, we call the linear functional $g_{z,r}^n = \otimes_{i=1}^{c_1 n}(l_{i,z}\vec{x}(h(l_{i,z}))^*)$ the functional
of ξ to the right of z. (Here h designates the truncating function which was
defined in definition 9.) Let $y \in [-3e^n, 3e^n]$ be such that $\xi(y) \neq \xi(y - 1)$.
Let $l_{i,y}$ designate the length of the i-th last block of $\xi|] - \infty, y]$ when we start
counting from y and go in the direction of $-\infty$. Let $g_{y,l}^{n\prime}$ be the linear functional
$g_{y,l}^{n\prime} = \otimes_{i=1}^{c_1 n}(l_{i,y}\vec{x}(h(l_{i,y}))^*)$. Let $g_{y,l}^n = (2e^{3n}p^{n17/ln2+2})g_{y,l}^{n\prime}$. We will call $g_{y,l}^n$
the functional of ξ to the left of y. Next we are going to define some more

events:

$E_{III,1}^n = \{$ for any integer interval $I \subset [-3e^n, 3e^n]$ of length $n13/(ln2)$ we have that there exists two points $y < z$ such that $I \subset [y,z]$ and $\xi(y-1) \neq \xi(y)$ and $\xi(z) \neq \xi(z+1)$ and $y - z \leq n17/(ln2)\}$. $E_{III,2}^n = \{$ the product of the lengths of any $c_1 n$ consecutive blocks of the scenery $\xi|[-3e^n, 3e^n]$ is smaller than $e^{c_3 n}.\}$. $E_{III,3}^n = \{|\varepsilon| \leqslant e^{-2c_4 n}\}$. $E_{III,4}^n = \{$for any integer $y \in [-3e^n, 3e^n]$ such that $\xi(y) \neq \xi(y-1)$ the left functional of ξ at y is also a left limiting functional of ξ at $y\} \cap \{$for any integers $z \in [-3e^n, 3e^n]$ such that $\xi(z) \neq \xi(z+1)$ the right functional of ξ at z is also a right limiting functional of ξ at $z\}$.

Next we are going to prove that whenever all the events $E_{III,1}^n$, $E_{III,2}^n$, $E_{III,3}^n$ and $E_{III,4}^n$ all hold, then E_{III}^n also holds. In other words, **we are going to prove that:** $E_{III,1}^n \cap E_{III,2}^n \cap E_{III,3}^n \cap E_{III,4}^n \subset E_{III}^n$. For this purpose let SET^n designate the collection of pieces of sceneries produced by $SUBALGIII^n$, in other words $SET^n = SUBALGIII^n(\tau, \xi \circ S||[\tau(1), \tau(e^{an})])$. We are first going to prove that the collection of pieces of sceneries SET^n **are all pieces of sceneries contained in** $\xi|[-e^{3n}, e^{3n}]$. Note that when $E_{III,3}^n$ holds we have $|\varepsilon| \leqslant e^{-2c_4 n}$. Furthermore, when we pick a word w for SET^n according to algorithm 13, we have two positive functionals g_1, g_2 satisfying jointly with w three conditions, the third condition to get picked by the algorithm being $|g_1 \otimes g_2|_2 \leqslant e^{c_4 n}$. Thus, when $E_{III,3}^n$ holds we have that $|g_1 \otimes g_2|_2 \cdot |\varepsilon| \leqslant e^{-c_4 n}$. The expression on the right side of the last inequality is (at least for n big enough) smaller than $\frac{1}{2}/n^2$ and thus w, g_1, g_2 satisfy jointly all the conditions for theorem 11. It follows that w is contained in $\xi|[-e^{3n}, e^{3n}]$. **Next we are going to show that we have enough pieces of sceneries in the set** SET^n. Let I be an integer interval of length $13n/ln2$ contained in $[-3e^n, 3e^n]$. Then by $E_{III,1}^n$ there exists $y < z$ such that $I \subset [y,z]$ and $\xi(y-1) \neq \xi(x_1)$ and $\xi(z) \neq \xi(z+1)$ and $z - y \leq n17/(ln2)$. Let $d = z - y$ and let $g_{y,l}$, resp. $g_{z,r}$ designate the functional of ξ to the left of y, resp. to the right of z. Let w denote the word of length d, $w = (\xi(y), \xi(y+1), ..., \xi(z))$. Next we are going to prove that the triple $(w, g_{y,l}, g_{z,r})$ satisfies all the criteria in order to get selected by $SUBALGIII^n$ which then finishes this proof. First note that both $g_{y,l}$, resp. $g_{z,r}$ are positive functionals: when we express the defective distribution $\mu(m, i) \circ h^{-1}$ for any $m > 1$ and any $i \in \{0, 1\}$ as linear combination of the basis $(\vec{x}(2), \vec{x}(3), \vec{x}(4), \vec{x}(5), \vec{x}(6))$ then no coefficients are negative. Thus for all $j \in \{2, 3, 4, 5, 6\}$ and all $i > 1$ we get that $(\vec{x}(j))^*(\mu(m, i) \circ h^{-1}) \geq 0$. Since however, $g_{y,l}$ and $g_{z,r}$ are tensor products of elements of $\{\vec{x}(2)^*, \vec{x}(3)^*, \vec{x}(4)^*, \vec{x}(5)^*, \vec{x}(6)^*\}$ we get that both $g_{y,l}$ and $g_{z,r}$ are positive. Now note that the $|.|_2$-norm has the property that the $|.|_2$-norm of the tensor product is equal to the product of the $|.|_2$-norm. Applying this, we get that $|g_{y,l} \otimes g_{z,r}|_2$ is smaller than $2e^{3n}p^{n17/ln2+2}(max\{|\vec{x}(i)^*|_2| i \in 2, 3, 4, 5, 6\})^{2c_1 n}$ times the product of the lengths of two sequences of $c_1 n$ consecutive blocks in $[-3e^n, 3e^n]$. By $E_{III,2}^n$ we get that the product of the lengths of $c_1 n$ consecutive blocks in $[-3e^n, 3e^n]$ is smaller than $e^{c_3 n}$. Thus, $|g_{y,l} \otimes g_{z,r}|_2$ is smaller than $2e^{n(3+17lnp/ln2+2c_1(max\{ln|\vec{x}(i)^*|_2| i \in 2,3,4,5,6\})+2c_3)}$. By our definitions of $c_4 n$ this is much smaller (at least for n big enough) than $e^{c_4 n}$. So

the third condition for the triple $w, g_{y,l}, g_{z,r}$ to get selected in $SUBALGIII^n$ is satisfied. (The next condition we are only going to prove to hold for the case $q \neq 0$. The other case is similar and left to the reader.) So assume that $q \neq 0$ until the end of the proof. Next note that when E^n_{d-1} holds then $f^n(d) - b^n = d - 1$ and thus $S(f^n(d-1)) - S(b^n) \leq d - 1$. Thus, when E^n_{d-1} holds we have that either $S(b^n)$ is strictly to the right of y or $S(f^n(d-1))$ is strictly to the left of z. When $E^n_{III,4}$ holds we have that $g_{y,l}$, resp. $g_{z,r}$ is a left limiting functional of ξ at y, resp. a right limiting functional of ξ at z. Thus by conditioning under $S(f^n(d-1))$ and $S(b^n)$, we get that $g_{y,l} \otimes g_{z,r}(P_\mu(E^n_{d-1})\mathcal{L}_\mu(B^n, F^n(d-1)|\xi, E^n_{d-1}) = 0$. Now $\hat{\mu} = \mu + \varepsilon$. Thus, $g_1 \otimes g_2(P_{\hat{\mu}}(E^n_{d-1})\mathcal{L}_{\hat{\mu}}(B^n, F^n(d-1)|\xi, E^n_{d-1}) \leq |g_{y,l} \otimes g_{z,r}|_2 \times |\varepsilon|$. However, we saw that when $E^n_{III,2}$ holds than $|g_{y,l} \otimes g_{z,r}|_2$ is smaller $e^{c_4 n}$. When $E^n_{III,3}$ holds, then $|\varepsilon| \leq e^{-2c_4 n}$. Thus, $g_{y,l} \otimes g_{z,r}(P_{\hat{\mu}}(E^n_{d-1})\mathcal{L}_{\hat{\mu}}(B^n, F^n(d-1)|\xi, E^n_{d-1}) \leq e^{-c_4 n}$. Since (at least for n big enough) $e^{-c_4 n}$ is much smaller than $1/n^2$ we have that the second condition for getting selected by $SUBALGIII^n$ holds. Because $|g_{y,l} \otimes g_{z,r}|_2 \times |\varepsilon| \leq e^{-c_4 n}$ and because $\hat{\mu} = \mu + \varepsilon$, we get that $g_{y,l} \otimes 1_w \otimes g_{z,r}(P_{\hat{\mu}}(E^n_d)\mathcal{L}_{\hat{\mu}}(B^n, OBS^n(d), F^n(d)|\xi, E^n_d)$ is bigger than $g_{y,l} \otimes 1_w \otimes g_{z,r}(P\mu(E^n_d)\mathcal{L}\mu(B^n, OBS^n(d), F^n(d)|\xi, E^n_d) - e^{-c_4 n}$. Let y_0, resp. z_0 designate the left end, resp. right end of the $c_1 n$-th block in ξ before y, resp. after z. Let b^n_0, resp. f^n_0 designate the left end, resp. the right end of the $c_1 n$-th block in χ before b^n, resp. after $f^n(d)$. Let $E^n_{x,y,z}$ designate the measurable event that $S_x(b^n_0) = y_0$ and $S_x(b^n) = y$ and $S_x(f^n) = z$ and $S_x(f^n_0) = z_0$. Now because $g_{y,l} \otimes g_{z,r}$ is positive we have that $g_{y,l} \otimes 1_w \otimes g_{z,r}(P_x(E^n_d)\mathcal{L}_x(B^n, OBS^n(d), F^n(d)|\xi, E^n_d)$ is bigger than $g_{y,l} \otimes 1_w \otimes g_{z,r}(P_x(E^n_d \cap E^n_{x,y,z})\mathcal{L}_x(B^n, OBS^n(d), F^n(d)|\xi, E^n_d, E^n_{x,y,z})$. Let $l_{i,z}$, resp. $l_{i,y}$ designate the length of the i-th block of $\xi||z, \infty[$, resp. the length of the last i-th bock of $\xi|| - \infty, y]$. Then, the defective distribution

$$P_x(E^n_d \cap E^n_{x,y,z})\mathcal{L}_x(B^n, OBS^n(d), F^n(d)|\xi, E^n_d, E^n_{x,y,z})$$

is equal to

$$pP_x(S_x(b^n_0) = y_0) \cdot ([\otimes^{c_1 n}_{i=1} \mu(l_{i,y}, 1) \circ h^{-1}] \otimes [p^d \delta_w] \otimes [\otimes^{c_1 n}_{i=1} \mu(l_{i,z}, 1) \circ h^{-1}]).$$

(Here δ_w designate the probability distribution for $OBS^n(d)$ with exactly one atom with mass one at w.) Now, remember that by definition for all $l \geq 2$, we always have $\bar{x}(h(l))^*(\mu(l,1)) \geq 1/l$. Thus we get that

$$g'_{y,l} \otimes 1_w \otimes g_{z,r}(P_x(E^n_d \cap E^n_{x,y,z})\mathcal{L}_x(B^n, OBS^n(d), F^n(d)|\xi, E^n_d, E^n_{x,y,z})$$

is bigger than $p^{d+1}P_x(S_x(b^n_0) = y_0)$. However, $P_x(S_x(b^n_0) = y_0) \geq P_x(S_x(e^{2n}) = y_0, S_x(e^{2n} + 1) = y_0 + 1) = pP_x(S_x(e^{2n}) = y_0)$, (here we assumed that $q \neq 0$ so that there are no parity problems, the other case is left to the reader.) Now by the local central limit theorem we get (at least for n big enough) that $P_x(S_x(e^{2n}) = y_0) \geq e^{-3n}$. Combining all this we get

$$g_{y,l} \otimes 1_w \otimes g_{z,r}(P_x(E^n_d \cap E^n_{x,y,z})\mathcal{L}_x(B^n, OBS^n(d), F^n(d)|\xi, E^n_d, E^n_{x,y,z})$$

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is bigger than $p^{d+2}e^{-3n}(2e^{3n}p^{n17/ln2+2}) = 2$. Since μ is a convex combination of the $\mathcal{L}_x(.|\xi)$ we get that

$$g_{y,l} \otimes 1_w \otimes g_{z,r}(P_\mu(E_d^n \cap E_{x,y,z}^n)\mathcal{L}_\mu(B^n, OBS^n(d), F^n(d)|\xi, E_d^n, E_{x,y,z}^n)$$

is bigger than 2. Thus, we get that

$$g_{y,l} \otimes 1_w \otimes g_{z,r}(P_{\hat\mu}(E_d^n)\mathcal{L}_{\hat\mu}(B^n, OBS^n(d), F^n(d)|\xi, E_d^n)$$

is bigger or equal than $2 - e^{-c_4 n} > 1$. (At least for n big enough.) This implies, that the first condition in the algorithm $SUBALGIII^n$ for getting picked is also meet by the triple $(w, g_{y,l}, g_{z,r})$ and so we are done with this proof.

Proof that $P(E_{III,1}^{nc})$ is negatively exponentially small in n: Let $E_{III,5}^n$ denote the event { the longest block of $\xi|[-3e^n, 3e^n]$ is not longer than $2n/ln2$.}. Then, $E_{III,5}^n \subset E_{III,1}^n$. Thus, $P(E_{III,1}^{nc}) \leq P(E_{III,5}^{nc})$. However, it is easy to check (and we thus leave it to the reader) that $P(E_{III,5}^{nc})$ is exponentially small in n.

Proof that $P(E_{III,2}^{nc})$ is negatively exponentially small in n. Note that the lengths of the blocks of ξ are i.i.d. (Except may be for the block at the origin, but we will not care about that detail.) Let l_i designate the length of the i-th block of ξ. Then by Chebycheff we get that $P(\Pi_{j=1}^{c_1 n} l_{i+j} \geq e^{c_3 n}) \leq E[\Pi_{j=1}^{c_1 n} l_{i+j}]/e^{c_3 n}$. Because the lengths of the blocks are i.i.d. we get $P(\Pi_{j=1}^{c_1 n} l_{i+j} \geq e^{c_3 n}) \leq E[l_1]^{c_3 n}/e^{c_3 n}$. Since there are at most $6e^n$ blocks in $\xi|[-3e^n, 3e^n]$ we get that $P(E_{III,2}^{nc}) \leq 6e^n E[l_1]^{c_3 n}/e^{c_3 n}$. Now, $E[l_1] = 2$ and thus $E[l_1]^{c_3 n}/e^{c_3 n} \leq e^{n(c_1 - c_3)}$. Because of how we defined c_1 and c_3, we get that $6e^n E[l_1]^{c_3 n}/e^{c_3 n}$ is smaller than $6e^{-n}$ and so is $P(E_{III,2}^{nc})$.

Proof that $P(E_{III,3}^{nc})$ is negatively exponentially small in n. According to lemma we can choose an $\alpha > 0$ such that there exists $\beta' > 0$ not depending on n, such that $P(\{|\varepsilon| \leq e^{-2c_4 n}\}) \leq e^{-\beta' n}$. Choose the α to be such and then $P(E_{III,3}^{nc}) \leq e^{-\beta' n}$. (Note that one can choose α as big as one wants, as long as it does not depend on n, our proof that $ALGORITHM^n$ works is not affected.)

Proof that $P(E_{III,4}^{nc})$ is negatively exponentially small in n. We are first going to define two events $E_{III,6}^n$ and $E_{III,7}^n$, show that $E_{III,6}^n \cap E_{III,7}^n \subset E_{III,4}^n$ and show that $E_{III,6}^n$ and $E_{III,7}^n$ both hold with high probability. Let $E_{III,6}^n$ be the event that in each consecutive sequence of $c_1 n$ blocks of $\xi|[-3e^n, 3e^n]$ there are at least $c_2 n$ blocks of length 3 or 5. Now the probability for a block of ξ to be of length 3 or 5 is equal to $\frac{5}{16}$. By a large deviation principle, there exists a constant $\beta_{21} > 0$ not depending on j, such that the probability for j i.i.d. Bernoulli variables with parameter $\frac{5}{16}$ to not contain at least $\frac{1}{16}$ of them equal to one is smaller than $e^{-\beta_{21} j}$. Thus, the probability for a sequence of $c_1 n$ consecutive blocks of ξ to not contain at least $c_2 n = c_1 n/5$ blocks of length 3 or 5 is smaller than $e^{-\beta_{21} c_1 n}$. There are at most $6e^n$ blocks in $\xi|[-3e^n, 3e^n]$. Thus, $P(E_{III,6}^{nc}) \leq 6e^n e^{-\beta_{21} c_1 n}$. Because of our definition of c_1 the expression on the right side of the last inequality is exponentially small in n. Next we need a few definitions: let ϕ be a piece of scenery and $\{b_1, b_2\}$ a block of ϕ. Then, we call $\phi(b)$ color of the block $\{b_1, b_2\}$, where b

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is any integer strictly between b_1 and b_2. Let r designate the number of blocks of length 3 or 5 of $\xi\|[-3e^n, 3e^n]$. For $i \in 1, 2, ..., r$ let l_i^{35} and s_i^{35} designate the length and the color of the i-th block of length 3 or 5 of $\xi\|[-3e^n, 3e^n]$. If $sequ$ is a sequence of elements in $\{30, 50, 31, 51\}$, then we write $sequ^T$ for the sequence obtained by taking $sequ$ and exchanging 30 with 31, 31 with 30, 50 with 51 and 51 with 50. Let Ψ be the following coloring of the integer unit intervals of $[0, r]$, (where $a < b$ are two integers) be defined as follows: $\Psi : \{\{i-1, i\}|i \in 1, 2, ..., r\} \longrightarrow \{30, 50, 31, 51\}$ where $\Psi(\{i-1, i\}) = l_i s_i$. Let $[j_0, j_1]$ be an integer interval. We call a function $R : [j_0, j_1] \longrightarrow [a, b]$ a nearest neighbor walk on $[a, b]$, (where $a < b$ are two integers) iff for all j such that $j_0 \leqslant j < j_1$ we have that $|R(j) - R(j+1)| = 1$. We call $R(j_0)$ the starting point of R and $j_1 - j_0$ the length of R and we call the sequence $\Psi(\{R(0), R(1)\}), \Psi(\{R(1), R(2)\}), \Psi(\{R(2), R(3)\}), ..., \Psi(\{R(r-1), R(r)\})$ the sequence generated by R on Ψ. Let $E_{III,7}^n$ be the event {for all integer $x \in [c_2 n, r]$ we have that for any nearest neighbor walk R on $[0, r]$ starting strictly to the right of x, the sequence generated by R on Ψ is different from $(l_x^{35} s_x^{35}, l_{x-1}^{35} s_{x-1}^{35}, l_{x-2}^{35} s_{x-2}^{35}, ..., l_{x-c_2 n}^{35} s_{x-c_2 n}^{35})$ and from $(l_x^{35} s_x^{35}, l_{x-1}^{35} s_{x-1}^{35}, l_{x-2}^{35} s_{x-2}^{35}, ..., l_{x-c_2 n}^{35} s_{x-c_2 n}^{35})^T\} \cap$ {for all integer $x \in [0, r - c_2 n]$ we have that for any nearest neighbor walk R on $[0, r]$ starting strictly to the left of x, the sequence generated by R on Ψ is different from

$$(l_x^{35} s_x^{35}, l_{x+1}^{35} s_{x+1}^{35}, l_{x+2}^{35} s_{x+2}^{35}, ..., l_{x+c_2 n}^{35} s_{x+c_2 n}^{35})$$

and from $(l_x^{35} s_x^{35}, l_{x+1}^{35} s_{x+1}^{35}, l_{x+2}^{35} s_{x+2}^{35}, ..., l_{x+c_2 n}^{35} s_{x+c_2 n}^{35})^T\}$.

Let us prove that $P(E_{III,7}^{nc})$ is exponentially small in n. Let l_i^{35} and s_i^{35} designate the length and the color of the i-th block of length 3 or 5 of $\xi\|[-3e^n, \infty]$. Then, the sequence $l_1^{35} s_1^{35}, l_2^{35} s_2^{35}, ...$ is a Markov chain with stationary transition probabilities. Furthermore, $\{l_k^{35}\}_{k \geq 0}$ is independent of $\{s_k^{35}\}_{k \geq 0}$ and the l_k^{35}'s are i.i.d.. The probability for a block of ξ given that it is of length 3 or 5 to have length 3 is equal to $\frac{4}{5}$. Furthermore, let p_{35} designate the probability that a block of ξ has length 3 or 5. We get that $p_{35} = \frac{1}{4} + \frac{1}{16} = \frac{5}{16}$. Let $q_{35} = 1 - p_{35}$. Now the probability $P(s_{k+1}^{35} \neq s_k^{35})$ is equal to $p_{35} + p_{35}(q_{35})^2 + p_{35}(q_{35})^4 + ... = p_{35}/(1 - (q_{35})^2) = 1/(1 + q_{35}) = 1/(1 + \frac{11}{16}) = \frac{16}{27}$. This implies that the maximum transition probability for the Markov process $\{l_k^{35} s_k^{35}\}_{k \geq 0}$ is $\frac{16}{27} \times \frac{4}{5} = \frac{64}{135} < \frac{1}{2}$. Let $x \in [c_2 n, r]$ and let R be a nearest neighbor walk (non random) on $[0, r]$ starting strictly to the right of x and of length $c_2 n$. Then the probability that the sequence generated by R on Ψ is equal to $(l_x^{35} s_x^{35}, l_{x-1}^{35} s_{x-1}^{35}, l_{x-2}^{35} s_{x-2}^{35}, ..., l_{x-c_2 n}^{35} s_{x-c_2 n}^{35})$ or to $(l_x^{35} s_x^{35}, l_{x-1}^{35} s_{x-1}^{35}, l_{x-2}^{35} s_{x-2}^{35}, ..., l_{x-c_2 n}^{35} s_{x-c_2 n}^{35})^T$ is smaller than $2(\frac{64}{135})^{c_2 n}$. (To see this, note that because the nearest neighbor walk can move at most one unit to the left at each step and because it starts strictly to the right of x we have that the first i-steps of R are strictly to the right of those unit intervals mapped by Ψ onto $l_x s_x, l_{x-1} s_{x-1}, l_{x-2} s_{x-2}, ..., l_{x-i} s_{x-i}$.) There are at most $2 \cdot 2^{c_2 n} e^n$ nearest neighbor walk on $[0, r]$ of length $c_2 n$. Thus, the probability that there exists a nearest neighbor walk on $[0, r]$ starting strictly to the right of x and of length $c_2 n$ and generating on Ψ the sequence $(l_x s_x, l_{x-1} s_{x-1}, l_{x-2} s_{x-2}, ..., l_{x-c_2 n} s_{x-c_2 n})$ or

$(l_x s_x, l_{x-1} s_{x-1}, l_{x-2} s_{x-2}, ..., l_{x-c_2 n} s_{x-c_2 n})^T$ is smaller than $2 \cdot 2^{c_2 n} e^n 2 (\frac{64}{135})^{c_2 n}$ $= 4(\frac{128}{135})^{c_2 n} e^n$. Because of how we defined c_2 the last expression is negatively exponentially small in n. A symmetric argument can be used for when R is a nearest neighbor walk starting strictly to the left of x, so that eventually one gets that $P(E_{III,7}^{nc}) \leqslant 8(\frac{128}{135})^{c_2 n} e^n$.

The only thing which **remains to be proven** is that $E_{III,6}^n \cap E_{III,7}^n \subset E_{III,4}^n$. This is what we are going to do next: we are going to do a proof by the absurd. Thus we assume that there exist $z \in [-e^n, e^n]$ and $z_0 \in [-e^n, z[$ such that $\xi(z) \neq \xi(z+1)$ and $not(\xi(z_0) = \xi(z_0+1) = \xi(z_0-1))$ if g_r designates the right functional at z of ξ then $g_r(\mathcal{L}_x(F^n|S_x(f^n) = z_0, \xi) \neq 0$. (Let us use the following notation: F^n, resp. f^n stands for $F^n(0)$, resp. for $f^n(0)$. Because of the strong Markov property of $\{S_x(k)\}_{k \geq 0}$ it does not matter how we choose d in $F^n(d)$ and $f^n(d)$.) We will only do the proof with g_r and leave the case with g_l to the reader. We need first the following definition: let $R : D \longrightarrow \mathbb{Z}$ be an admissible piece of path. Then we write l_i^R for the length of the block of ξ on with the i-th block of $\xi \circ R$ was generated. Let a_i, resp. b_i be the left end, resp. right end of the i-th block of $\xi \circ R$. Then, we define $s_i^R = 1$ one if $R(a_i) \neq R(b_i)$ and $s_i^R = 0$ otherwise. Thus, s_i^R determines whether R crossed a block of ξ or merely entered and exited a block of ξ from the same side whilst generating the i-th block of $\xi \circ R$. We call $R(a_1)$ the starting point of R on ξ. We say that $(l_1^R s_1^R, l_2^R s_2^R, ..., l_{c_1 n}^R s_{c_1 n}^R)$ is a possible sequence with starting point $R(a_1)$. We say that R generates j blocks iff there are j blocks in $\xi \circ R$. Now, with this definition the defective distribution $P(S(f^n) = z_0)\mathcal{L}_x(F^n|S(f^n) = z_0, \xi)$ can be written as the sum $\Sigma_{(l_1 s_1, l_2 s_2, ..., l_{c_1 n} s_{c_1 n})} \mu(l_1, s_1) \circ h^{-1} \otimes \mu(l_2, s_2) \circ h^{-1} \otimes ... \otimes \mu(l_{c_1 n}, s_{c_1 n}) \circ h^{-1}$ where the last sum is taken over all $(l_1 s_1, l_2 s_2, ..., l_{c_1 n} s_{c_1 n})$ possible sequences of length $c_1 n$ with starting point z_0. Now because, $g_r(\mathcal{L}_x(F^n|S(f^n) = z_0, \xi) \neq 0$ and by positivity of g_r we have that g_r must be different form zero on at least one of the terms of the last sum. Thus there exist an admissible path R which generates $c_1 n$ blocks and with starting point z_0 such that $g_r(\mu(l_1, s_1) \circ h^{-1} \otimes \mu(l_2, s_2) \circ h^{-1} \otimes ... \otimes \mu(l_{c_1 n}, s_{c_1 n}) \circ h^{-1}) \neq 0$. Thus, $\Pi_{i=1}^{c_1 n} \vec{x}(h(l_{z,i}))^*(\mu(l_i^R, s_i^R) \circ h^{-1}) \neq 0$. Thus, for all $i = 1, 2, ..., c_1 n$ we have that $\vec{x}(h(l_{z,i}))^*(\mu(l_i^R, s_i^R) \circ h^{-1}) > 0$. (Here $l_{z,i}$ designates the length of the i-th block of $\xi||[z, \infty[.)$ Now, recall that $\vec{x}(k)^*(\mu(3,1) \circ h^{-1}) \neq 0$ iff $k = 3$, and $\vec{x}(k)^*(\mu(5,1) \circ h^{-1}) \neq 0$ iff $k = 5$. For $l \geq 2$ and $s \in 0,1$, we have $\vec{x}(3)^*(\mu(l,s) \circ h^{-1}) \neq 0$ iff $l = 3$ and $s = 1$ and $\vec{x}(5)^*(\mu(l,s) \circ h^{-1}) \neq 0$ iff $l = 5$ and $s = 1$. Thus, $l_{z,i} \in 3,5$ iff $l_i^R \in 3,5$ and $s_i^R = 1$ and then $l_i^R = l_{z,i}$. This then implies that for $i \in 1,2,3,...,c_1 n$ the i-th block of $\xi||[z, \infty[$ has length 3, resp.5 iff the i-th block of $\xi \circ R$ corresponds to a crossing (i.e. not just entering a block and leaving it on the same side but leaving it on the other side) by R of a block of length 3, resp. 5. Note that in any scenery, or piece of scenery or observations the color of the blocks alternate between 0 and 1. This then implies that (assume that there r designates the number of blocks of length 3 or 5 in the first $c_1 n$ blocks of $\xi||[z, \infty[$), then either:

for each $i \leqslant r$ the color of the i-th block of length 3 or 5 of $\xi||[z, \infty[$ is equal to the color of the i-th block of ξ of length 3 or 5 crossed by R

or

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for each $i \leqslant r$ the color of the i-th block of length 3 or 5 of $\xi|[z,\infty[$ is opposite of the color of the i-th block of length 3 or 5 crossed by R

(Now let h_3 be any function from $\mathbb{Z}$ to $\mathbb{Z}$ mapping the closed integer interval between the right end of the i-th block and the left end of the $i+1$-th block of length 3 or 5 of $\xi|[-3e^n,3e^n]$ onto i, for each i smaller or equal then the total number of blocks $\xi|[-3e^n,3e^n]$. Then $h_3 \circ R$ is an admissible path (but not yet an nearest neighbor walk). However, we can take out the holdings from $h_3 \circ R$ and make it a nearest neighbor walk. This nearest neighbor walk will then be called the nearest neighbor walk induced by R on Ψ, and we will write for it $h_3 \circ R$modulo $hold$.) What we said before about the color of the blocks of length 3 or 5 crossed by R then implies that the nearest neighbor walk induced by R on Ψ generates on Ψ a sequence equal to either $(l_{z+1}^{35}s_{z+1}^{35}, l_{z+2}^{35}s_{z+2}^{35}, ..., l_r^{35}s_r^{35})$ or $(l_{z+1}^{35}s_{z+1}^{35}, l_{z+2}^{35}s_{z+2}^{35}, ..., l_r^{35}s_r^{35})^T$. Because of $E_{III,6}^n$ we have that $r \geq c_2 n$. Now because $z_0 < z$ we get that after the first step $h_3 \circ R$modulo $hold$ is strictly to the left of $h_3(z)$. Thus if we take away the first step only of the nearest neighbor walk $h_3 \circ R$modulo $hold$ we get a nearest neighbor walk starting strictly to the left of $h_3(z)$ of length at least $c_2 n - 1$ which generates on Ψ the sequence $(l_{z+1}^{35}s_{z+1}^{35}, l_{z+2}^{35}s_{z+2}^{35}, ..., l_r^{35}s_r^{35})$ or $(l_{z+1}^{35}s_{z+1}^{35}, l_{z+2}^{35}s_{z+2}^{35}, ..., l_r^{35}s_r^{35})^T$. This contradicts $E_{III,7}^n$ and so we are done with our proof.

5   The algorithm $SUBALGII^n$

This section is dedicated to defining $SUBALGII^n$ and proving theorem 5. In principle $SUBALGII^n$ is a slightly modified version of $SUBALGI^{n^{0.2}}$. The main differences between $SUBALGII^n$ and $SUBALGI^{n^{0.2}}$ are the following: a) $SUBALGII^n$ is not given in his input stopping times. Thus it has to construct stopping time itself. b) $SUBALGII^n$ is not given a little piece of scenery in its input: it has to construct that little piece of scenery itself. c) $SUBALGII^n$ is not trying to reconstruct an i.i.d. piece of scenery but the piece of scenery obtained by restricting ξ in an interval of radius $e^{n^{0.2}}$ around $\{x_1^n, x_2^n\}$. Now, we will see that except for the block $\{x_1^n, x_2^n\}$ the rest of the bits of that scenery is very close in distribution to an i.i.d. piece of scenery. d) We proved that $SUBALGI^n$ works with high probability. Here however we need to show a little bit more: we need to show that with high probability ξ is such that conditioned under ξ, $SUBALGII^n$ works with high probability (if it is given as input the $e^{n^{0.3}}$ first observations of a random walk starting at $\{x_1^n, x_2^n\}$).

Let us now define the algorithm $SUBALGII^n$. Let us recall that as entry $SUBALGII^n$ is only given $e^{n^{0.3}}$ bits. So, $SUBALGII^n$ is a map from $\{0,1\}^{e^{n^{0.3}}}$ to $\cup_k \{0,1\}^k$. Let Λ be an element of $\{0,1\}^{e^{n^{0.3}}}$ and let us define next what $SUBALGII^n(\Lambda)$ would be. We need the following definition: let χ represent $e^{3n^{0.2}}$ observations that is $\chi \in \{0,1\}^{e^{3n^{0.2}}}$. We already defined the functions $\chi \longmapsto B^{n^{0.2}}, OBS^{n^{0.2}}(d), F^{n^{0.2}}(d), f^{n^{0.2}}(d) - b^{n^{0.2}}(\chi)$. Let us now defined $F^{n^{0.2}}(d)^*(\chi)$. $F^{n^{0.2}}(d)^*(\chi)$ is defined to be the (non-truncated)

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length of the first block of $\chi|[b^{n^{0.2}}(\chi), e^{3n^{0.2}}]$. (WE will however always assume that length is $\leq n^4$.) Let $v(k)$ designate the right end of the k-th block of Λ of length $\geq n^2$ (if that block exists). Let $\chi_i \in \{0,1\}^{e^{3n^{0.2}}}$ be equal to $\Lambda|[v(ie^{3n^{0.2}}), v(ie^{3n^{0.2}}) + e^{3n^{0.2}}]$. Then, let $\hat{\mu}$ designate the empirical distribution based on the variables $(B^{n^{0.2}}, OBS^{n^{0.2}}(d), F^{n^{0.2}}(d), f^{n^{0.2}}(d) - b^{n^{0.2}}, F^{n^{0.2}}(d)^*)(\chi_i)$ where $i \in 1, 2, 3, ..., e^{\alpha n^{0.2}}$. Let $1_{>n^2}$ designate the linear functional on $\mathbb{R}^N$ defined in the following way: $1_{>n^2} = e^*_{n^2} + e^*_{n^2+1} + e^*_{n^2+2} + ...$ (Here e^*_i for all $i > 0$ designates the i-th canonical coordinate of $\mathbb{R}^N$.) We are now ready to define $SUBALGII^n(\Lambda)$:

Algorithm 14 *step a) If Λ contains less than $e^{\alpha n^{0.2}}$ blocks of length $\geq n^2$ then have $SUBALGII^n$ break down (or alternatively define then $SUBALGII^n(\Lambda)$ to be the trivial piece of scenery $0 \longmapsto 0$). Step b) If you have not been breaking down in step a, then for each $k \in 1, 2, 3, ..., e^{\alpha n^{0.2}}$ let $\nu^n_{II}(k)$ designate the right end of the k-th block of Λ of length longer than n^2. We will write ν^n_{II} for $(\nu^n_{II}(1), \nu^n_{II}(2), ..., \nu^n_{II}(e^{\alpha n^{0.2}}))$. Step c) Apply the algorithm $SUBALGIII$ $n^{0.2}$ to the input ν^n_{II} and $\Lambda|[\nu^n_{II}(1), \nu^n_{II}(e^{\alpha n^{0.2}})]$ and get as output the collection of pieces of sceneries denoted by SET^n_{II}. Step d) Take the color of the first block of Λ of length longer than n^2 call it s. Step e) Take the average of the lengths of the first $e^{n^{0.2}}$ blocks of Λ of length longer than i_n and call it l. Use it to estimate the length of the block $\{x^n_1, x^n_2\}$. (See lemma 10.) Step-f) Select any couple g, w where g is a positive functional and w is a word of length d such that $13n^{0.2}/ln2 \leq d \leq 17n^{0.2}/ln2$ and such that: Case where $q \neq 0$) $g \otimes 1_w \otimes 1_{>n^2}(P_{\hat{\mu}}(E^{n^{0.2}}_d)\mathcal{L}_{\hat{\mu}}(B^{n^{0.2}}, OBS^{n^{0.2}}(d), F^{n^{0.2}}(d)^*|\xi, E^{n^{0.2}}_d) > 1$ and $g_1 \otimes 1_{>n^2}(P_{\hat{\mu}}(E^{n^{0.2}}_{d-2})\mathcal{L}_{\hat{\mu}}(B^{n^{0.2}}, OBS^{n^{0.2}}(d-1)^*|\xi, E^{n^{0.2}}_{d-1}) \leq 1/(9n^{0.4})$ and c) $|g_1|_2 \leq e^{c_4 n^{0.2}}$. Case where $q = 0$) Same thing as for $q \neq 0$ but simply replace the second inequality by $g_1 \otimes 1_{>n^2}(P_{\hat{\mu}}(E^{n^{0.2}}_{d-2})\mathcal{L}_{\hat{\mu}}(B^{n^{0.2}}, OBS^{n^{0.2}}(d), F^{n^{0.2}}(d)^*|\xi, E^{n^{0.2}}_{d-2}) \leq 1/(9n^{0.4})$. Step g) Find a couple g_a, w_a satisfying the same conditions than in f, but such that on top w_a is different from w it its last $13n^{0.2}/ln2$ bits. Step h) Let ψ be the piece of scenery which starts with the word w and ends with w_a and such that in-between there is exactly a block of length corresponding to our estimate of the length of the block $\{x^n_1, x^n_2\}$. and of color s. Step i) Assemble the piece of sceneries from the set SET^n_{II} together with ψ to get the piece of scenery which is going to be the final output of this algorithm. That is place ψ at the origin and then select one after another pieces from SET^n_{II} which you move around on $\mathbb{Z}$ until they coincide on an interval of at least $13n^{0.2}/ln2$ with an already placed piece. (For precise instruction on this point simply follow algorithm with $n^{0.2}$.)*

We are now going to prove theorem 5. For this let $\{S^+_{x_1}(k)\}_{k \geq 0}$, resp. $\{S^+_{x_2}(k)\}_{k \geq 0}$ be a random walk starting at x^+_1, resp. at x^+_2 but having its increments independent of $\{\xi(k)\}_{k \in \mathbb{Z}}$. Let E^n_{II,x^+_1} be the event that $SUBALGII^n(\xi \circ S^+_{x_1}|[0, e^{n^{0.3}}])$ is a piece of scenery equivalent to $\xi|[x^+_1 - e^{n^{0.2}}, x^+_2 + e^{n^{0.2}}]$. We are going to prove that up to an exponentially small probability in $n^{0.2}$ we have

that ξ is such that conditioned under ξ, E^n_{II,x_1^+} holds with probability which up to an exponentially small probability in $n^{0.2}$ is close to one. A similar thing can be proven for x_2^+, x_2^- and x_1^-. (We will leave those proofs to the reader since they are similar.) This then implies theorem 5. So in what follows we will write E^n_{II} for the event E^n_{II,x_1^+}. Let $\mathcal{L}_1(.|\xi)$ designate the conditional distribution of $(B^{n^{0.2}}, OBS^{n^{0.2}}(d), F^{n^{0.2}}(d), f^{n^{0.2}}(d) - b^{n^{0.2}}, F^{n^{0.2}}(d)^*)(\xi \circ S_{x_1}^+|[0, e^{3n^{0.2}}])$ if we condition under ξ. Let $\mathcal{L}_2(.|\xi)$ designate the conditional distribution of $(B^{n^{0.2}}, OBS^{n^{0.2}}(d), F^{n^{0.2}}(d), f^{n^{0.2}}(d) - b^{n^{0.2}}, F^{n^{0.2}}(d)^*)(\xi \circ S_{x_2}^+|[0, e^{3n^{0.2}}])$ if we condition under ξ. Let $a(1)$ designate the proportion of i's where $i \in 1, 2, 3, ..., e^{\alpha n^{0.2}}/e^{3n^{0.2}}$ such that the right end of the $ie^{3n^{0.2}}$-th block of $\xi \circ S_{x_1}^+$ of length longer than n^2 stops $S_{x_1}^+$ at x_1^+, let $a(2)$ designate those for which $S_{x_1}^+$ is stopped at x_2^+. Let ε_{II} designate the signed measure $\hat{\mu} - a(1)\mathcal{L}_1(.|\xi) - a(2)\mathcal{L}_2(.|\xi)$. Define the following events:

$E^n_{II,1} = \{$ for any integer interval $I \subset [-3e^{n^{0.2}} + x_1^+, x_1^+] \cup [x_2^+, x_2^+ + 3e^{n^{0.2}}]$ of length $n^{0.2}13/(ln2)$ we have that there exists two points $y < z$ such that $I \subset [y, z]$ and $\xi(y - 1) \neq \xi(y)$ and $\xi(z) \neq \xi(z + 1)$ and $z - y \leq n^{0.2}17/(ln2)\}$. $E^n_{II,2} = \{$ the product of the lengths of any $c_1 n^{0.2}$ consecutive blocks of the scenery $\xi|[-3e^{n^{0.2}} + x_1^+, x_2^+ + 3e^{n^{0.2}}]$ is smaller than $e^{c_3 n^{0.2}}\}$. $E^n_{II,3} = \{|\varepsilon_{II}| \leq e^{-2c_4 n^{0.2}}\}$. $E^n_{II,4} = \{$for any integers $y \in [-3e^{n^{0.2}} + x_1^+, x_2^+ + 3e^{n^{0.2}}]$ such that $\xi(y) \neq \xi(y - 1)$ the left functional of ξ at y is also a left limiting functional of ξ at $y\} \cap \{$for any integers $z \in [-3e^{n^{0.2}} + x_1^+, x_2^+ + 3e^{n^{0.2}}]$ such that $\xi(z) \neq \xi(z + 1)$ the right functional of ξ at z is also a right limiting functional of ξ at $z\}$. Let $E^n_{II,6}$ be the event that in each consecutive sequence of $c_1 n^{0.2}$ blocks of $\xi|[-3e^{n^{0.2}} + x_1^+, x_2^+ + 3e^{n^{0.2}}]$ there are at least $c_2 n^{0.2}$ blocks of length 3 or 5. Let $E^n_{II,7}$ be the event that in the four color unit interval coloring associated with $\xi|[-3e^{n^{0.2}} + x_1^+, x_2^+ + 3e^{n^{0.2}}]$ no nearest neighbor walk of length $c_2 n^{0.2}$ can generate the same sequence then a sequence in the four unit interval coloring located strictly to the left or the right from where the nearest neighbor walk starts. Let $E^n_{II,8}$ be the event that except of the block $\{x_1^+, x_2^+\}$ there is no other block of length $\geq n^{0.4}$ in $\xi|[-e^{n^{0.3}} + x_1^+, x_2^+ + e^{n^{0.3}}]$. Let $E^n_{II,9}$ be the event that all the blocks of $\xi \circ S_{x_1}^+|[0, e^{n^{0.3}}]$ of length longer than n^2 are generated by $\xi \circ S_{x_1}^+$ on $\{x_1^+, x_2^+\}$ and there are more than $e^{\alpha n^{0.2}}$ blocks of $\xi \circ S_{x_1}^+|[0, e^{n^{0.3}}]$ of length longer than i^n (see lemma 6). Let $E^n_{II,10}$ be the event that $x_2^+ - x_1^+ < 2n$. Let $E^n_{II,11}$ be the event that the average of the lengths of the first $e^{n^{0.2}}$ blocks of $\xi \circ S_{x_1}^+$ which have been generated on $\{x_1^+, x_2^+\}$ and which are longer than i^n is less away than 0.5 from the value $E[T^{x_2^+ - x_1^+}|T^{x_2^+ - x_1^+} \geq i^n = \beta_{xx} n^2 lnn]$. Let $E^n_{II,12}$ be the event that for each interval I in $[-3e^{n^{0.2}} + x_1^+, x_1^+]$ or in $[x_2^+, x_2^+ + 3e^{n^{0.2}}]$ and of length $ln13n^{0.2}/ln2$, there exist an interval J such that $I \subset J$ and $\xi|J$ is equivalent to a piece of scenery in the collection SET^n_{II} and all the pieces of sceneries which are elements of SET^n_{II} are also contained in $\xi|[-e^{3n^{0.2}} + x_1^+, x_2^+ + e^{3n^{0.2}}]$. Let $E^n_{II,13}$ be

the event that in $\xi|[-3e^{n^{0.2}}+x_1^+, x_1^+]$ and in $\xi|[-3e^{n^{0.2}}+x_1^+, x_1^+]$ each piece of scenery of length $ln13n^{0.2}/ln2 - 1$ is contained at most once. Let $E_{II,14}^n$ be the event that there exists two points a and b (may be random) such that the piece of scenery ψ constructed in step h of our algorithm $SUBALGII^n$ is equivalent to $\xi|[a,b]$ and $a \in [x_1^+ - ln17n^{0.2}/ln2, x_1^+ - ln13n^{0.2}/ln2]$ and $b \in [x_2^+ + ln13n^{0.2}/ln2, x_2^+ + ln17n^{0.2}/ln2]$.

We have $E_{II,12}^n \cap E_{II,13}^n \cap E_{II,14}^n \subset E_{II}^n$. Furthermore, in a very similar way to what we did for the algorithm $SUBALGIII^n$, one can prove that $E_{II,1}^n \cap E_{II,2}^n \cap E_{II,3}^n \subset E_{II,12}^n$ and $E_{II,1}^n \cap E_{II,2}^n \cap E_{II,3}^n \cap E_{II,9}^n \cap E_{II,11}^n \subset E_{II,14}^n$ and eventually $E_{II,6}^n \cap E_{II,7}^n \subset E_{II,4}^n$. We get that $(E_{II,1}^n \cap E_{II,2}^n \cap E_{II,6}^n \cap E_{II,7}^n) \cap E_{II,3}^n \cap E_{II,9}^n \cap E_{II,11}^n \subset E_{II}^n$. Thus, $E_{II}^{nc} \subset E_{II,1}^{nc} \cup E_{II,2}^{nc} \cup E_{II,6}^{nc} \cup E_{II,7}^{nc} \cup E_{II,3}^{nc} \cup E_{II,9}^{nc} \cup E_{II,11}^{nc}$. Furthermore, E_{II}^{nc} is contained in $(E_{II,1}^{nc} \cup E_{II,2}^{nc} \cup E_{II,6}^{nc} \cup E_{II,7}^{nc} \cup E_{II,8}^{nc} \cup E_{II,10}^{nc}) \cup (E_{II,8}^n \cap E_{II,9}^{nc}) \cup (E_{II,9}^n \cap E_{II,3}^n) \cup (E_{II,10}^n \cap E_{II,11}^{nc})$. Now define $E_{II,\xi}^n$ to be equal to $E_{II,1}^n \cap E_{II,2}^n \cap E_{II,6}^n \cap E_{II,7}^n \cap E_{II,8}^n \cap E_{II,10}^n$. Then first note that $E_{II,\xi}^n$ only depends on ξ, i.e. is $\sigma(\xi(k)|k \in \mathbb{Z})$ measurable. Now, when $E_{II,\xi}^n$ holds (or alternatively when we condition under ξ where $\xi \in E_{II,\xi}^n$) we get that $E_{II}^{nc} \subset E_{II,9}^{nc} \cup (E_{II,9}^n \cap E_{II,3}^n) \cup E_{II,11}^{nc}$. Now when we condition under ξ and we have a $\xi \in E_{II,\xi}^n$ we get that $E_{II,9}^{nc}$ has exponentially small probability in $n^{0.3}$. More precisely there exist $\beta_{25}, \beta_{26} > 0$ not depending on n and ξ such that $P(E_{II,9}^{nc}|\xi) \leqslant \beta_{25}e^{-\beta_{26}n^{0.3}}$ for all $\xi \in E_{II,\xi}^n$. (To see that this is true see the proof in section 2 that E_3^n and E_5^n both hold with high probability.) Next note that when we condition under ξ and we have a $\xi \in E_{II,\xi}^n$ we get that $(E_{II,9}^{nc} \cap E_{II,3}^n)$ has exponentially small probability in $n^{0.2}$. To see this note in our proof that $P(E_{III,3}^{nc})$ is negatively exponentially small in n we did not use any assumptions on ξ. We only needed the fact that the stopping times all stop the random walk in the interval $[-e^n, e^n]$. Thus, the same upper bound which holds for $P(E_{III,3}^{nc})$ holds also for $P(E_{II,3}^{nc}|\xi)$ no matter what ξ. Thus the same upper bound but with $n^{0.2}$ instead of n holds also for $P(E_{II,9}^{nc} \cap E_{II,3}^n|\xi)$ no matter what ξ. Eventually, by a large deviation principle it is easy to get an exponentially small upper bound in $n^{0.2}$ which does not depend on n or ξ for $P(E_{II,11}^{nc}|\xi)$ where $\xi \in E_{II,8}^n$. This then implies (since we saw that when $\xi \in E_{II,\xi}^n$ then $E_{II}^{nc} \subset E_{II,9}^{nc} \cup (E_{II,9}^n \cap E_{II,3}^n) \cup E_{II,11}^{nc}$) that there exists $\beta_{26}, \beta_{27} > 0$ not depending on n or on ξ, such that for all $\xi \in E_{II,\xi}^n$ we have $P(E_{II}^{nc}|\xi) \leqslant \beta_{26}e^{-\beta_{27}n^{0.2}}$. Now, $P(E_{II,\xi}^{nc})$ is exponentially small in $n^{0.2}$. Since the same kind of result can be proven if we take x_1^-, x_2^- or x_2^+ instead of x_2^+, we have that theorem 5 holds. Now, let us give us a few more details about why $P(E_{II,\xi}^{nc})$ is exponentially small in $n^{0.2}$. Note first that $P(E_{II,\xi}^{nc}) \leqslant P(E_{II,1}^{nc}) + P(E_{II,2}^{nc}) + P(E_{II,6}^{nc}) + P(E_{II,7}^{nc}) + P(E_{II,8}^{nc}) + P(E_{II,10}^{nc})$. Now let $E_{II,12}^n$ be the event that $\{x_1^+ > e^{3n^{0.2}}\}$. Note that the lengths of the blocks of $\xi|[0, x_1^+]$ are i.i.d. with each one of them having the distribution of a block of ξ conditioned under the event that block has length $< n$. It follows, that when we condition under the event $\{x_1^+ > e^{3n^{0.2}}\}$ then the distribution of $\xi|[x_1^+ - e^{3n^{0.2}}, x_1^+]$ is the same as the distribution of $\xi|[0, x_1^+]$ conditioned under the event $\{x_1^+ > e^{3n^{0.2}}\}$. It follows that in total variation the distribution of

$\xi||[x_1^+ - e^{3n^{0.2}}, x_1^+]$ is different from the i.i.d. scenery distribution by a quantity which in absolute value is smaller or equal to $P(E_{II,12}^{nc})$. It is easy to check that $P(E_{II,12}^{nc})$ is exponentially small in n. Now if $\xi||[x_1^+ - e^{3n^{0.2}}, x_1^+]$ would have the i.i.d. distribution, then $P(E_{II,1}^{nc})$, $P(E_{II,6}^{nc})$, resp. $P(E_{II,7}^{nc})$ would have the same upper bound as $P(E_{III,1}^{nc})$, $P(E_{III,6}^{nc})$, resp. $P(E_{III,7}^{nc})$ but with $n^{0.2}$ instead of n. Thus, $P(E_{II,1}^{nc})$, $P(E_{II,6}^{nc})$, resp. $P(E_{II,7}^{nc})$ would be exponentially small in $n^{0.2}$. Since, however $\xi||[x_1^+ - e^{3n^{0.2}}, x_1^+]$ is not i.i.d. we need to add to these upper bounds the value $P(E_{II,12}^{nc})$. Since $P(E_{II,12}^{nc})$.is exponentially small in n, we get that $P(E_{II,1}^{nc})$, $P(E_{II,6}^{nc})$, resp. $P(E_{II,7}^{nc})$ would be exponentially small in $n^{0.2}$. It is easy to check that $P(E_{II,1}^{nc})$ is exponentially small in $n^{0.4}$, whilst $P(E_{II,8}^{nc})$ is exponentially small in n. For $P(E_{II,2}^{nc})$ consider the following thing: let $E_{II,2}^{n\prime}$ be the event { the product of the lengths of any $c_1 n^{0.2} - 1$ consecutive blocks , where we take out the block $\{x_1^+, x_2^+\}$, of the scenery $\xi||[-3e^{n^{0.2}} + x_1^+, x_2^+ + 3e^n]$ is smaller than $\frac{1}{2n} e^{c_3 n}$.} Now it is easy to see that if $\xi||[x_1^+ - e^{3n^{0.2}}, x_1^+]$ would be i.i.d. then we could an get an exponential upper bound in $n^{0.2}$ for $P(E_{II,2}^{n\prime c})$. (This can be done in almost the same way, then we found the upper bound for $P(E_{II,2}^{nc})$ since the quantity $\frac{1}{2n}$ has little effect in comparison to the exponentially quantities we are dealing with.) Thus, $P(E_{II,2}^{n\prime c})$ can be bound from above by an exponentially small quantity in $n^{0.2}$ plus $P(E_{II,12}^{nc})$. Thus, $P(E_{II,2}^{n\prime c})$ is exponentially small in $n^{0.2}$. Now we have that $E_{II,2}^{n\prime} \cap E_{II,8}^{n\prime} \subset E_{II,2}^{n}$. Thus, $P(E_{II,2}^{nc}) \leqslant P(E_{II,2}^{n\prime c}) + P(E_{II,8}^{n\prime c})$. Thus, $P(E_{II,2}^{nc})$ is exponentially small in $n^{0.2}$.

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