A scaling limit theorem for a class of superdiffusions

János Engländer * Dmitry Turaev

EURANDOM
PO Box 513
NL-5600 MB Eindhoven, The Netherlands
and
Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstrasse 39
D-10117 Berlin, Germany

E-mail: jengland@euridice.tue.nl, turaev@wias-berlin.de

AMS classification: primary 60J80; secondary 60J60, 60G57

Keywords: measure-valued process, superprocess, super-Brownian motion, scaling limit, single point source, invariant curve

typeset by LaTeX printed March 9, 2000

* Partially supported by the EU grant to Stochastic Analysis via Humboldt University, Berlin. The hospitality of the Weierstrass Institute is also gratefully acknowledged.

Abstract

Consider the σ-finite measure-valued diffusion corresponding to the evolution equation $u_t = Lu + \beta(x)u - f(x,u)$, where

$$f(x,u) = \alpha(x)u^2 + \int_0^\infty (e^{-ku} - 1 + ku)\, n(x, \mathrm{d}k)$$

and n is a smooth kernel satisfying an integrability condition. We assume that $\beta, \alpha \in C^\eta(\mathbb{R}^d)$ with $\eta \in (0,1]$, and $\alpha > 0$. Under appropriate spectral theoretical assumptions we prove the existence of the random measure

$$\lim_{t\uparrow\infty} e^{-\lambda_c t} X_t(\mathrm{d}x)$$

(with respect to the vague topology), where λ_c is the principal eigenvalue of $L+\beta$ on $\mathbb{R}^d$ and it is assumed to be finite and positive, completing a result of Pinsky on the expectation of the rescaled process. Moreover we prove that this limiting random measure is a nonnegative nondegenerate random multiple of a deterministic measure related to the operator $L + \beta$.

When β is bounded from above, X is finite measure-valued. In this case, under an additional assumption on $L + \beta$, we can actually prove the existence of the previous limit with respect to the weak topology.

As a particular case, we show that if L corresponds to a positive recurrent diffusion Y and β is a positive constant, then

$$\lim_{t\uparrow\infty} e^{-\beta t} X_t(\mathrm{d}x)$$

exists and equals to a nonnegative nondegenerate random multiple of the invariant measure for Y.

Taking $L = \frac{1}{2}\Delta$ on $\mathbb{R}$ and replacing β by δ_0 (super-Brownian motion with a single point source), we prove a similar result with λ_c replaced by $1/2$ and with the deterministic measure $e^{-|x|}\,\mathrm{d}x$, giving an answer in the affirmative to a proposed problem in [EF99].

The proofs are based upon two new results on invariant curves of strongly continuous nonlinear semigroups.

1 Introduction and statement of results

1.1 Motivation

In [Pin96] it has been proven that the superdiffusion corresponding to the semilinear operator $Lu + \beta u - \alpha u^2$ tends to a nonzero limit in expectation if and only if the linear operator $L+\beta$ satisfies a certain spectral assumption. Although the statement was proved for the case when α and β are positive constants, it is easy to check that the proof works just as well in the variable coefficient case. A similar result has been presented in [EF99] for a non-regular setting (super-Brownian motion with a single point source).

In this paper we replace the expectations by the superdiffusions themselves, and prove that the rescaled superdiffusions tend to a limit in law. For the case of the super-Brownian motion with a single point source this will give a positive answer to a proposed problem in [EF99].

1.2 Preparation

We begin with a number of notations. Let $\mathcal{M}=\mathcal{M}(\mathbb{R}^d)$ denote the set of finite measures μ on $\mathbb{R}^d$ endowed with the topology of weak convergence and with $\|\mu\|$ denoting the total mass of μ; and let $\mathcal{M}_c=\mathcal{M}_c(\mathbb{R}^d)$ denote the subset of all compactly supported measures. Write $C^{k,\eta}=C^{k,\eta}(\mathbb{R}^d)$ for the usual Hölder spaces of index $\eta\in(0,1]$ including derivatives of order k, and set $C^\eta:=C^{0,\eta}$. Let $C_b=C_b(\mathbb{R}^d)$ and $C_b^+=C_b^+(\mathbb{R}^d)$ denote the space of bounded continuous functions on $\mathbb{R}^d$ and the space of nonnegative bounded continuous functions respectively; and $\|\cdot\|$ denote the sup-norm for bounded functions. Furthermore, $C=C(\mathbb{R}^d)$ and $C_0=C_0(\mathbb{R}^d)$ refer to continuous functions on $\mathbb{R}^d$ and continuous functions on $\mathbb{R}^d$ decaying to zero, respectively. Finally, C_c (C_c^+) denotes the space of continuous (nonnegative continuous) functions on $\mathbb{R}^d$ with compact support.

We now continue with recalling the definition of the $(L,\beta,\alpha;\mathbb{R}^d)$-superdiffusion. Let L be an elliptic operator on $\mathbb{R}^d$ of the form

$$L=\frac{1}{2}\nabla\cdot a\nabla+b\cdot\nabla \quad \text{on } \mathbb{R}^d, \tag{1}$$

where $a_{ij},\ b_i\in C^{1,\eta}$, $i,j=1,...,d$, for some $\eta\in(0,1]$ and the symmetric matrix $a=\{a_{ij}\}$ satisfies

$$\sum_{i,j=1}^d a_{ij}(x)v_iv_j>0 \text{ for all } v\in\mathbb{R}^d\backslash\{0\} \text{ and all } x\in\mathbb{R}^d. \tag{2}$$

In addition, let $\alpha,\beta\in C^\eta$ where β is bounded from above (we will later relax this condition) and α is positive.

Notation 1 (superdiffusion) Let $(X,P_\mu,\ \mu\in\mathcal{M})$ denote the $(L,\beta,\alpha;\mathbb{R}^d)$-*superdiffusion.* That is, X is the unique $\mathcal{M}$-valued continuous (time-homogeneous) Markov process which satisfies, for any bounded continuous $g:\mathbb{R}^d\mapsto\mathbb{R}_+$,

$$E_\mu\exp\langle X_t,-g\rangle=\exp\langle\mu,-u(\cdot,t)\rangle, \tag{3}$$

where u is the minimal nonnegative solution to

$$\left.\begin{aligned} u_t = Lu + \beta u - \alpha u^2 \quad \text{on } \mathbb{R}^d \times (0,\infty), \\ \lim_{t\to 0+} u(\cdot,t) = g(\cdot) \end{aligned}\right\} \tag{4}$$

(see [EP99]). Here $\langle \nu, f\rangle$ denotes the integral $\int_{\mathbb{R}^d} \nu(\mathrm{d}x)\, f(x)$. $\diamond$

Here is an equivalent way of replacing the word *minimal* in the definition of u in Notation 1 (cf. [EP99]): u is the nonnegative solution to (4) obtained as a limit of solutions with Dirichlet boundary condition: $u = \lim_{n\to+\infty} u_n$ where $u_n(x,t)$ is the solution to (4) for $|x| \le n$ with $u_n(x) = 0$ at $|x| = n$.

Remark 2 We note that this definition will later be extended to a more general class of $\beta' s$ and a more general class of nonlinearities (see the last subsection of this section). $\diamond$

Remark 3 (mild equation with linear semigroup) In fact the parabolic semilinear pde under (4) can be rewritten as an integral-equation (or *mild*-equation) as follows: u is the unique function which solves

$$u(\cdot,t) = T_t g - \int_0^t \mathrm{d}s\; T_{t-s}\left(\alpha u^2(\cdot,s)\right), \tag{5}$$

with $\sup_{0\le s\le t} \|u(\cdot,s)\| < \infty$ for all $t > 0$. Here $\{T_t\}_{t\ge 0}$ denotes the semigroup corresponding to the operator $L+\beta$ and acting on C_b. That is, for bounded and continuous g,

$$T_t g := \mathbb{E}_x\left[\exp\left(\int_0^t \beta(Y_s)\,\mathrm{d}s\right) g(Y_t)\ ;\ \tau > t\ \right], \tag{6}$$

where Y denotes the diffusion corresponding to L on $\mathbb{R}^d$ living on $\mathbb{R}^d \cup \{\Delta\}$, the one-point compactification of $\mathbb{R}^d$ (with expectations $\{\mathbb{E}_x\}_{x\in\mathbb{R}^d}$), and τ denotes its lifetime:

$$\tau := \inf\{t \ge 0 \mid Y_t \notin \mathbb{R}^d\}.$$

We mention that the mild equation under (5) is usually written in a slightly different form: $\{T_t\}_{t\ge 0}$ is replaced by the semigroup corresponding to the operator L on $\mathbb{R}^d$ and the nonlinearity αu^2 is replaced by $-\beta u + \alpha u^2$ (see e.g. formula (1.3) in [EP99]). The advantage of that formulation is that the semigroup then describes the spatial motion (the diffusion corresponding to L on $\mathbb{R}^d$), while the nonlinear term refers to the branching mechanism built in the construction of X. In this paper we chose to include β in the linear semigroup as in (6) for technical reasons. For example, we do not have to assume that β is bounded from below, the semigroup under (6) makes sense whenever β is bounded from above. $\diamond$

Remark 4 (formula for expectation) Using the stochastic representation formula for solutions of parabolic pde's (see formula 5.15 in [Fri64]) it is easy to show that $u(x,t) := T_t g(x)$ is the minimal nonnegative solution for (4) with $\alpha = 0$. From this, it is standard to verify that

$$E_{\delta_x}\langle X_t, g\rangle = T_t g(x). \tag{7}$$

$\diamond$

In the sequel we will use concepts and facts from the so-called 'criticality-theory' of second order elliptic operators (see Chapter 4 in [Pin95]) without further reference. The definitions for *subcritical, critical* and *product-critical* operators, for the *ground-state* of a critical operator and its adjoint, and for the *generalized principle eigenvalue* of $L+\beta$ on $\mathbb{R}^d$ are presented in Appendix 2. The reader should consult that section from time to time, where a review is given on criticality-theory.

We will also use the notation $\langle f, g\rangle$ with nonnegative f and g for the (possibly infinite) integral $\int_{\mathbb{R}^d} \mathrm{d}x\, f(x)g(x)$. In [Pin96] the following result has been proved (though formally for a somewhat more restricted case — see the note after the theorem):

Theorem P *Let $\mu \in \mathcal{M}_c$ and $g \in C_c^+$. Let $\lambda_c \in \mathbb{R}$ denote the generalized principal eigenvalue of $L+\beta$ on $\mathbb{R}^d$. In the case when $L+\beta-\lambda_c$ is critical we denote the corresponding ground state by ϕ. (The ground state for the formal adjoint of $L+\beta-\lambda_c$ will be denoted by $\tilde{\phi}$.) Finally, let $\rho \in \mathbb{R}$.*

(i) $\lim_{t\uparrow\infty} e^{-\rho t} E_\mu\langle X_t, g\rangle = 0$ if $\rho > \lambda_c$, and $\lim_{t\uparrow\infty} e^{-\rho t} E_\mu\langle X_t, g\rangle = \infty$ if $\rho < \lambda_c$.

$(ii-a)$ If $L+\beta-\lambda_c$ is subcritical or if $L+\beta-\lambda_c$ is critical but $\langle \phi, \tilde{\phi}\rangle = \infty$, then

$$\lim_{t\uparrow\infty} e^{-\lambda_c t} E_\mu\langle X_t, g\rangle = 0.$$

$(ii-b)$ If $L+\beta-\lambda_c$ is critical and $\langle \phi, \tilde{\phi}\rangle < \infty$, then

$$\lim_{t\uparrow\infty} e^{-\lambda_c t} E_\mu\langle X_t, g\rangle = \langle \mu, \phi\rangle\langle \tilde{\phi}, g\rangle,$$

where ϕ and $\tilde{\phi}$ are normalized by $\langle \phi, \tilde{\phi}\rangle = 1$.

The condition in $(ii-b)$ of Theorem P is sometimes called '*product-criticality* ' (see Appendix A.2 for more explanation).

Although this result was stated for the case when L is a conservative diffusion (that is, a diffusion having an infinite lifetime) on $\mathbb{R}^d$ with a corresponding C_0-preserving semigroup and β and α are positive constants, it is easy to check that its proof never uses these assumptions and consequently it is valid for our general notion of the $(L,\beta,\alpha;\mathbb{R}^d)$-superdiffusion as well. (Note that if β is constant, we have $\lambda_c=\beta+\lambda_c(L)$, where $\lambda_c(L)$ denotes the generalized principal eigenvalue of L on $\mathbb{R}^d$.)

In a recent paper [EF99] a non-regular setting, namely a super-Brownian motion with a *single point source* has been studied and a result analogous to Theorem P has been proved for this process. In this case the additional mass production is zero everywhere except at a single point (the origin, say) where the mass production is infinite (in a δ-function sense). In other words, consider the superdiffusion $X^{\sin}$ corresponding to the *formal* evolution equation

$$u_t=\frac{1}{2}\Delta u+\delta_0 u-\alpha u^2 \text{ on } \mathbb{R}\times(0,\infty),$$
$$u(\cdot,0)=g(\cdot),$$

where δ_0 denotes the Dirac δ-function at zero. The precise meaning of the above evolution equation is that u is the unique (nonnegative) solution to the *integral equation*

$$\begin{aligned}u(\cdot\,,t)=&\int_{-\infty}^{\infty}\mathrm{d}y\;p(t,\cdot\,,y)g(y)+\int_0^t\mathrm{d}s\;p(t-s,\cdot\,,0)u(0,s)\\&-\int_0^t\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\;p(t-s,\cdot\,,y)\alpha(y)u^2(y,s),\qquad t>0,\end{aligned}\tag{8}$$

with $\sup_{0\le s\le t}\|u(\cdot,s)\|<\infty$ for all $t>0$, where $\{p(t,x,y)=p(t,x-y);\ t>0,\ x,y\in\mathbb{R}\}$ denote the Brownian transition densities. $X^{\sin}$ is then determined by its Laplace-functional as in (3), but with u from (8). The corresponding expectations will be denoted by $\{E^{\sin}_\mu,\ \mu\in\mathcal{M}_f\}$.

In [EF99] the following result is proved for $\alpha=1$ (the proof for general $\alpha>0$ is virtually identical to the proof given in [EF99]):

Theorem EF *For all bounded continuous $g:\mathbb{R}\mapsto\mathbb{R}_+$ and $\mu\in\mathcal{M}(\mathbb{R})$,*

$$\lim_{t\uparrow\infty}\mathrm{e}^{-t/2}E^{\sin}_\mu\left\langle X^{\sin}_t,g\right\rangle=\langle\mathrm{e}^{-|x|},\mu\rangle\langle\mathrm{e}^{-|x|},g\rangle.\tag{9}$$

Note that in this (non-regular) setting, the number $1/2$ and the function $x\mapsto e^{-|x|}$ play the role of λ_c and $\phi\ (=\tilde{\phi})$. Note also that $\langle e^{-2|x|},1\rangle=1$, that is $x\mapsto e^{-|x|}$ has already been 'normalized'.

An obvious but important fact is recorded in the following remark.

Remark 5 ('overscaling') By Theorem P(i) and the Markov-inequality, for the $(L,\beta,\alpha;\mathbb{R}^d)$-superdiffusion X we have $\lim_{t\uparrow\infty}\langle e^{-\rho t}X_t,g\rangle=0$ in probability if $\rho>\lambda_c$, provided $X_0\in\mathcal{M}_c$. Similarly, using Theorem EF, $\lim_{t\uparrow\infty}\langle e^{-\rho t}X^{\sin}_t,g\rangle=0$ in probability if $\rho>1/2$, provided $X_0\in\mathcal{M}(\mathbb{R})$. $\diamond$

Motivated by these results and a proposed problem in [EF99] (see Remark 3 in that paper), we ask the following natural questions: Let the $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion X satisfy the condition in (ii-b) of Theorem P. Does the rescaled process $\mathrm{e}^{-\lambda_c t} X_t$ have itself a limit in law for any $X_0 \in \mathcal{M}_c$? Is the same true for the rescaled process $\mathrm{e}^{-t/2} X_t^{\sin}$ for any $X_0 \in \mathcal{M}(\mathbb{R})$?

In order to answer the question, we first invoke the definition of *local extinction.*

Definition 6 (local extinction) A measure-valued path X *exhibits local extinction* if $X_t(B) = 0$ for all sufficiently large t, for each ball B. The measure-valued process X corresponding to P_μ is said to possess this property if it is true with P_μ-probability one. $\diamond$

Roughly speaking, local extinction means that the support of the measure-valued process leaves any given compact set in finite time.

Remark 7 (process property) In [Pin96, EP99] it was shown that, for fixed L, β and α, if the property in Definition 6 holds for some P_μ, $\mu \in \mathcal{M}_c$ with $\mu \neq 0$, then it in fact holds for every P_μ, $\mu \in \mathcal{M}_c$. $\diamond$

Local extinction can be characterized in terms of L and β (see Theorem 6 and Remark 1 in [Pin96]):

Lemma 8 (spectral condition for local extinction) *The $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion X exhibits local extinction if and only if there exists a (strictly) positive solution u to the equation $(L + \beta)u = 0$ on $\mathbb{R}^d$ that is if and only if $\lambda_c \leq 0$.*

Remark 9 (ergodicity and local extinction) Let $f : \mathbb{R}_+ \to \mathbb{R}_+$. Using Lemma 8, it immediately follows that if $\lambda_c \leq 0$, we have $f(t)\langle X_t, g\rangle \to 0$ as $t \to \infty$ a.s. for any $g \in C_c^+$ and $X_0 \in \mathcal{M}_c$, no matter how 'large' f is.

Nevertheless, the situation is completely different when replacing $g \in C_c^+$ by $g \in C_b^+$. For the case when $\mu \in \mathcal{M}_c$ but $g = 1$, the condition $\lambda_c \leq 0$ (local extinction) does not contain enough information about the behavior of the total mass. To elucidate this point, consider the following example. Fix $\beta, \alpha > 0$ and take an L with $\lambda_c(L) \leq -\beta$ corresponding to a conservative diffusion. Let X denote the corresponding superdiffusion and let X^* denote the superdiffusion where L is replaced by $\frac{1}{2}\Delta$ (supercritical super-Brownian motion). Then $\lambda_c(\frac{1}{2}\Delta + \beta) = \beta$ but for X we have $\lambda_c(L + \beta) \leq 0$. Nevertheless, the processes $\|X\|$ and $\|X^*\|$ have the same law, because the branching is independent from the motion process and 'no mass is lost' due to the conservativeness of the diffusion corresponding to L. (See the argument preceding formula (1.4) in [Pin96].) Therefore $\|X\|$ grows exponentially in expectation in this case. On the other hand, the (sub)critical super-Brownian motion exhibits local extinction too but its total mass is constant (resp. tends to zero) in expectation.

Last, we mention that the case when $\lambda_c \le 0$ and μ does not belong to $\mathcal{M}_c$ but rather σ-finite, has also been studied in the literature. The simplest case is the critical super-Brownian motion, that is $L = \frac{1}{2}\Delta, \beta = 0$ and $0 < \alpha =$const. In this case $\lambda_c = 0$. For the ergodic behavior of this process under different, and even mixed starting measures, see [BCG93]. For $(L, \beta, \alpha, \mathbb{R}^d)$-superdiffusions see [Pin99]. $\diamond$

In the sequel *we will always assume that* $\lambda_c > 0$, that is that the $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion under consideration does not exhibit local extinction. (As already mentioned in this subsection, in the singular setting the number 1/2 plays the role of λ_c.)

1.3 Scaling limits for superdiffusions

In this paper we will prove the existence of the scaling limits in the case of $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusions and in the case of the single point source as well, under the assumption that $\lambda_c(L+\beta) > 0$ and that the condition in (ii-b) of Theorem P (product-criticality) holds. In addition, we will assume that α is not 'too large'. In fact we will be able to replace $\mathcal{M}_c$ and $\mathcal{M}(\mathbb{R})$ by two families of measures, each satisfying an integrability assumption only. (See Theorems 1 and 2 below.)

As it is usual in the analysis of nonlinear phenomena, we use a geometric approach to the equation (5). For a continuous function u define the weighted norm $\|u\|_{\phi^{-1}} = \sup_x \|u(x)\phi^{-1}(x)\|$ where ϕ is the ground state of $L+\beta-\lambda_c$. Under certain conditions guaranteed by Theorem 1 or 2 below, we prove in Lemma 20 of section 3 the existence of a special smooth curve $u = \psi(\sigma),\ \sigma \in [0, \infty)$, in the space of nonnegative functions bounded in the norm $\|\cdot\|_{\phi^{-1}}$, such that $\psi(0) = 0$ and $\psi'(0) = \phi$ and that the curve is invariant under the positive time shift $u(0) \mapsto u(t)$ defined by (5). Thus, the curve emanates from zero and is tangent at zero to the one-dimensional invariant (with respect to the semigroup $\{T_t\}_{t\ge 0}$) subspace, spanned by ϕ. We prove that this curve is uniquely defined by the condition that for any point $u(0) = g = \psi(\sigma_0)$ on the curve we have

$$u(t) = \psi(\sigma_0 e^{\lambda_c t}), \tag{10}$$

where $u(t)$ is the unique nonnegative solution to (5), bounded in the $\|u\|_{\phi^{-1}}$-norm at all t. This condition means that the curve is parametrized in such a way that the equation (5) restricted to the invariant curve becomes linear: $\dot{\sigma} = \lambda_c \sigma$.

Since our invariant curve $u = \psi(\sigma)$ is defined uniquely by the nonlinear equation (5), it is quite legitimate to formulate the results in terms of the function ψ, as we do below (note that our proof of existence of the invariant curve in Lemma 20 is constructive and gives an algorithm for the computation of the function ψ). In essence, Theorems 1 and 2 illustrate one of the standard ideas of local nonlinear analysis: the analogy between invariant subspaces of linearized evolution equations and invariant curves of nonlinear equations.

Before stating our main result we introduce an additional notation.

Notation 10 For $0 \le g$ measurable, define the following space of measures:

$$\mathcal{M}^{(g)} := \{\mu \text{ is a measure on } \mathbb{R}^d : \langle \mu, g \rangle < \infty\}.$$

$\diamond$

We now state our main result.

Theorem 1 (scaling limit for $(L, \beta, \alpha; \mathbb{R}^d)$ -superdiffusions) *Let X be the $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion with L, β, α as in the paragraph preceding Notation 1. Let $0 < \lambda_c$ where λ_c denotes the generalized principal eigenvalue of $L + \beta$ on $\mathbb{R}^d$. Assume that the condition in (ii-b) of Theorem P (product-criticality) holds. In addition, assume that $\alpha\phi$ is bounded from above.*

Then for any $X_0 = \mu \in \mathcal{M}^{(\phi)}$, there exists a nonnegative non-degenerate random variable N_μ such that for all $g \in C_c^+$,

$$\lim_{t\uparrow\infty} e^{-\lambda_c t}\langle X_t, g\rangle = N_\mu \cdot \langle \tilde{\phi}, g\rangle \text{ in law.} \tag{11}$$

Moreover, under the normalization $\langle \phi, \tilde{\phi}\rangle = 1$, the law of N_μ is determined via its Laplace-transform as follows:

$$\mathbf{E}e^{-\sigma N_\mu} = \exp\langle \mu, -\psi(\sigma)\rangle \ , \sigma > 0 \tag{12}$$

where $\sigma \mapsto \psi(\sigma)$ is the invariant curve defined by (10). Furthermore,

$$\mathbf{E}N_\mu = \langle \mu, \phi\rangle. \tag{13}$$

In particular, $\mathbf{P}(N_\mu < \infty) = 1$.

If we assume in addition that ϕ is bounded away from zero, then

$$\lim_{t\uparrow\infty} e^{-\lambda_c t} X_t(\mathrm{d}x) = N_\mu \cdot \tilde{\phi}(x)\,\mathrm{d}x \text{ in law.} \tag{14}$$

An interpretation of the above theorem will be given in the next subsection.

Remark 11 It is not hard to show that (11) implies that

$$\mathbf{P}(N_\mu = 0) \geq P_\mu(\langle X_t, 1\rangle = 0 \text{ for all large } t's).$$

(We defer the proof to the next subsection, because we will need the concept of the h-transform for superprocesses defined in that subsection.) The rightmost probability, that is the probability of finite time extinction is positive for all $\mu \in \mathcal{M}_c$ (see Theorem 3.1 in [EP99]), and consequently $\mathbf{P}(N_\mu = 0) > 0$ for all $\mu \in \mathcal{M}_c$. $\diamond$

We continue with two proposed problems:

Problem 12 Is it true in general, that

$$\mathbf{P}(N_\mu = 0) = P_\mu(\langle X_t, 1\rangle = 0 \text{ for all large } t's) \ ?$$

(Cf. Theorem III.7.2 in [AN72] for non-spatial branching processes.) $\diamond$

Problem 13 What can we say about the asymptotic behavior of X in the case when $L+\beta-\lambda_c$ is subcritical or $L+\beta-\lambda_c$ is critical but $\langle \phi, \tilde{\phi}\rangle = \infty$ (case $(ii-a)$ in Theorem P)? $\diamond$

Finally, we state a theorem analogous to Theorem 1 for the superdiffusion $X^{\sin}$ of Theorem EF (super-Brownian motion with an additional single point source).

Theorem 2 (scaling limit in the case of a single point source) *Let $X^{\sin}$ be the superdiffusion corresponding to the integral equation (8), and assume that $\alpha(x) \leq K \cdot e^{|x|}$, $K > 0$. For any $X(0) = \mu \in \mathcal{M}^{(\exp -|x|)}$, there exists a nonnegative non-degenerate random variable N_μ with $\mathbf{E}N_\mu = \langle \mu, e^{-|x|}\rangle$ satisfying that*

$$\lim_{t\uparrow\infty} e^{-t/2} X_t^{\sin}(\mathrm{d}x) = N_\mu \cdot e^{-|x|}\,\mathrm{d}x \text{ in law.} \tag{15}$$

Furthermore, the law of N_μ is determined via its Laplace-transform as in (12), where $\sigma \mapsto \psi(\sigma)$ is the invariant curve defined by (10) when replacing the nonlinear equation (5) with (8), and using the formal substitution $\lambda_c = 1/2$.

1.4 An interpretation of our main theorem via reducing it to a particular case

Before presenting an interpretation of Theorem 1 , first recall the definition of the h–transformed superdiffusion. (The h-transform for $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusions was developed in [EP99].)

Definition 14 (h–transformed superdiffusion X^h) Let $0 < h \in C^{2,\eta}$ and consider the $(L,\beta,\alpha;\mathbb{R}^d)$–superdiffusion X. Define

$$X_t^h \;:=\; hX_t \quad \left(\text{that is, } \frac{\mathrm{d}X_t^h}{\mathrm{d}X_t} = h\right), \qquad t \geq 0. \tag{16}$$

Then X^h is the $(L_0^h, \beta^h, \alpha^h; \mathbb{R}^d)$–superdiffusion, where

$$L_0^h \;:=\; L + a\,\frac{\nabla h}{h}\cdot\nabla, \quad \beta^h \;:=\; \frac{(L+\beta)h}{h}, \quad \text{and} \quad \alpha^h := \alpha h. \tag{17}$$

X^h makes sense even if β^h is unbounded from above (see [EP99, Section 2] for more elaboration). X^h is called the *h–transformed superdiffusion.* $\diamond$

Remark 15 (h–transforms) **(i)** L_0^h is just the diffusion part of the usual *linear* h–transformed operator L^h (see [Pin95, Chapter 4]).
(ii) The operators $\mathcal{A}(u) := Lu + \beta u - \alpha u^2$ and $\mathcal{A}^h(u) := L_0^h u + \beta^h u - \alpha^h u^2$ are related by $\mathcal{A}^h(u) = \frac{1}{h}\mathcal{A}(hu)$. $\diamond$

Remark 16 (invariance under h–transforms) An obvious but important property of the h–transform is that it leaves invariant the *support process* $t \mapsto \operatorname{supp}(X_t)$ of X. $\diamond$

We now give an interpretation of Theorem 1 using the transformed process $X^\phi = \phi X$ as follows. First note that ϕ and $\tilde{\phi}$ transform into 1 and $\phi\tilde{\phi}$ respectively. Hence, Theorem 1 states that for $X_0^\phi = \nu \in \mathcal{M}$,

$$\lim_{t\uparrow\infty} e^{-\lambda_c t} X_t^\phi(\mathrm{d}x) = N_\nu^\phi \cdot \phi\tilde{\phi}\,\mathrm{d}x \text{ in law} \tag{18}$$

(cf. Theorem III.7.1 in [AN72] for non-spatial branching processes). Recall that X^ϕ is the $(L_0^\phi, \lambda_c, \alpha\phi; \mathbb{R}^d)$-superdiffusion. (Note that $\beta^\phi = \lambda_c$ is no more spatially dependent.)

Next, note that integrating against the function 1 in (18) yields

$$\lim_{t\uparrow\infty} e^{-\lambda_c t}\|X_t^\phi\| = N_\nu^\phi \text{ in law,} \tag{19}$$

that is, the total mass behaves like $e^{\lambda_c t}N_\nu^\phi$ as $t \to \infty$. Recall that λ_c is the average *mass creation* at each point of $\mathbb{R}^d$ and note that since ϕ transforms into 1, we have

$$\mathbf{E}N_\nu^\phi = \|\nu\|.$$

By (12) (applied for the ϕ-transformed setting) N_ν^ϕ depends on the whole branching term $\lambda_c u - \alpha\phi u^2$, where $\alpha\phi$ can be identified with the *variance* of the offspring distribution (see Appendix 1 in [EP99]). It depends also on L_0^ϕ, that is on the motion process, which fact comes of course from the spatial dependence of the branching.

Note also (see Appendix A.2) that by the product-criticality assumption, and by the invariance of this property under h-transforms, L_0^ϕ corresponds to a *positive recurrent* diffusion (loosely speaking, positive recurrence means that the diffusion hits any fixed ball in finite expected time) which ergodizes with invariant density $\phi\tilde{\phi}\,\mathrm{d}x$ (see Theorem 4.9.9. in [Pin95]). Putting this together with (19), the righthand side of the approximating formula

$$X_t^\phi(\mathrm{d}x) \sim e^{\lambda_c t} N_\nu^\phi \cdot \phi\tilde{\phi}\,\mathrm{d}x$$

can be interpreted as $e^{\lambda_c t} N_\nu^\phi$ being the *total mass* and $\phi\tilde{\phi}\,\mathrm{d}x$ being the *limiting distribution of the individual particle.*

We close this section with the

Proof of Remark 11. It is enough to prove the inequality for X^ϕ, because the probability of extinction is the same for X (starting with μ) and X^ϕ (starting with $\nu = \phi\mu$), and also $\mathbf{P}(N_\mu = 0) = \mathbf{P}(N_\nu^\phi = 0)$. Using (18), we have

$$\begin{aligned}\mathbf{P}(N_\nu^\phi = 0) \quad &= \quad \lim_{s\uparrow\infty} \mathbf{E} e^{-sN_\nu^\phi\langle\phi\tilde{\phi},1\rangle} = \lim_{s\uparrow\infty}\lim_{t\uparrow\infty} \mathbf{E} e^{-s\langle e^{-\lambda_c t}X_t^\phi,1\rangle} \\ &\geq P_\mu(\langle X_t^\phi, 1\rangle = 0 \text{ for all large } t's).\end{aligned}$$

This completes the proof of the remark. □

1.5 More general branching

In this subsection we will consider superdiffusions with more general branching mechanisms and generalize our main theorem for that setup. To this end, first recall that in [EP99] the definition of the $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion has been extended for β's which are not necessarily bounded from above but rather satisfy the more general condition

$$\lambda_c = \lambda_c(L + \beta) < \infty. \tag{20}$$

This extension relies on the fact that the h-transform with $h = \phi$ transforms formally the quadruple $(L, \beta, \alpha; \mathbb{R}^d)$ into the quadruple $(L_0^\phi, \lambda_c, \alpha\phi; \mathbb{R}^d)$, which corresponds to a superdiffusion X (since $\beta^h = \lambda_c < \infty$). Then the $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion $\hat{X}$ can be defined by $\hat{X} := \frac{1}{\phi} X$ (where X starts at

$X_0 = \mu \in \mathcal{M}_c$ if and only if $\hat{X}$ starts at $\hat{X}_0 = \frac{1}{\phi}\mu \in \mathcal{M}_c$). $\hat{X}$, however, is not $\mathcal{M}$-valued in general but rather σ-finite measure-valued. (See [EP99] for more elaboration.) In particular, the appropriate topology for measures becomes the *vague topology* in place of the weak one.

In fact, this construction can easily be generalized for (time-independent) *local branching*, that is for the case when instead of the quadratic nonlinearity in (5) we have the more general nonlinearity of the form:

$$f(x,u) = \alpha(x)u^2(x) + \int_0^\infty [e^{-ku(x)} - 1 + ku(x)]\, n(x, \mathrm{d}k). \tag{21}$$

Here n is a kernel from $\mathbb{R}^d$ to $[0,\infty)$, that is $n(x, \mathrm{d}k)$ is a measure on $[0,\infty)$ for each $x \in \mathbb{R}^d$, and $n(\cdot, B)$ is a continuous[1] function on $\mathbb{R}^d$ for every measurable $B \subseteq [0,\infty)$ (cf. subsections 1.7-1.8 in [Dyn93]). In order to be able to define the superdiffusion $\hat{X}$ corresponding to L, β and f via an h-transform, we assume that $0 < \alpha\phi$ is bounded from above and that n satisfies

$$\sup_{x\in\mathbb{R}^d} \int_0^\infty [k \wedge \phi(x)k^2]\, n(x, \mathrm{d}k) < \infty. \tag{22}$$

Moreover, we assume that the convergence to the limit

$$\lim_{K\to+\infty} \int_K^\infty kn(x, \mathrm{d}k) = 0 \tag{23}$$

is uniform with respect to x on every compact subset of $\mathbb{R}^d$. (This condition will guarantee that the map $x \mapsto f(x, u(x))$ is continuous whenever $u \in C$.)

The h-transform with $h = \phi$ takes the operator $L+\beta$ into $L_0^\phi + \lambda_c$, while $f(x,u)$ transforms into

$$f^\phi(x,u) = \alpha\phi(x)u^2(x) + \int_0^\infty [e^{-ku(x)} - 1 + ku(x)]\, n^\phi(x, \mathrm{d}k),$$

where

$$n^\phi(x, \mathrm{d}k) := \frac{1}{\phi(x)}\, n\left(x, \frac{\mathrm{d}k}{\phi(x)}\right).$$

Note that by (22), n^ϕ satisfies

$$\sup_{x\in\mathbb{R}^d} \int_0^\infty (k \wedge k^2)\, n^\phi(x, \mathrm{d}k) < \infty \tag{24}$$

[1] In the original setting of [Dyn93] only the measurability was required. We, however, prefer to work in this paper with the spaces of continuous functions.

(and this integral converges uniformly with respect to x). Using this, along with the fact that $\alpha\phi$ is bounded from above by assumption, the ϕ-transformed mild equation uniquely defines a superdiffusion X (see subsections 1.6-1.8 [Dyn93]). Then the superdiffusion $\hat{X}$ can be defined in the usual way: $\hat{X} := \frac{1}{\phi}X$. (Note that $\hat{X}$ is $\mathcal{M}^{\phi}$-valued for every starting measure in $\mathcal{M}^{\phi}$. In particular, if ϕ is bounded away from zero then $\hat{X}$ is $\mathcal{M}$-valued for every starting measure in $\mathcal{M}^{\phi}$.) Denote the semigroup corresponding to $L_0^{\phi} + \lambda_c$ by $\{T_t^{\phi}\}_{t\geq 0}$. It is immediately seen that $\hat{X}$ corresponds to the mild equation

$$u(\cdot,t) = T_t g - \int_0^t \mathrm{d}s\ T_{t-s}\left(f(u(\cdot,s))\right), \tag{25}$$

where the linear semigroup $\{T_t\}_{t\geq 0}$ is defined by

$$T_t(u) := \phi\, T_t^{\phi}(u/\phi),\ \ t \geq 0,$$

and the nonlinearity f is defined by

$$f(x,u) := \phi(x) f^{\phi}\left(x, u/\phi\right).$$

(The h-transformed mild-equation is defined whenever the initial function at $t = 0$ belongs to C_c^{+} – see [EP99] for further explanation for the case when $n \equiv 0$.)

In fact, Theorem P and the remark preceding it are still true for this more general setup. Our proof of Theorem 1 still works for this more general setup if (in addition to (20), the boundedness of $\alpha\phi$ and the product-criticality assumption of the theorem) one requires that

$$\sup_{x\in\mathbb{R}^d} \int_0^{\infty} [\phi^{\delta}(x)k^{1+\delta} \wedge \phi^2(x)k^2]\, n(x,\mathrm{d}k) < \infty \text{ for some } \delta > 0. \tag{26}$$

This will guarantee that the Hölder-type condition (32) in Lemma 22 is satisfied for the nonlinearity f^{ϕ}. Then Lemma 22 yields the existence of a unique smooth invariant curve defined by (10) for the nonlinear equation (25).

We summarize the above in a proposition. Let us call the superdiffusion described in this section the $(L, \beta, f; \mathbb{R}^d)$ *-superdiffusion.*

Proposition 17 (scaling limit for $(L, \beta, f; \mathbb{R}^d)$ -superdiffusions) *Let X be the $(L, \beta, f; \mathbb{R}^d)$-superdiffusion with L as in the paragraph preceding Notation 1, and the nonlinearity $f(x, u)$ given by (21) where (26) is satisfied. Let $0 < \lambda_c < \infty$ where λ_c denotes the generalized principal eigenvalue of $L + \beta$ on $\mathbb{R}^d$. Assume that the condition in (ii-b) of Theorem P (product-criticality) holds. In addition, assume that $\alpha\phi$ is bounded from above.*

Then for any $X_0 = \mu \in \mathcal{M}^{(\phi)}$, there exists a nonnegative non-degenerate random variable N_μ such that

$$\lim_{t\uparrow\infty} e^{-\lambda_c t} X_t(\mathrm{d}x) = N_\mu \cdot \tilde{\phi}(x)\,\mathrm{d}x \text{ in law.} \tag{27}$$

(Here the convergence is with respect to the vague topology.)

Moreover, under the normalization $\langle \phi, \tilde{\phi} \rangle = 1$, the law of N_μ is determined via its Laplace-transform as follows:

$$\mathbf{E}e^{-\sigma N_\mu} = \exp\langle \mu, -\psi(\sigma)\rangle \; \sigma > 0 \tag{28}$$

where $\sigma \mapsto \psi(\sigma)$ is the invariant curve defined by (10) for the nonlinear equation (25).

Furthermore,

$$\mathbf{E}N_\mu = \langle \mu, \phi \rangle, \tag{29}$$

and in particular, $\mathbf{P}(N_\mu < \infty) = 1$.

If we assume in addition that ϕ is bounded away from zero, then X is $\mathcal{M}$-valued and (27) holds with respect to the weak topology.

Letting $\alpha \equiv 0$ and choosing an appropriate n (see subsection 1.8 in [Dyn93]), (21) has the form

$$f(x, u) = c(x)u^{1+p},\ 0 < p < 1,$$

with some nonnegative, nonzero continuous function c. In this case (23) and (26) will be satisfied (with $\delta = p$) if we assume that $c\phi^p$ is bounded from above. (Alternatively, one can slightly modify the proof of Theorem 1 by writing u^{1+p} in place of u^2 everywhere. Since f transforms into $c\phi^p u^{1+p}$ under an h-transform with $h = \phi$, the proof goes through when assuming the boundedness of $c\phi^p$.)

1.6 Outline

In Section 2 we will present examples for Theorem 1. In Section 3 we will state and prove two lemmas on invariant curves which play a key role in the proofs. In Section 4 some preparations are made before turning to the proofs, and we also state Theorem 3, an auxiliary result on the recurrence of diffusion processes which we will use in the proof of our main theorem and which may be of independent interest. Section 5 will be devoted to the proofs of Theorems 1 and 2 and of Proposition 17. The first appendix presents the proof of Theorem 3. Finally, our second appendix will collect some known auxiliary material on the criticality-theory of second order elliptic operators.

2 Examples

In this section we present applications of our main result for three families of superdiffusions. In the first two examples the underlying motion process (corresponding to the operator L) is recurrent, in the last example, it is transient.

Our first example has actually been discussed in Subsection 1.4. In fact, as we have seen, every superdiffusion X satisfying the conditions of Theorem 1 can be h-transformed (with $h = \phi$) into the type of superdiffusion of the following example.

Example 18 (positive recurrent motion process, $0 < \beta$=const) Let L correspond to a positive recurrent diffusion and let $0 < \beta$ =const. Finally, let α be bounded from above. Then $L + \beta - \lambda_c = L$, because $\lambda_c(L) = 0$ by the recurrence property; and $\phi = 1$. Furthermore, since the diffusion process is positive recurrent, the operator L is product-critical (that is, $\tilde{\phi} \in L^1$). Therefore, (14) holds for any finite starting measure with $\lambda_c = \beta$.

To give a concrete example for a positive recurrent diffusion, let L correspond to an Ornstein-Uhlenbeck process:

$$L = \frac{1}{2}\Delta - kx \cdot \nabla \text{ on } \mathbb{R}^d \ , \ d \geq 1,$$

where $k > 0$. (It is easy to see (cf. Example 3 in [Pin96] on p.248) that $\tilde{\phi}(x) = \left(\frac{k}{\pi}\right)^{d/2} \exp(-k|x|^2)$.) $\diamond$

The next example can be considered as a smooth version of our Theorem 2. (Recall that formally $\lambda_c = 1/2$ in that theorem.)

Example 19 (super-Brownian motion with compactly supported β) Let $L = \frac{1}{2}\Delta$ on $\mathbb{R}^d$, $d \leq 2$. Let $\beta \in C_c^+$ be not identically zero. By the recurrence of the one and two dimensional Brownian motions and Theorem 4.6.3. in [Pin96], we have $\lambda_c > 0$. The criticality of $L - \lambda_c$ follows by the recurrence of the Brownian motion and Theorem 4.6.7 in [Pin96]. We now prove that $\phi \in L^2(\mathbb{R}^d)$ (product-criticality). To see this, first let $d = 1$. Note that ϕ satisfies $(\frac{1}{2}\Delta - \lambda_c)\phi = 0$ outside a compact set and therefore $\phi(x) = \text{const} \cdot \exp(\pm\sqrt{2\lambda_c}\, x)$ for large $|x|$. By the so-called *minimal growth property at infinity* (see Theorem 7.3.8. in [Pin96]) it follows that in fact $\phi(x) = \text{const} \cdot \exp(-\sqrt{2\lambda_c}\, |x|)$ for large $|x|$. The proof for $d = 2$ is similar: using polar coordinates, it is easy to check that $f(x) := \exp(-\sqrt{2\lambda_c}\, |x|)$ satisfies $(\frac{1}{2}\Delta - \lambda_c)f \leq 0$ outside a compact set. Putting this together with the fact that ϕ satisfies $(\frac{1}{2}\Delta - \lambda_c)\phi = 0$ outside a compact set and the minimal growth property of ϕ at infinity, we have that $\phi \leq K \cdot f$ for K large enough. Therefore, for both $d = 1$ and $d = 2$, (11) holds in the present case, provided

$$\alpha(x) \leq K \cdot \exp\left(\sqrt{2\lambda_c}\,|x|\right), \; K > 0,$$

and the starting measure $\mu = X_0$ satisfies $\left\langle \mu, \exp\left(-\sqrt{2\lambda_c}\,|x|\right)\right\rangle < \infty$. $\diamond$

Last, we present an example where L corresponds to a *transient* diffusion process on $\mathbb{R}^d$.

Example 20 Let

$$L = \frac{1}{2}\Delta + kx \cdot \nabla \text{ on } \mathbb{R}^d \; d \geq 1,$$

where $k > 0$. (Note that the diffusion corresponding to L is transient.) Let β be a constant satisfying $\beta > kd$. It is easy to see (cf. Example 2 in [Pin96] on p.247 and p.266) that $\lambda_c = \beta - kd$ and that $L + \beta - \lambda_c = L + kd$ is product-critical with $\phi(x) = \exp(-k|x|^2/2)$ and $\tilde{\phi}(x) = 1$. Therefore, (11) holds with $\lambda_c = \beta - kd$, whenever the starting measure $\mu = X_0$ satisfies $\langle \mu, \exp(-k|x|^2/2)\rangle < \infty$ and

$$\alpha(x) \leq K \cdot \exp(k|x|^2/2), \; K > 0.$$

Note that if $\beta \leq kd$, the superdiffusion X_t exhibits local extinction for any $\mu \in \mathcal{M}_c$. $\diamond$

3 Two results concerning invariant curves

Let $\mathcal{X}$ be a Banach space and let $\{T_t\}_{t\geq 0}$ be a continuous semigroup of bounded linear operators acting on $\mathcal{X}$. Let $\mathcal{X}^+ \subseteq \mathcal{X}$ be a cone. Consider the equation

$$u(t) = T_t u(0) + \int_0^t T_{t-s} \circ f(u(s))\mathrm{d}s \tag{30}$$

for which we assume that it defines for any $u(0) \in \mathcal{X}^+$ its semiorbit - a curve $u(t)$, $t \geq 0$ in $\mathcal{X}^+$. We assume that $f : \mathcal{X} \to \mathcal{X}$ is smooth, i.e. it is differentiable and its derivative is bounded and uniformly continuous on bounded subsets of $\mathcal{X}$. It is easy to see in this case that the semiorbit $u(t)$ defined by (30) is continuous with respect to t and it is smooth with respect to the initial condition $u(0)$.

We will also assume that

$$f(0) = 0, \qquad f'(0) = 0 \tag{31}$$

and that for the derivative map $F(u) : \mathrm{d}u \to f'(u)\mathrm{d}u$ we have

$$\|F(u)\| \leq K\|u\|^{\delta} \tag{32}$$

(in the usual operator-norm) for some positive constants K and δ and all small u. It follows, in particular, that

$$\|f(u)\| \leq K\|u\|^{1+\delta}. \tag{33}$$

Concerning the linear semigroup T_t, we assume that it has an eigenvector ϕ:

$$T_t\phi = e^{\lambda t}\phi \tag{34}$$

for some $\lambda > 0$, and that $\phi \in \mathrm{int}\,(\,\mathcal{X}^+\,)$ (here $\mathrm{int}\,(\,\mathcal{X}^+\,)$ denotes the interior of the cone $\mathcal{X}^+$ in norm-topology) . Since the vector ϕ is defined only modulo a scalar factor, we normalize it by $\|\phi\| = 1$. We also assume that for some constant $M > 0$

$$\|T_t\| \leq Me^{(\lambda+\varepsilon)t} \tag{35}$$

where (and this is a crucial assumption)

$$\varepsilon < \lambda\delta, \tag{36}$$

and δ is the exponent in the Hölder-type estimate (32).

Definition 21 A curve $\mathcal{Q}$ in $\mathcal{X}$ is called *invariant* with respect to the system (30), if for any point $u(0)$ on $\mathcal{Q}$ its positive semiorbit $u(t)$ lies in $\mathcal{Q}$. $\diamond$

Lemma 22 (the existence of a particular invariant curve) *Under (31)–(36), there exists a unique smooth invariant curve $\mathcal{Q}$ lying in $\mathcal{X}^+$, parametrically written as $u = \psi(\sigma)$, $\sigma \in [0,\infty)$, where $\psi(0) = 0$, $\psi'(0) = \phi$ (that is, $\mathcal{Q}$ starts at zero and it is tangent at zero to the eigenvector ϕ of the linear semigroup), such that for any σ_0, for the point $u(0) = \psi(\sigma_0)$ on $\mathcal{Q}$, its semiorbit is given by:*

$$u(t) = \psi(e^{\lambda t}\sigma_0). \tag{37}$$

Remark 23 Note that for any point $u(0) = \psi(\sigma_0)$ on $\mathcal{Q}$ there exists a negative semiorbit defined (just by formula (37)) for any $t \leq 0$, such that it tends to zero and is tangent at zero to ϕ as $t \to -\infty$. $\diamond$

Remark 24 Note that we parametrize the curve $\mathcal{Q}$ in such a way that the system becomes linear on $\mathcal{Q}$: $\dot{\sigma} = \lambda\sigma$. $\diamond$

Remark 25 Although our proof is more or less standard (see [SSTC98] for a comparison), our invariant curve result itself is not a standard one because we do not require the usual spectral gap assumption (note that $\varepsilon \geq 0$ in (35)). $\diamond$

Proof of Lemma 22. It is enough to define the function ψ at small σ only and show that $\psi(\sigma)$ lies in $\mathcal{X}^+$ for small σ's: given any point $u(0) = \psi(\sigma_0)$ on the curve $\mathcal{Q}$ with an arbitrarily small σ_0 the function ψ is defined at all larger σ by formula (37), because the positive semiorbit $u(t)$ of $u(0)$ is defined at all $t \geq 0$ by assumption.

So, take any sufficiently small σ and consider the equation

$$v(t) = \sigma\phi + e^{-\lambda t} \int_{-\infty}^{t} T_{t-s} \circ f(e^{\lambda s} v(s))\, \mathrm{d}s \tag{38}$$

where $t \leq 0$. Here, the unknown is a bounded continuous function $v : [-\infty, 0] \to \mathcal{X}$. We will find it as a fixed point of the operator $v \mapsto \bar{v}$ defined by

$$\bar{v}(t) = \sigma\phi + e^{-\lambda t} \int_{-\infty}^{t} T_{t-s} \circ f(e^{\lambda s} v(s))\, \mathrm{d}s,\ \ t \in [-\infty, 0]. \tag{39}$$

Conditions (31)-(36) imply (see below) that for all sufficiently small σ it is a smooth, contracting operator which maps the set V of continuous functions $v(t)$ bounded, say, as $\|v(t)\| \leq 2|\sigma|$, into V itself. Therefore, by the Banach principle of contraction mappings, it has a uniquely defined fixed point in V, which depends on σ smoothly. Equivalently, equation (38) has a unique solution v^* for all small σ which is uniformly bounded for all $t \leq 0$:

$$\sup_{t \leq 0} \|v^*(t; \sigma)\| \leq 2|\sigma|. \tag{40}$$

Note that $v \equiv 0$ solves equation (38) at $\sigma = 0$. Hence, by uniqueness,

$$v^*(t; 0) \equiv 0. \tag{41}$$

Since $v^*(t; \sigma)$ is a fixed point of a smooth contracting operator, its derivative $\frac{\partial}{\partial\sigma} v^*$ is found as the unique solution of the equation obtained by the formal differentiation of (38):

$$\frac{\partial}{\partial\sigma} v(t) = \phi + \int_{-\infty}^{t} T_{t-s} \circ e^{\lambda s} f'(e^{\lambda s} v^*(s; \sigma)) \frac{\partial}{\partial\sigma} v(s) \mathrm{d}s. \tag{42}$$

By (41), (42) we immediately have

$$\frac{\partial}{\partial\sigma} v^*(t; 0) \equiv \phi. \tag{43}$$

We define now the function $u^*(t;\sigma) \equiv e^{\lambda t} v^*(t;\sigma)$. By uniqueness of v^*, the function u^* is defined as the unique (bounded by $2|\sigma|e^{\lambda t}$) solution of

$$u(t) = \sigma e^{\lambda t}\phi + \int_{-\infty}^{t} T_{t-s} \circ f(u(s))\mathrm{d}s. \tag{44}$$

(compare this with (38)). Recall that we define the function v^* at non-positive t only, so the function u^* is, by now, defined only at $t \le 0$ as well. We define $u^*(t;\sigma)$ at $t \ge 0$ as the positive semiorbit of the point $u^*(0;\sigma)$ defined by the system (30). Comparing formulas (30) and (44) shows that the function u^* satisfies (44) at all t (we take into account that $T_t\phi = e^{\lambda t}\phi$ by assumption).

Now take any $\tau > 0$ and consider the function $u^{**}(t;\sigma) = u^*(t+\tau; e^{-\lambda\tau}\sigma)$. It is immediately seen that once u^* satisfies (44), the function u^{**} satisfies (44) as well. Therefore, by uniqueness, $u^{**} \equiv u^*$ at all non-positive t and, in particular,

$$u^*(0,\sigma) \equiv u^*(\tau; e^{-\lambda\tau}\sigma) \tag{45}$$

for any $\tau \ge 0$. By definition, this means that the time τ shift (by the semiflow defined by (30)) of the point $u^*(0; e^{-\lambda\tau}\sigma)$ is the point $u^*(0,\sigma)$. Thus, if we define the sought function ψ as $\psi(\sigma) = u^*(0,\sigma)(\equiv v^*(0,\sigma)$), we will have that the smooth curve $u = \psi(\sigma)$ is invariant with respect to system (30) and satisfies (37).

Note also that $\psi(0) = 0$ and $\psi'(0) = \phi$, according to (41), (43). Thus, this invariant curve will indeed be tangent at zero to the eigenvector ϕ. Since $\phi \in \mathrm{int}\,(\mathcal{X}^+)$ by assumption, it also follows that $\psi(\sigma)$ lies in $\mathcal{X}^+$ for all small σ's.

To show the uniqueness of the curve $\mathcal{Q} : u = \psi(\sigma)$ satisfying (37) and $\psi'(0) = \phi$, note that if we take any point $u(0)$ on $\mathcal{Q}$ and consider its negative semiorbit $u(t)_{t\le 0}$ defined by (37), then $u(t)$ must satisfy equation (44) whose solution is unique as we just have shown (the required boundedness of $u(t)$ by $2\sigma e^{\lambda}t$ follows from (37) due to the assumed boundedness of $\psi'(0)$).

To complete the proof it remains to check that the operator (39) is smooth and contracting on the set $V : \{v(s)_{s\in(-\infty,0]}, \|v(s)\| \le 2|\sigma|\}$ and maps this set into itself. First, note that in (39)

$$\|\bar{v}(t)\| \le |\sigma| + \int_{-\infty}^{t} \|T_{t-s}\| \cdot e^{-\lambda t}\|f(e^{\lambda s}v(s))\|\mathrm{d}s$$

(recall that $\|\phi\| = 1$) and, by virtue of (33) and (35),

$$\|\bar{v}(t)\| \le |\sigma| + MKe^{\varepsilon t}\int_{-\infty}^{t} e^{(\lambda\delta-\varepsilon)s}\|v(s)\|^{1+\delta}\mathrm{d}s.$$

Hence,

$$\sup_{t\le 0}\|\bar{v}(t)\| \le |\sigma| + \frac{MK}{\lambda\delta - \varepsilon}\left(\sup_{s\le 0}\|v(s)\|\right)^{1+\delta}$$

(recall that $\varepsilon < \lambda\delta$ by assumption). It is clear from this estimate that for all σ small enough, if $\sup_{s\leq 0}\|v(s)\| \leq 2|\sigma|$, then $\|\bar{v}(t)\| \leq 2|\sigma|$ at all $t \leq 0$, which means that the operator under consideration indeed maps the set V into itself.

The smoothness of this operator with respect to σ is obvious. To prove the smoothness with respect to v we must check that the linear operator

$$\Delta v(t) \mapsto \int_{-\infty}^{t} T_{t-s} e^{-\lambda(t-s)} f'(e^{\lambda s} v(s)) \cdot \Delta v(s) \mathrm{d}s \tag{46}$$

obtained by formal differentiation of (39) is well defined and bounded on the space of uniformly bounded $\Delta v(s)_{s\in(-\infty,0]}$, provided $v(s) \in V$. This is straightforward. In fact, by (35) and (32), we obtain that

$$\begin{aligned} &\left\| \int_{-\infty}^{t} T_{t-s} e^{-\lambda(t-s)} f'(e^{\lambda s} v(s)) \cdot \Delta v(s) \mathrm{d}s \right\| \\ &\quad \leq M \int_{-\infty}^{t} e^{\varepsilon(t-s)} \cdot K(2|\sigma|)^{\delta} e^{\lambda\delta s} \cdot \|\Delta v(s)\| \mathrm{d}s \\ &\quad \leq \frac{MK}{\lambda\delta - \varepsilon} (2|\sigma|)^{\delta} \sup_{s\in(-\infty,0]} \|\Delta v(s)\|, \end{aligned}$$

and we see that formula (46) for the derivative of (39) defines a bounded linear operator indeed (one may also check in the same way that the higher order derivatives of (39) are bounded multi-linear operators). Moreover, the norm of this operator is small (less than 1) for small σ, giving the required contraction. $\square$

The following result is a version of the well-known λ-*lemma* (see [SSTC98]) from the theory of finite-dimensional dynamical systems. The advantage of our result is that we do not assume the spectral gap condition.

Lemma 26 (the existence of the scaling limit) *Let for some initial condition u_0 the following limit relation hold*

$$\lim_{t\uparrow\infty} e^{-\lambda t} T_t u_0 = \sigma\phi. \tag{47}$$

Then there exists the limit

$$\lim_{t\uparrow\infty} u(t; e^{-\lambda t} u_0) = \psi(\sigma) \tag{48}$$

where $u(t;\xi)$ denotes the solution of (30) starting with the initial condition $u(0) = \xi$ and $\sigma \mapsto \psi(\sigma)$ is the equation of the invariant curve $\mathcal{Q}$ constructed in Lemma 22.

Proof of Lemma 26. By continuity of the nonlinear semigroup defined by (30), it is enough to prove that for some small $\rho > 0$

$$\lim_{t\uparrow\infty} u(t; \rho e^{-\lambda t} u_0) = \psi(\sigma\rho), \tag{49}$$

because if we denote $\theta = -\frac{1}{\lambda}\ln\rho > 0$, then $u(t+\theta; e^{-\lambda(t+\theta)}u_0)$ is the time θ shift of $u(t;\rho e^{-\lambda t}u_0)$ and $\psi(\sigma)$ is the time θ shift of $\psi(\sigma\rho)$ (see (37)).

Denote

$$v(t) = e^{-\lambda t} u(t+\tau; e^{-\lambda\tau}\rho u_0),\ t \in [-\tau, 0].$$

By (30)

$$v(t) = \rho e^{-\lambda(t+\tau)} T_{t+\tau} u_0 + \int_{-\tau}^{t} e^{-\lambda t} T_{t-s} \circ f(e^{\lambda s} v(s))\,\mathrm{d}s. \tag{50}$$

Let $v^*(t;\rho\sigma)$ be the solution of (38), i.e.

$$v^*(t) = \rho\sigma\phi + \int_{-\infty}^{t} e^{-\lambda t} T_{t-s} \circ f(e^{\lambda s} v^*(s))\,\mathrm{d}s \tag{51}$$

We will prove that

$$v(t) - v^*(t) \to 0 \tag{52}$$

as $\tau \to +\infty$, for any fixed $t \le 0$. Then putting $t = 0$ in (52) will give (49) and finish the proof of the lemma. In fact, we will prove that

$$\sup_{t\in[-\tau',0]} \|v(t)\| \to 0, \tag{53}$$

for an appropriately chosen τ' which tends to $+\infty$ as $\tau \to +\infty$.

First, note that it follows from the existence of the finite limit (47) that $e^{-\lambda s} T_s u_0$ is uniformly bounded for all $s \ge 0$:

$$\sup_{s\ge 0} \|e^{-\lambda s} T_s u_0\| \le L \tag{54}$$

for some finite L. It is now easy to show that

$$\|v(t)\| \le 2L\rho \tag{55}$$

for all $\tau \ge 0$ and $t \in [-\tau, 0]$, provided ρ is small enough. Indeed, this holds true at $t = -\tau$ for any τ, and let $t_0 \le 0$ be the maximal value of t for which (55) is still valid. If $t_0 < 0$, this means that $\|v(t_0)\| = 2L\rho$. Now, by (54), using estimates (35) and (33), we have from (50)

$$\|v(t_0)\| \leq L\rho + MK(2L\rho)^{1+\delta} e^{\varepsilon t_0} \int_{-\tau}^{t_0} e^{(\lambda\delta-\varepsilon)s} \mathrm{d}s \leq L\rho(1 + \frac{2MK}{\lambda\delta - \varepsilon}(2L\rho)^{\delta}).$$

If ρ was taken small enough, we get that $\|v(t_0)\|$ is strictly less than $2L\rho$, hence $t_0 = 0$ which proves the claim.

Now take any $\tau' < \tau$ such that $\tau' \to +\infty$ as $\tau \to +\infty$. We have

$$\left\| \int_{-\tau}^{-\tau'} e^{-\lambda t} T_{t-s} \circ f(e^{\lambda s} v(s)) \mathrm{d}s \right\| \leq MK \left(\sup_{s \leq 0} \|v(s)\| \right)^{1+\delta} e^{\varepsilon t} \int_{-\tau}^{-\tau'} e^{(\lambda\delta-\varepsilon)s} \mathrm{d}s. \tag{56}$$

By (36),(55), this integral tends to zero as $\tau' \to +\infty$, uniformly for any $t \leq 0$. The same conclusion can be made with respect to the integral

$$\int_{-\tau}^{-\tau'} e^{-\lambda t} T_{t-s} \circ f(e^{\lambda s} v^*(s)) \mathrm{d}s :$$

the estimate like (56) follows from (35) and (33), and the uniform boundedness of v^* was proven in Lemma 1 (see (40); note that the upper bound on the norm on v^* is also linear in ρ in present notations, i.e. v^* also satisfies (55) with an appropriately chosen L).

Hence, for any $t \in [-\tau', 0]$ we have from (50), (51) (we use estimates (32), (55) and (35)):

$$\begin{aligned} \|v(t) - v^*(t)\| \quad &\leq \xi(\tau') + MK(2L\rho)^{\delta} \left(\int_{-\tau'}^{0} e^{(\lambda\delta-\varepsilon)s} \mathrm{d}s \right) \\ &\cdot \sup_{s \in [-\tau', 0]} \|v(s) - v^*(s)\| + o(1)_{\tau' \to +\infty} \end{aligned} \tag{57}$$

where

$$\xi(\tau') = \rho \sup_{s \in [\tau - \tau', \tau]} \|e^{-\lambda s} T_s u_0 - \sigma\phi\|.$$

Since $\xi(\tau') \to 0$ as $\tau - \tau' \to +\infty$ (see (47)), it immediately follows from (57) that at sufficiently small ρ the sought relation (53) holds, provided τ' is chosen such that $\tau' \to +\infty$, $\tau - \tau' \to +\infty$. $\square$

Note that we never used in the proof of Lemma 26 (unlike in the proof of Lemma 22) the completeness of the space $\mathcal{X}$. Therefore, we may change Lemma 26 (in order to adopt it to the particular problem we consider in this paper) as follows.

Lemma 27 (the scaling limit in a weaker norm) *For any norm $\|\cdot\|_1$ which is weaker than the original norm $\|\cdot\|_0$ in $\mathcal{X}$, if the (linear) limit relation (47) holds in the norm $\|\cdot\|_1$ for some initial condition u_0, then the (nonlinear) limit relation (48) holds in the same norm, provided the following estimates are valid:*

$$\|F(u)\|_0 \le K\|u\|_0^\delta, \tag{58}$$

$$\|F(u)\|_1 \le K\|u\|_0^\delta, \tag{59}$$

$$\|T_t\|_0 \le Me^{\lambda t}, \tag{60}$$

$$\|T_t\|_1 \le Me^{(\lambda+\varepsilon)t} \tag{61}$$

with $\varepsilon < \lambda\delta$, where $F(u)$ is the derivative operator from (32).

Proof. The proof repeats the proof of Lemma 26 with the following modification: the estimate (55) (in the original $\|\cdot\|_0$-norm) follows now directly from (60). Then, it follows from (55), (59) and (61) that all the estimates of Lemma 26 remain unchanged in the norm $\|\cdot\|_1$. Finally, the required existence and uniform boundedness (in the original norm $\|\cdot\|_0$ and, hence, in the weaker norm $\|\cdot\|_1$) of the solution v^* of the integral equation (38) are given by Lemma 22. $\square$

4 Some preliminary results for the proof of the main theorem

The proof of Theorem 1 and Proposition 17 will be based on two propositions (see Propositions 29 and 31 below) and on two lemmas stated and proved in Section 3 (Lemmas 22 and 27). We will also use the following simple fact.

Lemma 28 *For any $0 < \gamma : \mathbb{R}^d \to \mathbb{R}$ continuous define the γ-*norm *by*

$$\|f\|_\gamma := \|\gamma f\|,$$

on $\{f$ continuous $: \gamma f$ is bounded$\}$. If $\gamma \in C_0$ and if $\mathcal{F}$ is a uniformly bounded family of functions, then the norm $\|\cdot\|_\gamma$ restricted to $\mathcal{F}$ is compatible with the topology of uniform convergence on compacts.

Proof. First, assume that f_n tends to zero uniformly on compacts as $n \uparrow \infty$. Since $\gamma \in C_0$ and by assumption $\|f_n\| \le K,\ n \ge 1$ for some $K > 0$, one can take a large ball $B \subset \mathbb{R}^d$ (depending on ε) such that

$$\sup_{x \in \mathbb{R}^d \setminus B} \gamma(x) f_n(x) < \varepsilon,\ n \ge 1.$$

Since γf_n also tends to zero uniformly on compacts as $n \uparrow \infty$, we can pick an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\sup_{x \in B} \gamma(x) f_n(x) < \varepsilon,\ n > N.$$

Then, altogether we have

$$\sup_{x \in \mathbb{R}^d} \gamma(x) f_n(x) < \varepsilon,\ n > N,$$

proving the γ-norm convergence for f_n.

Conversely, assume that f_n tends to zero in γ-norm and fix an arbitrary nonempty ball $B \subset \mathbb{R}^d$. We have

$$\sup_{x \in B} f_n(x) \le C(\gamma, B) \sup_{x \in B} \gamma(x) f_n(x)$$

with some $C(\gamma, B) > 0$. The righthand side of the last formula tends to zero as $n \uparrow \infty$ by assumption, thus the same is true for the lefthand side. This proves uniform convergence on compacts for f_n. $\square$

Let $\{S_t\}_{t>0}$ denote the semigroup corresponding to the operator $L + \beta - \lambda_c$ on $\mathbb{R}^d$ (and acting on C_b). Note that

$$S_t = e^{-\lambda_c t} T_t,$$

where $\{T_t\}_{t>0}$ is the semigroup defined in (6).

Proposition 29 (convergence for $S_t^\phi g$ in γ-norm) *Assume that the condition in (ii-b) of Theorem P is satisfied, and furthermore let* $0 < \gamma \in C_0$. *Then for any* $g \in C_b$,

$$\lim_{t \uparrow \infty} S_t^\phi g = \langle g, \phi\tilde{\phi} \rangle \text{ in } \|\cdot\|_\gamma.$$

Proof. Since $L + \beta - \lambda_c$ is critical on $\mathbb{R}^d$, so is the h-transformed ($h = \phi$) operator $(L + \beta - \lambda_c)^\phi$. Let $0 < \chi$ and $\tilde{\chi}$ denote the eigenfunctions corresponding to the latter operator and to its adjoint respectively. It is easy to see that $\chi = 1$ and $\tilde{\chi} = \phi\tilde{\phi}$. In particular $\langle \chi, \tilde{\chi} \rangle = \langle \phi, \tilde{\phi} \rangle$. Note that the ϕ-transformed operator

$$(L + \beta - \lambda_c)^\phi = L + a\frac{\nabla \phi}{\phi} \cdot \nabla$$

has no zeroth order part (it is a diffusion generator). Using this along with the second part of [Pin95, Theorem 4.4.9], we have that for any $g \in C_b$ given,

$$\lim_{t\uparrow\infty} S_t^\phi g = \langle g, \phi\tilde{\phi}\rangle,$$

in the topology of uniform convergence on compacts. Our goal is to verify that this convergence holds also in $\|\cdot\|_\gamma$. Using Lemma 28, it is enough to show that for any $g \in C_b$ given,

$$\mathcal{F} := \{(S_t^\phi)g\}_{t\geq 0}$$

is a uniformly bounded family of functions. Recalling, that the ϕ-transformed operator has no zeroth order part and denoting the corresponding expectations by $\{\mathbb{E}_x^\phi\}_{x\in\mathbb{R}^d}$ we have

$$(S_t^\phi g)(x) = \mathbb{E}_x^\phi g(Y_t)$$

where Y_t is the corresponding diffusion process. It then follows that

$$\|S_t^\phi(g)\| \leq \|g\|.$$

This completes the proof of the proposition. $\square$

We now choose a particular function γ in the following way:
Let h be a positive function satisfying

1) $(L+\beta-\lambda_c)^\phi h \leq 0$ outside some compact set,
2) $h(x) \to \infty$ as $|x| \to \infty$.

The existence of such an h follows by the recurrence of the diffusion corresponding to the operator $(L+\beta-\lambda_c)^\phi$ and from the following theorem which we feel is of independent interest. (For the proof see Appendix A.1)

Theorem 3 (necessary condition for recurrence) *Let L be as in (1), and assume that it corresponds to a recurrent diffusion process Y. Given any positive R_1 and any function $p(x)$ which tends to infinity as $|x| \to +\infty$, there exists a supersolution on $|x| \geq R_1$, that is, a positive $C^{2,\eta}$-function $U(x)$ such that*

$$LU \leq 0 \text{ on } |x| \geq R_1, \tag{62}$$

converging to infinity as $|x| \to +\infty$, asymptotically slower than p:

$$\lim_{r\to+\infty} \inf_{|x|=r} U(x) = \infty, \quad \lim_{r\to+\infty} \sup_{|x|=r} \frac{U(x)}{p(x)} = 0.$$

The existence of such growing to infinity supersolution is known as a sufficient condition for the recurrence of L (see Theorem 6.1.2 in [Pin95]). Our result here shows that this is also a necessary condition for recurrence, (earlier it was known only in the one-dimensional case – then the statement follows easily from Theorem 5.1.1(i) in [Pin95]).

Remark 30 By the previous theorem, h can be chosen *arbitrarily slowly growing.* This fact will be used later, in the proof of Theorem 1. $\diamond$

Using the above h, we define γ as follows. Let

$$\gamma := 1/\hat{h}, \quad \text{where} \quad \hat{h} = h + K \tag{63}$$

and K is a positive constant to be fixed later. Then, obviously, $0 < \gamma \in C_0$.

Proposition 31 (estimate for S_t^ϕ in γ-norm) *Assume that $L + \beta - \lambda_c$ is critical with the ground state ϕ and let $\{S_t\}_{t>0}$ be as in Proposition 29. For any $\varepsilon > 0$*

$$\| \, S_t^\phi \|_\gamma \leq e^{\varepsilon t}, \; t > 0, \tag{64}$$

if $K = K_\varepsilon$ is large enough (K is defined in (63)).

Proof. By a simple computation, the statement is equivalent to

$$\| \, S_t^{\phi \hat{h}} \| \leq e^{\varepsilon t}, \; t > 0. \tag{65}$$

Recall that $(L + \beta - \lambda_c)^\phi$ has no zeroth order part. Since the zeroth order term of $(L + \beta - \lambda_c)^{\phi \hat{h}}$ is

$$\frac{1}{\hat{h}}(L + \beta - \lambda_c)^\phi \hat{h} = \frac{1}{\hat{h}}(L + \beta - \lambda_c)^\phi h =: V,$$

we have that

$$V \leq 0$$

outside a compact set by the first assumption on h. Also, if K is large enough, we can obviously guarantee that

$$V \leq \varepsilon \text{ on } \mathbb{R}^d.$$

The estimate under (65) now follows from this and (6) with $g = 1$ and β replaced by V (but now with $\mathbb{E}$ corresponding to $L_0^{\phi \hat{h}}$). $\square$

5 Proof of Theorems 1 and 2 and Proposition 17

Proof of Theorem 1. The strategy of the proof is as follows. We will show that the scaling limit exists in law for X^ϕ. More precisely, we will prove that, for $\mu \in \mathcal{M}^{1/\gamma}$ with γ given by (63),

$$\lim_{t \uparrow \infty} E_\mu^\phi \exp \left\langle e^{-\lambda_c t} X_t^\phi, -g \right\rangle = E \exp \langle Z_\mu, -g \rangle \, , \; g \in C_b^+, \tag{66}$$

with some random measure Z_μ having the form $Z_\mu = N_\mu^\phi \cdot \phi\tilde{\phi}\, \mathrm{d}x$, where the random variable N_μ^ϕ is determined by (12) (or by (28) for a general nonlinearity) and enjoys the properties stated in the theorem (note that in (12) or (28) the curve $\sigma \mapsto \psi(\sigma)$ is now replaced by a new curve corresponding to the ϕ-transformed dynamics, that is, to T_t^ϕ and α^ϕ or f^ϕ.) Having shown this, it will follow from the definition of the h-transform that (11) holds for X starting with the measure $\nu := \frac{1}{\phi} \cdot \mu$ (a simple computation shows that (12) holds for the original curve $\sigma \mapsto \psi(\sigma)$ when going back to X). That is, when $\phi(h+K)\cdot\nu$ (where h, K are from (63)) is a finite measure. Putting this together with the fact that h can be chosen arbitrarily slowly growing by Theorem 3, we will have that (11) holds true whenever $\nu \in \mathcal{M}^\phi$. It will also follow that (66) is satisfied for X in place of X^ϕ and C_b^+ replaced by the class of all continuous g's with $g \leq \mathrm{const} \cdot \phi\hat{h} = \mathrm{const} \cdot \phi(h+K)$. In particular, (66) will hold for X^ϕ replaced by X, provided that ϕ is bounded away from zero (recall that $h(x) \to \infty$ as $|x| \to \infty$). This will prove (14).

Now we are going to show (66). To do this, let us summarize what we already know about the nonlinear semigroup corresponding to X^ϕ. First, concerning the linear part of the semigroup, T_t^ϕ, we know that the rescaled semigroup S_t^ϕ corresponding to $(L+\beta-\lambda_c)^\phi$ has the following properties:

a) $(L+\beta-\lambda_c)^\phi$ is a diffusion generator, i.e. $(\beta-\lambda_c)^\phi = 0$, and the ground state ϕ transforms into 1.

b) By Proposition 29, for any $g \in C_b$, $S_t^\phi g$ has the limit $\langle g, \phi\tilde{\phi}\rangle$ in $\|\cdot\|_\gamma$.

c) By Proposition 31, S_t^ϕ satisfies the exponential estimate under (64). Also, $\|S_t^\phi\| \leq 1$ since $\{S_t^\phi\}_{t\geq 0}$ is a diffusion-semigroup (see the end of the proof of Proposition 29).

In addition to the linear part of the semigroup, we have to control the nonlinear term

$$f^\phi(u) = \alpha^\phi u^2.$$

Here $\alpha^\phi = \alpha\phi$. Thus, for the derivative map

$$F(u) : \mathrm{d}u \mapsto 2\alpha\phi u \cdot \mathrm{d}u,$$

we have (recall that $\|\cdot\|$ denotes the supremum norm):

$$\|2\alpha\phi u \cdot \mathrm{d}u\|_\gamma \leq \|2\alpha\phi u\| \cdot \|\mathrm{d}u\|_\gamma.$$

That is,

$$\|F(u)\|_\gamma \leq \|2\alpha\phi u\| \leq 2\|\alpha\phi\| \cdot \|u\|.$$

By the same computation, also

$$\|F(u)\| \leq 2\|\alpha\phi\| \cdot \|u\|.$$

Altogether, working with the nonlinear dynamics corresponding to X^ϕ and with $\|\cdot\|_\gamma$, we are in the position to implement the invariant curve method of Section 3. More precisely, we are going to apply Lemma 27 with $\mathcal{X} = C_b$, $\mathcal{X}^+ = C_b^+$, $\|\cdot\|_0 = \|\cdot\|$ and $\|\cdot\|_1 = \|\cdot\|_\gamma$; where furthermore T_t and ϕ are replaced by T_t^ϕ and the function 1. (Clearly, $1 \in \mathrm{int} C_b^+$ in sup-norm topology.) Let $0 \le u(t, g, \cdot)$ denote the solution of (5) or (25) (but L, β and f replaced with $(L + \beta - \lambda_c)^\phi, (\beta - \lambda_c)^\phi = 0$ and f^ϕ, respectively) with $u(0, \cdot) = g(\cdot)$. Let furthermore $\sigma \mapsto \psi(\sigma)$ be the invariant curve constructed in Section 3. Working with $\|\cdot\|_\gamma$ and using the discussion at the beginning of this paragraph along with Lemma 27 of Section 3, (3) and (5) or (25) applied to the ϕ-transformed setting yields

$$E_\mu^\phi \exp\langle e^{-\lambda_c t} X_t^\phi, -g\rangle = \exp\langle \mu, -u(t, e^{-\lambda_c t} g)\rangle =$$

$$= \exp\langle \mu/\gamma, -\gamma u(t, e^{-\lambda_c t} g)\rangle \longrightarrow \exp\langle \mu/\gamma, -\gamma\psi(\langle g, \phi\tilde{\phi}\rangle)\rangle \quad \text{as } t \to \infty,$$

provided $\mu \in \mathcal{M}^{1/\gamma}$, $\gamma g \in C_b^+$ (and in particular for $g \in C_b^+$). That is,

$$E_\mu^\phi \exp\langle e^{-\lambda_c t} X_t^\phi, -g\rangle \longrightarrow \exp\langle \mu, -\psi(\langle g, \phi\tilde{\phi}\rangle)\rangle \quad \text{as } t \to \infty.$$

Let us fix now a $\mu \in \mathcal{M}^{1/\gamma}$. Note, that the functional

$$\Psi_\mu(g) := \exp\langle \mu, -\psi(\langle g, \phi\tilde{\phi}\rangle)\rangle$$

defined on C_b^+ is positive definite (for the definition of positive definiteness see e.g. the proof of Theorem A in [EP99]), because it is the pointwise limit of functionals possessing this property. Moreover, Ψ_μ is continuous with respect to bounded pointwise convergence, since $\phi\tilde{\phi}\,\mathrm{d}x \in \mathcal{M}$ by assumption. Also, $\Psi_\mu(0) = 1$, because $\psi(0) = 0$. It follows from these properties by a standard result (see the proof of Theorem A1 in [EP99]; see also Lemma 3.1 in [Dyn91]), that Ψ_μ is the Laplace functional of a random measure, that is, there exists a random measure Z_μ such that

$$E e^{\langle Z_\mu, -g\rangle} = \exp\langle \mu, -\psi(\langle g, \phi\tilde{\phi}\rangle)\rangle, \tag{67}$$

for $g \in C_b^+$. Therefore, altogether,

$$E_\mu \exp\langle e^{-\lambda_c t} X_t^\phi, -g\rangle \longrightarrow E e^{\langle Z_\mu, -g\rangle}, \text{ as } t \to \infty,$$

whenever $g \in C_b^+$. That is, $e^{-\lambda_c t} X_t^\phi$ converges to Z_μ in law.

In order to identify Z_μ, note that if N_μ^ϕ is a nonnegative random variable satisfying (12) (the Laplace transform in (12) defines uniquely N_μ^ϕ — again, because of the positive definiteness and continuity of $s \mapsto \exp\langle \mu, \psi(s)\rangle)$, then the random variable

$$Z_\mu^* := N_\mu^\phi \cdot \phi\tilde{\phi}(x)\,\mathrm{d}x$$

clearly satisfies (67) and thus by uniqueness $Z_\mu = Z_\mu^*$.

Using the fact that $\psi'(0) = \phi$, it follows (13). (To do this rigorously, recall that $\psi'(0) = \phi$ means that $\lim_{s\to 0}\psi(s)/s = \phi$ in $\|\cdot\|_\gamma$. Since $\mu \in \mathcal{M}^{1/\gamma}$, we can use uniform convergence to conclude (13).)

Finally, we show that N_μ^ϕ is non-degenerate. Suppose to the contrary that $N_\mu^\phi = \mathbf{E}N_\mu^\phi = \langle \mu, \phi\rangle$ with P_μ-probability one. By (12) this would imply that $\psi(s) = s\phi$ for $s > 0$. But this is impossible because ψ is invariant with respect to the *nonlinear* system (75). Consequently N_μ^ϕ is indeed non-degenerate. This completes the proof of Theorem 1. $\square$

Proof of Proposition 17. The proof is the same as the proof of Theorem 1 except the following. For the general nonlinearity (21) we have

$$f^\phi(u) = \alpha^\phi u^2 + \int_0^\infty (e^{-ku} - 1 + ku)\, n^\phi(x, \mathrm{d}k)$$

where $n^\phi(x, \mathrm{d}k) = \phi(x)^{-1} n(x, \phi(x)^{-1}\mathrm{d}k)$. The derivative map is

$$F(u) : \mathrm{d}u \mapsto \left[2\alpha\phi u + \int_0^\infty k(1 - e^{-ku})\, n^\phi(x, \mathrm{d}k)\right]\, \mathrm{d}u.$$

Here, we have

$$\begin{aligned} &\left\| \int_0^\infty k(1 - e^{-ku})\, n^\phi(x, \mathrm{d}k)\cdot \mathrm{d}u \right\|_\gamma \\ \le \quad & \sup_{x\in\mathbb{R}^d} \int_0^\infty \left[u(x)\phi(x)k^2 \wedge u^\delta(x)\phi^\delta(x)k^{1+\delta}\right]\, n(x, \mathrm{d}k)\cdot \|\mathrm{d}u\|_\gamma. \end{aligned}$$

By (26),

$$\|F(u)\|_\gamma = O(\|u\| + \|u\|^\delta),$$

and, analogously,

$$\|F(u)\| = O(\|u\| + \|u\|^\delta).$$

These estimates are enough to obtain the results of Section 3, so the rest of the proof for the general nonlinearity goes exactly the same way as in the case $f(u) = \alpha u^2$. $\square$

Proof of Theorem 2. The proof of Theorem 2 will be very similar to that of Theorem 1. We will use the results of Section 3 exactly in the same way as in the case of

Theorem 1, but we have to replace the 'linear result' with an analogous result for the singular setting and moreover to replace the pde setting of Propositions 29 and 31 by using the integral equation (8). Fix a bounded continuous g, and set

$$u(x,t) := E^{\sin}_{\delta_x} \langle X_t \,, g \rangle \,, \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{68}$$

Using the equation (8), it is standard to verify the following integral equation for the expectations (see formula 91 in [EF99]):

$$u(x,t) = \int_{\mathbb{R}} \mathrm{d}y \; p(t, y-x) g(y) + \int_0^t \mathrm{d}s \; p(t-s, x) u(0,s), \tag{69}$$

$x \in \mathbb{R}, \ t \geq 0$. (Symbolically, $u_t = \frac{1}{2}\Delta u + \delta_0 u$ with $u(x,0) = g$.) Analogously to the section preceding (7), let us define now the semigroup $\{T_t\}_{t \geq 0}$ by

$$(T_t g)(\cdot) := u(\cdot, t).$$

(The semigroup property can be checked by direct calculation.) By Theorem EF then, we know that $e^{-t/2} T_t g$ has a pointwise limit as $t \to \infty$ for any bounded continuous $g : \mathbb{R} \mapsto \mathbb{R}_+$.

Let $\phi(x) := e^{-|x|}$ (recall that the function $x \mapsto e^{-|x|}$ plays the role of the ground state, this justifies our notation.) Define the ϕ-transformed semigroup by

$$T_t^{\phi}(g) := e^{|x|} T_t(e^{-|x|} g), \text{ for } e^{-|x|} g \in C_b^+.$$

Define also $S_t^{\phi}(g) := e^{-t/2} T_t^{\phi}(g)$. Let $\mu = \delta_x$ and rewrite (9):

$$\lim_{t \uparrow \infty} (S_t g)(x) = e^{-|x|} \langle e^{-|x|}, g \rangle, \ g \in C_b^+.$$

Let $G := e^{|x|} g$. Then

$$\lim_{t \uparrow \infty} (S_t^{\phi} G)(x) = \langle e^{-2|x|}, G \rangle. \tag{70}$$

Now (70) holds for every G satisfying $e^{-|x|} G \in C_b^+$. In particular, (70) holds for every $G \in C_b^+$. We now show that this convergence is uniform on compacts. Let us fix a $K \subset \mathbb{R}$ compact. We must show that for $g \in C_b^+$,

$$e^{-t/2} e^{|x|} u(x,t) \to C(g) \text{ as } t \uparrow \infty \tag{71}$$

uniformly for $x \in K$, where $C(g) := \langle e^{-|x|}, g \rangle$. Exploiting the notations $u_x(t) := u(x,t)$ and $p_x(t) := p(t,x)$, the Laplace-transform of (69) (with respect to t) is

$$\widehat{u_x}(\lambda) = \int_{\mathbb{R}} \mathrm{d}y \, \widehat{p_{y-x}}(\lambda) g(y) + \widehat{p_x}(\lambda) \widehat{u_0}(\lambda) \tag{72}$$

where $\widehat{u_x}$ and $\widehat{p_x}$ denote the Laplace-transforms of u_x and p_x respectively. Using (72), the Laplace-transform of the lefthand side of (71) is

$$e^{|x|}\widehat{u_x}\left(\lambda+\frac{1}{2}\right) = e^{|x|}\int_{\mathbb{R}} \mathrm{d}y\, \widehat{p_{y-x}}\left(\lambda+\frac{1}{2}\right) g(y) + e^{|x|}\widehat{p_x}\left(\lambda+\frac{1}{2}\right)\widehat{u_0}\left(\lambda+\frac{1}{2}\right)$$
$$=: M(x,\lambda) + N(x,\lambda)\cdot \widehat{u_0}\left(\lambda+\frac{1}{2}\right).$$

By continuity, M is bounded on $K\times[0,\varepsilon]$. Let

$$a := \inf_{\substack{x\in K\\ \lambda<\varepsilon}} M(x,\lambda) \text{ and } A := \sup_{\substack{x\in K\\ \lambda<\varepsilon}} M(x,\lambda).$$

In the proof of Theorem 4(b) in [EF99] we have shown that

$$\widehat{u_0}\left(\lambda+\frac{1}{2}\right) \sim C(g)\frac{1}{\lambda} \text{ as } \lambda\downarrow 0, \tag{73}$$

and that

$$N(x,\lambda)\to 1 \text{ as } \lambda\downarrow 0, \tag{74}$$

for each $x\in\mathbb{R}$. We now show that in fact the convergence in (74) is uniform on K. To see this, note that $N(x,\lambda)$ is continuous in x by monotone convergence. The uniformity of the limit in (74) thus follows by Dini's theorem. Let

$$b(\lambda) := \inf_{x\in K} N(x,\lambda) \text{ and } B(\lambda) := \sup_{x\in K} N(x,\lambda).$$

Then we have

$$a + b(\lambda)\cdot\widehat{u_0}\left(\lambda+\frac{1}{2}\right) \le e^{|x|}\widehat{u_x}\left(\lambda+\frac{1}{2}\right) \le A + B(\lambda)\cdot\widehat{u_0}\left(\lambda+\frac{1}{2}\right),$$

with

$$\lim_{\lambda\downarrow 0} b(\lambda) = \lim_{\lambda\downarrow 0} B(\lambda) = 1.$$

Using this, (73) and a well known Tauberian theorem ([Fel71, formula (13.5.22)]) along with the monotonicity of the Laplace-transform, it follows that (71) holds uniformly on K.

Similarly to the proof of Theorem 1, in order to conclude convergence in γ-norm, we have to show that $\{S_t^\phi G,\ t\ge 0\}$ is a uniformly bounded family, for every given $G\in C_b^+$. Let $G\in C_b^+$ with $\|G\|=K$. Since $\langle e^{-2|x|},1\rangle = 1$, we have

$$\lim_{t\uparrow\infty}(S_t^\phi G)(x) \le K.$$

Consequently,

$$\|S_t^\phi G\| \leq K^* \text{ for all } t \geq 0,$$

with some $K^* > K$, that is, $\{S_t^\phi G,\ t \geq 0\}$ is a uniformly bounded family, for every given $G \in C_b^+$. Thus, we have shown the convergence in γ-norm for any $\gamma \in C_0$.

Now choose

$$\gamma := \phi = e^{-|x|}.$$

We look for a substitute of Proposition 31 for the non-regular setting. By Theorem 4(a) in [EF99] we have that

$$\lim_{t\uparrow\infty} e^{-t/2}\|T_t\| = 2.$$

A simple calculation reveals that

$$\|T_t^\phi\|_\phi = \|T_t\|.$$

Therefore, also

$$\lim_{t\uparrow\infty} e^{-t/2}\|T_t^\phi\|_\phi = 2,$$

and consequently

$$e^{-t/2}\|T_t^\phi\|_\phi \leq K, \text{ for all } t \geq 0,$$

with some $K > 2$. This gives the required estimate for the ϕ-transformed linear semigroup.

Finally, the ϕ-transformed superdiffusion X^ϕ can be defined in the usual way: it will correspond to the integral equation

$$u(\cdot,t) = T_t^\phi g - \int_0^t \mathrm{d}s T_{t-s}^\phi(\alpha\phi u^2(\cdot,s)).$$

The rest of the proof is virtually identical with the last part of the proof of Theorem 1 (by setting $\lambda_c = 1/2$ and $\phi = e^{-|x|}$ in that proof), except that the convergence of the ϕ-transformed Laplace-functional now holds for all $g's$ with $\phi g \in C_b^+$ (recall that $\gamma = \phi$), thus yielding convergence far all nonnegative bounded continuous functions when going back to the original Laplace-functional. □

A Appendices

A.1 Proof of Theorem 3

Proof of Theorem 3. Let Y denote the diffusion corresponding to L on $\mathbb{R}^d$ with probabilities $\{\mathbb{P}_x,\ x \in \mathbb{R}^d\}$. Let $\tau_R := \inf\{t \geq 0 \mid |Y_t| = R\}$. Using Itô's formula, it is immediate that for any fixed $R_0 > 0$, $\bar{U}(x, R_0, R) := \mathbb{P}_x(\tau_{R_0} > \tau_R)$ is the unique solution to the boundary value problem

$$Lu = 0 \text{ at } R_0 \le |x| \le R, \qquad u = 0 \text{ at } |x| = R_0 \text{ and } u = 1 \text{ at } |x| = R. \tag{75}$$

By the recurrence of Y, $\bar{U}(x, R_0, R)$ tends to zero in the layer $|x| \in [R_0, R_0 + C]$, as $R \to +\infty$, for any fixed finite $C > 0$.

Note that

$$0 < \bar{U} < 1 \text{ for } |x| \in (R_0, R). \tag{76}$$

Let (r, φ) denote spherical coordinates; i.e. $r = |x|$. By the Hopf maximum principle (see Theorem 3.2.5 in [Pin95]),

$$\bar{U}'_r(x; R_0, R) > 0 \text{ both at } r = R_0 \text{ and } r = R. \tag{77}$$

Next, we show that

$$\bar{U}'_r\big|_{r=R_0} \le K(R_0) \sup_{\varphi} \bar{U}\big|_{r=R_0+1} \tag{78}$$

where the constant K depends (continuously) only on the coefficients of L at $r \in [R_0, R_0 + 1]$; i.e. it is independent of the position of the outer boundary $(r = R)$. Hence,

$$\bar{U}'_r\big|_{r=R_0} \to 0 \text{ as } R \to +\infty.$$

To prove inequality (78), just note that

$$U^*(x) = U^*(r) = \frac{1 - e^{-K(r-R_0)}}{1 - e^{-K}}$$

is a supersolution for a sufficiently large K:

$$LU^* = -K^2 e^{-K(r-R_0)}(\nabla r, a\nabla r) + O(K) < 0,$$

and, by construction, $U^*(r = R_0) = 0$, $U^*(r = R_0 + 1) = 1$. Hence, the product $U^*(x) \cdot \left(\sup_\varphi \bar{U}\big|_{r=R_0+1}\right)$ is a supersolution with the boundary values at $r = R_0$ and $r = R_0 + 1$ not smaller than those of $\bar{U}$. By the elliptic comparison principle, this implies that

$$U^*(x) \cdot \left(\sup_{\varphi} \bar{U}\big|_{r=R_0+1}\right) \ge \bar{U}(x) \text{ at } r \in [R_0, R_0 + 1]$$

and, in particular, $\bar{U}'_r(r = R_0) \leq U^{*\prime}_r(r = R_0) \cdot \left(\sup_\varphi \bar{U}\big|_{r=R_0+1}\right)$, which proves (78). When using this inequality we will always assume that $K(R_0)$ grows monotonically with R_0.

To prove our theorem on the existence of supersolutions, we will use an inductive construction: we will produce an increasing to infinity sequence $R_1 < R_2 < \dots$ and, having built a supersolution $U^{(q)}$ defined at $R_1 \leq r \leq R_q$ we will continue it to the domain $r \leq R_{q+1}$ where $R_{q+1} > R_q$ may be taken arbitrarily large (though finite). The new supersolution $U^{(q+1)}$ will coincide with $U^{(q)}$ at $r \leq R_q - \delta_q$ where δ_q can be taken arbitrarily small. So, this procedure, indeed, gives in the limit a supersolution defined at all $r \geq R_1$ (recall that (r, φ) denote spherical coordinates: $r = |x|$).

At the first step ($q = 2$) we take

$$U^{(2)}(x) = \bar{U}(x; R_1, R_2),$$

i.e. it is the solution of the boundary-value problem (75) for an arbitrary $R_2 > R_1$.

Let us now assume that we have the supersolution $U^{(q)}$ defined at $R_1 \leq r \leq R_q$ such that

$$U^{(q)}(R_q, \varphi) \equiv u_q = \text{const} \tag{79}$$

and

$$\inf_\varphi U_r^{(q)\prime}(R_q, \varphi) > 0. \tag{80}$$

By construction (see (77)), these two requirements are satisfied at $q = 2$, with $u_2 = 1$.

Denote

$$\alpha(\varphi) \equiv U_r^{(q)\prime}(R_q, \varphi).$$

Take any $R_{q+1} > R_q + 1$ such that

$$K(R_q) \sup_{|x| \in [R_q - 1, R_q + 1]} \bar{U}(x; R_q - 1, R_{q+1}) < \frac{1}{\sup_\varphi \alpha(\varphi)} \inf_\varphi \alpha^2(\varphi). \tag{81}$$

Choose a sufficiently small $\delta_q > 0$ (arbitrarily small, in fact) and take the solution $\bar{U}(x; R_q - \delta_q, R_{q+1})$ of the boundary-value problem (75). For brevity, we will denote $\bar{U}(x) \equiv \bar{U}(x; R_q - \delta_q, R_{q+1})$ below. We will also use the notation

$$\beta(\varphi) \equiv \bar{U}'_r(R_q - \delta_q, \varphi).$$

Since $\bar{U}(x;R_q-\delta_q,R_{q+1}) \le \bar{U}(x;R_q-1,R_{q+1})$ for any $\delta_q\in[0,1]$, it follows (see (77), (78) and (81)) that

$$0<\inf_{\delta_q\in[0,1]}\inf_{\varphi}\beta(\varphi)\le\sup_{\delta_q\in[0,1]}\sup_{\varphi}\beta(\varphi)<\frac{1}{\sup_{\varphi}\alpha(\varphi)}\inf_{\varphi}\alpha^2(\varphi). \tag{82}$$

This inequality allows us to find such constants λ and A that

$$\lambda>\sup_{\delta_q\in[0,1]}\sup_{\varphi}\frac{\beta(\varphi)}{\alpha(\varphi)} \tag{83}$$

and

$$\inf_{\varphi}\alpha(\varphi)>A>\lambda\sup_{\varphi}\alpha(\varphi). \tag{84}$$

Let us now define

$$U^{(q+1)}(x)=\begin{cases} U^{(q)}(x) & \text{for } |x|\le R_q-\delta_q,\\ \bar{U}(x)+u_q-A\delta_q & \text{for } R_{q+1}\ge|x|\ge R_q,\\ u_q(U^{(q)}(x)-u_q)(1-\xi_1)+ & \\ +(\bar{U}(x)-A\delta_q)\xi_2 & \text{for } |x|\in[R_q-\delta_q,R_q], \end{cases} \tag{85}$$

where A is the constant from (84) and $\xi_{1,2}$ are some $C^{2,\eta}$-functions of $z\equiv(r-R_q+\delta_q)/\delta_q$ such that

$$\xi(z)\equiv 0 \ \text{ at } \ z\le 0, \quad \xi(z)\equiv 1 \ \text{ at } \ z\ge 1 \tag{86}$$

and

$$0<\xi(z)<1 \ \text{ at } \ z\in(0,1). \tag{87}$$

Moreover,

$$\xi'(z)>0 \ \text{ at } \ z\in(0,1). \tag{88}$$

In the rest of this section, any $C^{2,\eta}$-function satisfying (86)-(88) will be called *nice*.

Obviously, the function $U^{(q+1)}$ defined by (85) is $C^{2,\eta}$ and it is a supersolution (i.e. it satisfies (62)) for $r\le R_q-\delta_q$ and $r\ge R_q$. So, we must check that it is a supersolution in the layer $R_q-\delta_q\le r\le R_q$ too, for an appropriate choice of the 'gluing' functions $\xi_{1,2}$. In this layer, the inequality to check is

$$-(U^{(q)}(x)-u_q)L\xi_1-2(\nabla U^{(q)},a\nabla\xi_1)+(\bar{U}(x)-A\delta_q)L\xi_2+2(\nabla\bar{U}(x),a\nabla\xi_2)\le 0. \tag{89}$$

Note that at $|x| \in [R_q - \delta_q, R_q]$ we have

$$\nabla U^{(q)}|_{x=(r,\varphi)} = \alpha(\varphi)\nabla r + O(\delta_q),$$

$$\nabla \bar{U}|_{x=(r,\varphi)} = \beta(\varphi)\nabla r + O(\delta_q),$$

$$U^{(q)}(r,\varphi) = u_q - [\alpha(\varphi) + O(\delta_q)]\,(R_q - r),$$

$$\bar{U}(r,\varphi) = [\beta(\varphi) + O(\delta_q)]\,(r - R_q + \delta_q).$$

Also, it is easy to see that

$$\nabla \xi = \frac{1}{\delta_q}\xi' \cdot \nabla r$$

and

$$L\xi = \frac{1}{\delta_q^2}\xi'' \cdot (\nabla r, a\nabla r) + O\left(\frac{1}{\delta_q}\right)\xi'.$$

Plugging this into (89) we arrive at the following condition which must be fulfilled at all φ and at all $z \in [0,1]$:

$$[(1-z)\xi_1'' - 2(1+O(\delta_q))\xi_1'](z) \le \left[\frac{A - \beta(\varphi)z + O(\delta_q)}{\alpha(\varphi) + O(\delta_q)}\xi_2'' - 2\frac{\beta(\varphi) + O(\delta_q)}{\alpha(\varphi) + O(\delta_q)}\xi_2'\right](z). \tag{90}$$

Since $\xi_{1,2}'$ is nonnegative by assumption, and since δ_q may be taken as small as necessary, it is sufficient that for some small enough ν

$$(1-z)\xi_1''(z) - (2-\nu)\xi_1'(z) \le (1 - \nu\,\mathrm{sign}(\xi_2''(z)))\frac{A - \beta(\varphi)z}{\alpha(\varphi)}\xi_2''(z) - (2-\nu)\lambda\xi_2'(z), \tag{91}$$

where λ is the constant from (83), (84) (recall that $A > \beta(\varphi)$ by (83),(84)). Denote

$$\xi_0(z) = \frac{\xi_1(z) - \lambda\xi_2(z)}{1-\lambda}. \tag{92}$$

By (83),(84), if ν is sufficiently small, then to satisfy the inequality (91) it is enough to require that

$$(1-z)\xi_0''(z) - (2-\nu)\xi_0'(z) \le \chi(z)\xi_2''(z), \tag{93}$$

where

$$\chi(z) = \begin{cases} \chi_+ & \text{for } \xi_2''(z) > 0 \\ \chi_- & \text{for } \xi_2''(z) < 0 \end{cases} \tag{94}$$

for some appropriately chosen constants $\chi_\pm$ which may be taken such that

$$0 < \chi_+ < \chi_- < 1. \tag{95}$$

Let us now take a smooth function $\psi(z)$ with zeros at 0, 1 and at some $\zeta \in (0,1)$. Let $\psi(z) > 0$ at $0 < z < \zeta$ and $\psi(z) < 0$ at $\zeta < z < 1$. Also, let

$$\int_0^\zeta \psi(z)dz = -\int_\zeta^1 \psi(z)dz = 1. \tag{96}$$

Denote

$$I_\nu^+ = \int_0^\zeta \psi(z)(1-z)^{(1-\nu)}dz, \qquad I_\nu^+ = -\int_\zeta^1 \psi(z)(1-z)^{(1-\nu)}dz.$$

Let

$$\xi_2(z) == \frac{1}{I_0^+ + I_0^-}\int_0^z (z-s)\psi(s)ds \tag{97}$$

at $z \in [0,1]$. It is easy to see that this defines a nice function ξ_2 for any ψ satisfying (96). Moreover,

$$\xi_2''(z) = \frac{1}{I_0^+ + I_0^-}\psi(z). \tag{98}$$

We will assume now that ξ_2 is given by (98) where the choice of ψ will be specified below. Note that the inequality (93) which must be satisfied by the function ξ_0 is rewritten as

$$(1-z)\xi_0''(z) - (2-\nu)\xi_0'(z) \le \frac{1}{I_0^+ + I_0^-}\chi(z)\psi(z). \tag{99}$$

We will look for a nice function ξ_0 which satisfies the equation

$$(1-z)\xi_0''(z) - (2-\nu)\xi_0'(z) = \kappa(z)\chi(z)\frac{\psi(z)}{I_0^+ + I_0^-},\ z \in [0,1]. \tag{100}$$

Here we denote

$$\kappa(z) = \begin{cases} \kappa_+ & \text{for } z \in [0,\zeta] \\ \kappa_- & \text{for } z \in [\zeta,1] \end{cases} \tag{101}$$

for some constant $\kappa_\pm$ such that

$$\kappa_+ < 1 < \kappa_-. \tag{102}$$

The integration of (100) gives

$$\xi_0(z) = \begin{cases} \frac{\kappa_+\chi_+}{(I_0^+ + I_0^-)(1-\nu)} \int_0^z \psi(s)(\left(\frac{1-s}{1-z}\right)^{1-\nu} - 1)\mathrm{d}s & \text{for } z \in [0,\zeta] \\ 1 + \frac{\kappa_-\chi_-}{(I_0^+ + I_0^-)(1-\nu)} \int_z^1 \psi(s)(1 - \left(\frac{1-s}{1-z}\right)^{1-\nu})\mathrm{d}s & \text{for } z \in [\zeta,1] \end{cases} \tag{103}$$

It is seen that $\xi_0(0) = 0$, $\xi_0(1) = 1$. We also have

$$\xi_0'(z) = \begin{cases} \frac{\kappa_+\chi_+}{(I_0^+ + I_0^-)(1-z)^{2-\nu}} \int_0^z \psi(s)(1-s)^{1-\nu}\mathrm{d}s & \text{for } z \in [0,\zeta] \\ -\frac{\kappa_-\chi_-}{(I_0^+ + I_0^-)(1-z)^{2-\nu}} \int_z^1 \psi(s)(1-s)^{1-\nu}\mathrm{d}s & \text{for } z \in [\zeta,1] \end{cases} \tag{104}$$

Thus, $\xi_0'(z) > 0$ at $z \in (0,1)$ and $\xi_0'(0) = 0$, $\xi_0'(1) = -\frac{\kappa_-\chi_-}{(I_0^+ + I_0^-)(2-\nu)}\psi(1) = 0$. One can also check that $\xi_0''(0) = \frac{\kappa_+\chi_+}{I_0^+ + I_0^-}\psi(0) = 0$ and $\xi_0''(1) = -\frac{\kappa_-\chi_-}{(I_0^+ + I_0^-)(3-\nu)}\psi'(1)$. It follows that in order to have a nice function ξ_0 we must assume additionally that $\psi'(1) = 0$ and that the continuity conditions

$$\xi_0(\zeta - 0) = \xi_0(\zeta + 0), \qquad \xi_0'(\zeta - 0) = \xi_0'(\zeta + 0)$$

are fulfilled (the continuity of the second derivative would then follow from equation (100) since $\psi(\zeta) = 0$ by assumption). By (103) and (104) the continuity conditions are written as

$$\kappa_+\chi_+ I_\nu^+ = \kappa_-\chi_- I_\nu^-$$

and

$$1 - \frac{\kappa_-\chi_-}{(1-\nu)(I_0^+ + I_0^-)} = -\frac{\kappa_+\chi_+}{(1-\nu)(I_0^+ + I_0^-)}$$

(note that we took into account the equality (96)). This leads to the following formula

$$\kappa_\pm = \frac{(1-\nu)}{\chi_\pm I_\nu^\pm} \cdot \frac{I_0^+ + I_0^-}{\frac{1}{I_\nu^-} - \frac{1}{I_\nu^+}}. \tag{105}$$

To fulfill (102) at a sufficiently small ν, it is enough to have

$$\frac{\chi_-}{I_0^+} < \frac{I_0^+ + I_0^-}{I_0^+ - I_0^-} < \frac{\chi_+}{I_0^-}. \tag{106}$$

By (95), this will be satisfied if I_0^+ is close enough to 1 and I_0^- is close enough to zero. To this aim, just take ψ sufficiently closely approximating the sum of the delta-function near zero and the minus delta-function near 1.

So, fixing the choice of a smooth function ψ such that (106) and (96) were satisfied (along with the requirements $\psi(0) = 0$, $\psi(1) = 0$, $\psi'(1) = 0$, $\psi(\zeta) = 0$

and $\psi(z) > 0$ at $z \in (0, \zeta)$ and $\psi(z) < 0$ at $z \in (\zeta, 1)$), we obtain the nice functions ξ_2 and ξ_0 satisfying (100) and (98), respectively. By (102), this means that the inequality (93) holds for these two functions.

It follows that the function ξ_1 recovered from (92) is also nice and satisfies (91). Hence, for such chosen functions $\xi_{1,2}$ the function $U^{(q+1)}$ given by (85) is a supersolution indeed at all $r \in [R_1, R_{q+1}]$.

As required, $U^{(q+1)}$ is constant at $r = R_{q+1}$:

$$U^{(q+1)}(R_{q+1}, \varphi) = u_q + 1 - C\delta_q \equiv u_{q+1}.$$

Hence,

$$u_q + 1 \geq u_{q+1}$$

and, by taking δ_q small, we may always ensure

$$u_{q+1} \geq u_q + \frac{1}{2}.$$

Therefore,

$$q \geq u_q \geq \frac{q}{2}.$$

By construction,

$$U^{(q+1)}(x) \leq u_{q+1}$$

and

$$U^{(q+1)}(x) \geq u_{q+1} - 1 \ \text{ at } \ r \geq R_q.$$

It follows, first, that the supersolution U which we obtain in the limit of this procedure as $q \to +\infty$ grows to infinity:

$$\lim_{r \uparrow \infty} \inf_{|x|=r} U(x) = +\infty.$$

On the other hand, this growth can be made arbitrarily slow: it is seen that $U(x) \leq q$ at $|x| \leq R_q$ but R_q may be taken growing as fast as necessary. $\square$

A.2 A review on criticality theory

Let L be as in (1). Then there exists a corresponding diffusion process Y on $\mathbb{R}^d$ that solves the generalized martingale problem for L on $\mathbb{R}^d$ (see Chapter 1 in [Pin95]). The process lives on $\mathbb{R}^d \cup \Delta$ with Δ playing the role of a cemetery state. We denote by $\mathbb{P}_x$ and $\mathbb{E}_x$ the corresponding probabilities and expectations, and define the transition measure $p(t, x, \mathrm{d}y)$ for $L + \beta$ by

$$p(t, x, B) = \mathbb{E}_x \left(\exp \left(\int_0^t \beta(Y_s)\, \mathrm{d}s \right) ; Y_t \in B \right),$$

for measurable $B \subseteq \mathbb{R}^d$.

Definition 32 If

$$\int_0^\infty p(t,x,B)\,\mathrm{d}t = \mathbb{E}_x \int_0^\infty \exp\left(\int_0^t \beta(Y_s)\,\mathrm{d}s\right) 1_B(Y_t)\,\mathrm{d}t < \infty,$$

for all $x \in \mathbb{R}^d$ and all bounded $B \subset \mathbb{R}^d$, then

$$G(x,\mathrm{d}y) = \int_0^\infty p(t,x,\mathrm{d}y)\,\mathrm{d}t$$

is called the *Green's measure* for $L+\beta$ on $\mathbb{R}^d$. If the above condition fails, then the Green's measure for $L+\beta$ on $\mathbb{R}^d$ is said not to exist. $\diamond$

In the former case, $G(x,\mathrm{d}y)$ possesses a density, $G(x,\mathrm{d}y) = G(x,y)\mathrm{d}y$, which is called the *Green's function* for $L+\beta$ on $\mathbb{R}^d$.

For $\lambda \in \mathbb{R}$ define

$$C_{L+\beta-\lambda} \;=\; \{\; u \in C^2 \;:\; (L+\beta-\lambda)u = 0 \text{ and } u > 0 \text{ in } \mathbb{R}^d \;\}.$$

The operator $L+\beta-\lambda$ on $\mathbb{R}^d$ is called *subcritical* if the Green's function exists for $L+\beta-\lambda$ on $\mathbb{R}^d$; in this case $C_{L+\beta-\lambda} \neq \emptyset$. If the Green's function does not exist for $L+\beta-\lambda$ on $\mathbb{R}^d$, but $C_{L+\beta-\lambda} \neq \emptyset$, then the operator $L+\beta-\lambda$ on $\mathbb{R}^d$ is called *critical.* In this case $C_{L+\beta-\lambda}$ is one-dimensional. The unique function (up to a constant multiple) in $C_{L+\beta-\lambda}$ is called the *ground state* of $L+\beta$ on $\mathbb{R}^d$. The formal adjoint of the operator $L+\beta-\lambda$ on $\mathbb{R}^d$ is also critical with ground state $\tilde{\phi}$. If furthermore $\phi\tilde{\phi} \in L^1(\mathbb{R}^d)$, we call $L+\beta-\lambda$ on $\mathbb{R}^d$ *product-critical* . (For $\phi = \tilde{\phi}$ this means that ϕ is an L^2-eigenfunction.) Finally, if $C_{L+\beta-\lambda} = \emptyset$, then $L+\beta-\lambda$ on $\mathbb{R}^d$ is called *supercritical.*

If $\beta \equiv 0$, then $L+\beta$ is not supercritical on $\mathbb{R}^d$ since the function $f \equiv 1$ satisfies $Lf = 0$ on $\mathbb{R}^d$. In this case $L+\beta = L$ is subcritical or critical on $\mathbb{R}^d$ according to whether the corresponding diffusion process, Y, is transient or recurrent on $\mathbb{R}^d$. Product criticality in this case is equivalent to positive recurrence for Y. If $\beta \leq 0$ and $\beta \not\equiv 0$, then $L+\beta$ is subcritical on $\mathbb{R}^d$.

In terms of the solvability of inhomogeneous Dirichlet problems, subcriticality guarantees that the equation $(L+\beta)u = -f$ in $\mathbb{R}^d$ has a positive solution u for every $0 \lneq f \in C_c^\eta$. (Here $C_c^\eta = C_c \cap C^\eta$.) If subcriticality does not hold, then there are no positive solutions for any $0 \lneq f \in C_c^\eta$.

One of the two following possibilities holds :

1) There exists a number $\lambda_c \in \mathbb{R}$ such that $L-\lambda$ on $\mathbb{R}^d$ is subcritical for $\lambda > \lambda_c$, supercritical for $\lambda < \lambda_c$, and either subcritical or critical for $\lambda = \lambda_c$.

2) $L-\lambda$ on $\mathbb{R}^d$ is supercritical for all $\lambda \in \mathbb{R}$, in which case we define $\lambda_c = \infty$.

Definition 33 The number $\lambda_c \in (-\infty,\infty]$ is called the *generalized principal eigenvalue* for L on $\mathbb{R}^d$. $\diamond$

Note that $\lambda_c = \inf \{\lambda \in \mathbb{R} : C_{L+\beta-\lambda} \neq \emptyset\}$. Also, if β is bounded from above, then case 1) holds.

If $L+\beta$ is symmetric with respect to a reference measure $\rho\,\mathrm{d}x$, then λ_c equals the supremum of the spectrum of the self-adjoint operator on $L^2(\mathbb{R}^d, \rho\,\mathrm{d}x)$ obtained from $L+\beta$ via the Friedrichs' extension theorem.

Let $h \in C^{2,\eta}$ satisfy $h > 0$ in $\mathbb{R}^d$. The operator $(L+\beta)^h$ defined by

$$(L+\beta)^h f = \frac{1}{h}(L+\beta)(hf)$$

is called the *h-transform* of the operator $L+\beta$. Written out explicitly, one has

$$(L+\beta)^h f = L_0 + a\frac{\nabla h}{h}\cdot\nabla + \beta + \frac{Lh}{h},$$

where $L_0 = \frac{1}{2}\nabla\cdot a\nabla + b\cdot\nabla$.

All the properties defined above are invariant under h-transforms.

For further elaboration and proofs see Chapter 4 in [Pin95].

Acknowledgements. We are grateful to M. Effendiev, K. Fleischmann, J. Gärtner, R. Pinsky, K. Schneider and M. Wolfrum for helpful discussions.

References

[AN72] K.B. Athreya and P.E. Ney. *Branching Processes.* Springer-Verlag, Berlin-Heildelberg-New York, 1972.

[BCG93] M. Bramson, J.T. Cox and A. Greven. Ergodicity of critical spatial branching processes in low dimensions. *Ann. Probab.*, 21:1946–1957, 1993.

[Daw93] D.A. Dawson. Measure-valued Markov processes. In P.L. Hennequin, editor, *École d'été de probabilités de Saint Flour XXI–1991*, volume 1541 of *Lecture Notes in Mathematics*, pages 1–260. Springer-Verlag, Berlin, 1993.

[Dyn91] E.B. Dynkin. Branching particle systems and superprocesses. *Ann. Probab.*, 19:1157–1194, 1991.

[Dyn93] E.B. Dynkin. Superprocesses and partial differential equations (The 1991 Wald memorial lectures). *Ann. Probab.*, 21(3):1185–1262, 1993.

[EF99] J. Engländer and K. Fleischmann Extinction properties of super-Brownian motions with additional spatially dependent mass production. *Stoch. Proc. Appl.*, to appear, 1999.

[EP99] J. Engländer and R.G. Pinsky. On the construction and support properties of measure-valued diffusions on $D \subset R^d$ with spatially dependent branching. *Ann. Probab.*, 27:684-730, 1999

[Fel71] W. Feller. *An Introduction to Probability Theory and its Applications*, volume II. John Wiley and Sons, New York, 2nd edition, 1971.

[Fri64] A. Friedman *Partial Differential Equations of Parabolic Type*, Prentice Hall, 1964.

[Pin95] R.G. Pinsky. *Positive Harmonic Functions and Diffusion.* Cambridge University Press, 1995.

[Pin96] R.G. Pinsky. Transience, recurrence and local extinction properties of the support for supercritical finite measure-valued diffusions. *Ann. Probab.*, 24(1):237–267, 1996.

[Pin99] R.G. Pinsky. Invariant probability distributions for measure-valued diffusions. *Preprint.*

[SSTC98] L. Shilnikov, A. Shilnikov, D. Turaev, L. Chua. *Methods of qualitative theory in nonlinear dynamics,* Part I. World Scientific, Singapore, 1998.