# Krawtchouk Polynomials and Iterated Stochastic Integration

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**Abstract**

We show that for the binomial process $(S_n)_{n\in\mathbb{N}}$, the orthogonal functionals constructed in Kroeker [8] for Markov chains can be expressed using the Krawtchouk polynomials, and by iterated stochastic integrals. This allows to construct a chaotic calculus based on gradient and divergence operators and structure equations, and to establish a Clark representation formula.

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## 1 Introduction

In the classical deterministic integration theory the polynomials $\{p_n(x) = x^n, n \geq 0\}$, and the exponential function $\exp(x) = \sum_{n=0}^{\infty} p_n(x)/n!$ play a special role because they satisfy $\int_0^t p_n(x)dx = \frac{1}{n+1}p_{n+1}(t)$ and $\int_0^t \exp(x)dx = \exp(t) - \exp(0)$. In the stochastic case it turns out that the role of $p_n$ is taken up by orthogonal polynomials related to the distribution of the integrator. The most studied stochastic case is integration with respect to Brownian motion $\{B_t, t \geq 0\}$, where $B_t$ has a Normal distribution $\mathcal{N}(0,t)$, i.e. with mean zero and variance $t \geq 0$. The notion of multiple stochastic integration for this process was first introduced by Wiener. As is well known, in stochastic Itô integration theory with respect to standard Brownian motion, the Hermite polynomials play the role of the $p_n$, as such

$$\int_0^t H_n(B_s; s)dB_s = \frac{H_{n+1}(B_t; t)}{n+1},$$

where $H_n(x;t)$ is the monic (with leading coefficient equal to one) Hermite polynomial with parameter $t$. The monic Hermite polynomials $H_n(x,t)$ are orthogonal with respect to the Normal distribution $\mathcal{N}(0,t)$, the distribution of $B_t$. Note also that because stochastic integral are martingales, that $\{H_n(B_t;t)\}$ are martingales, see also Schoutens and Teugels [14] (1998). Using the generating function $\sum_{n=0}^{\infty} H_n(x;t)z^n/n! = \exp(-tz^2/2 + zx)$, one can easily see that the role of the exponential function is now taken by the function

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$\exp(-t/2+B_t)$ because we have $\int_0^t \exp(-s/2+B_s)dB_s = \exp(-t/2+B_t)-1$. The transformation $\exp(B_t - t/2)$ of the Brownian motion is sometimes called *geometric Brownian motion* or the *stochastic exponent* of the Brownian motion. There is a similar result for the *compensated Poisson process* $M_t = N_t - t$: the monic orthogonal polynomials with respect to the Poisson distribution $\mathrm{P}(t)$ are the monic Charlier polynomials, $C_n(x;t)$, defined by the generating function (Koekoek and Swarttow, [7], 1998): $W(x,t,z) = \sum_{n=0}^{\infty} C_n(x;t)z^n/n! = \exp(-tz)(1+x)^x$. We have:

$$\int_0^t C_n(N_{s-};s)dM_s = \frac{C_{n+1}(N_t;t)}{n+1}. \tag{1}$$

In terms of the generating function, this is equivalent with

$$\int_0^t W(N_{s-},s,w)dM_s = \frac{W(N_t,t,w)-1}{w}.$$

This result goes back to Ogura [10] (1972) and Engel [3] (1982), and also implies that the monic Charlier polynomials $\{C_n(N_t;t)\}$ are martingales, see also [14] (1998).

Chaos expansions for Markov chains have been constructed in Kroeker [8] (1980) via orthogonal functionals that are the analogs of multiple stochastic integrals with respect to martingales. A natural question for investigation is the determination of martingales whose multiple stochastic integrals can be expressed as polynomials. In continuous time, Privault, Solé and Vives [12] (1997) proved that the only normal martingales solutions of structure equations which have an associated family of polynomials are the Poisson process and the Brownian motion. In the i.i.d. case it is shown in [5] that such polynomials have to be Meixner polynomials. In this paper we will show that the Markov chain approach to multiple stochastic integrals coincides with the i.i.d. approach of [5] only for the binomial process, and that the binomial process is the only i.i.d. discrete time process for which the multiple stochastic integrals of [5] can be expressed with polynomials, namely the Krawtchouk polynomials. Moreover, in this case these functionals can be also expressed as discrete iterated integrals with respect to the compensated binomial process.

This paper is organised as follows. In Sect. 2 we reformulate the construction of [8] in the language of tensor products. We give a particular attention to this construction because it is valid for processes with non-independent increments, and the non-independence of increments is always a non-trivial problem in chaotic representation, cf. Emery [2] and Biane [1] in continuous time. Sect. 3 deals with the i.i.d. case. Sect. 4 is devoted to the representation of orthogonal functionals of the binomial process as Krawtchouk polynomials, and Sect. 5 presents the iterated stochastic integrals and the relation between the

Krawtchouk polynomials and the binomial process. In Sect. 6 we obtain a Clark representation formula for functionals of the binomial process, using gradient and divergence operators.

## 2 Orthogonal expansions for Markov chains

In this section we formulate the construction of orthogonal functionals of Markov chains due to [8], in the language of tensor products. This construction does not seem to be related to the chaos expansions defined in [1] (1989) for finite Markov chains in continuous time. The notion of tensor product makes the construction significantly different, but leads to the same objects. The tensor product, resp. symmetric tensor product of functions in $l^2(\mathbb{N}^*)$ with $\mathbb{N}^* = \mathbb{N}\backslash\{0\} = \{1,2,\ldots\}$ will be denoted as "$\otimes$", resp. "$\circ$" and $(e_k)_{k\in\mathbb{N}}$ denotes the canonical basis of $l^2(\mathbb{N}^*)$. In particular, $e_{i_1}^{\circ n_1}\circ\cdots\circ e_{i_d}^{\circ n_d}$ is the symmetrization in $n_1+\cdots+n_d=n$ variables of $e_{i_1}^{\otimes n_1}\otimes\cdots\otimes e_{i_d}^{\otimes n_d}$, and

$$\langle e_{i_1}^{\circ n_1}\circ\cdots\circ e_{i_d}^{\circ n_d}, e_{i_1}^{\circ n_1}\circ\cdots\circ e_{i_d}^{\circ n_d}\rangle_{l^2(\mathbb{N}^*)^{\circ n}} = \frac{n_1!\cdots n_d!}{n!}. \tag{2}$$

The symmetric tensor product $l^2(\mathbb{N}^*)^{\circ n}$ is by definition the set of all square-summable symmetric functions in $n$ strictly positive integer variables. Let $(S_n)_{n\in\mathbb{N}}$ be a Markov chain with state space $\mathbb{N}$ and transition matrix $(P(x,y))_{x,y\in\mathbb{N}}$, starting from 0, on a probability space $\Omega$. Let $\mu(k)\in\mathbb{N}\cup\{\infty\}$, denote the dimension of $l^2(\mathbb{N};P(k,\cdot))$, and let $(\phi^n(\cdot\mid k))_{0\le n\le\mu}$ be a complete orthogonal set of polynomials in $l^2(\mathbb{N};P(k,\cdot))$, with $\phi^n(\cdot\mid k)$ of degree $n$, $\|\phi^n(\cdot\mid k)\|^2_{l^2(\mathbb{N},P(k,\cdot))}=n!$, $0\le n\le\mu(k)$, $k\ge 0$, and $\phi^n(x\mid k)=0$, $n>\mu(k)$, $x,k\in\mathbb{N}$.

**Remark:** In the construction of [8], the functional $\phi^n(x\mid y)$ is not constrained to be a polynomial in $x\in\mathbb{N}$, and the choice of the family $(\phi^n(\cdot\mid y))_{n\in\mathbb{N}}$ is not unique. $\square$

Also, in [8], the data of the initial distribution $\pi$ of $(S_n)_{n\in\mathbb{N}}$ is also considered. In our notation this can be easily taken into account by letting $P(0,\cdot)=\pi(\cdot)$.

**Definition 1** *With $1\le i_1<\cdots<i_n$, and $n_1+\cdots+n_d=n\ge 1$, let*

$$J_n(e_{i_1}^{\circ n_1}\circ\cdots\circ e_{i_d}^{\circ n_d}) = \prod_{k=1}^{k=d}\phi^{n_k}(S_{i_k}\mid S_{i_k-1}).$$

We let $J_0(f_0)=1$, $f_0\in\mathbb{R}$, i.e. $l^2(\mathbb{N})^{\circ 0}$ is identified to $\mathbb{R}$. A Wick type product $\diamond$ of random variables may also be defined as

$$\left(\prod_{i=1}^{i=d}\phi^{n_i}(S_{k_i}\mid S_{k_i-1})\right)\diamond\left(\prod_{i=1}^{i=d}\phi^{m_i}(S_{k_i}\mid S_{k_i-1})\right) = \prod_{i=1}^{i=d}\phi^{n_i+m_i}(S_{k_i}\mid S_{k_i-1}).$$

(By induction on $d \in \mathbb{N}$, $\prod_{i=1}^{i=d} \phi^{n_i}(S_{k_i} \mid S_{k_i-1})$ is the unique functional representing $J_n(e_{i_1}^{\circ n_1} \circ \cdots \circ e_{i_d}^{\circ n_d})$). For all symmetric function $f_n \in l^2(\mathbb{N}^*)^{\circ n}$ of $n$ variables with finite support written as

$$f_n = \sum_{d=1}^{d=n} \sum_{\substack{1 \le i_1 < \cdots < i_d \\ n_1 + \cdots + n_d = n}} a_{i_1,\ldots,i_d} e_{i_1}^{\circ n_1} \circ \cdots \circ e_{i_d}^{\circ n_d}, \tag{3}$$

we let

$$J_n(f_n) = \sum_{d=1}^{d=n} \sum_{\substack{1 \le i_1 < \cdots < i_d \\ n_1 + \cdots + n_d = n}} a_{i_1,\ldots,i_d} J_n(e_{i_1}^{\circ n_1} \circ \cdots \circ e_{i_d}^{\circ n_d}).$$

The functional $J_n$ is the analog of the multiple stochastic integral in the case of continuous time martingales. In terms of the Wick product $\diamond$ we have for $f_n$ and $g_m$ as above:

$$J_n(f_n) \diamond J_m(g_m) = J_{n+m}(f_n \circ g_m).$$

**Proposition 1** *The functional $J_n(f_n)$ is orthogonal to $J_m(g_m)$ in $L^2(\Omega)$ if $n \neq m$, and $J_n : l^2(\mathbb{N}^*)^{\circ n} \longrightarrow L^2(\Omega)$ extends as a linear continuous operator with*

$$E[J_n(f_n)^2] \le n! \|f_n\|^2_{l^2(\mathbb{N}^*)^{\otimes n}}, \quad f_n \in l^2(\mathbb{N}^*)^{\otimes n}, \ n \in \mathbb{N}. \tag{4}$$

*The equality $E[J_n(f_n)^2] = n! \|f_n\|^2_{l^2(\mathbb{N}^*)^{\otimes n}}$, $f_n \in l^2(\mathbb{N}^*)^{\circ n}$, $n \in \mathbb{N}$, holds if $\mu(k) = \infty$, $k \in \mathbb{N}$.*

**Proof:** By construction we have

$$E[\phi^{n_k}(S_{i_k} \mid S_{i_k-1}) \phi^{m_k}(S_j \mid S_{j-1}) \mid S_{i_0}, \ldots, S_{i_k-1}] = n_k! 1_{\{i_k=j\}} 1_{\{n_k=m_k\}} 1_{\{n_k \le \mu(S_{i_k-1})\}},$$

and for $n_k \ge 1$:

$$E[\phi^{n_k}(S_{i_k} \mid S_{i_k-1}) \mid S_{i_0}, \ldots, S_{i_k-1}] = 0,$$

hence by induction on $k = 1, \ldots, d$,

$$E[J_n(e_{i_1}^{\circ n_1} \circ \cdots \circ e_{i_d}^{\circ n_d}) J_m(e_{j_1}^{\circ m_1} \circ \cdots \circ e_{j_l}^{\circ m_l})] = 0$$

if $\{i_1, \ldots, i_d\} \neq \{j_1, \ldots, j_d\}$ or $n \neq m$, and

$$0 \le E[J_n(e_{i_1}^{\circ n_1} \circ \cdots \circ e_{i_d}^{\circ n_d})^2] \le n_1! \cdots n_d!.$$

With $f_n \in l^2(\mathbb{N}^*)^{\circ n}$ and $g_m \in l^2(\mathbb{N}^*)^{\circ m}$ as in (3) we have $E[J_n(f_n) J_m(g_m)] = 0$ if $n \neq m$, and

$$E[J_n(f_n)^2] \quad \le \quad n! \sum_{d=1}^{d=n} \sum_{\substack{1 \le i_1 < \cdots < i_d \\ n_1 + \cdots + n_d = n}} \frac{n_1! \cdots n_d!}{n!} a^2_{i_1,\ldots,i_d} = n! \langle f_n, f_n \rangle_{l^2(\mathbb{N}^*)^{\circ n}},$$

from (2). $\square$

**Remark:** The isometry formula $E[J_n(f_n)J_m(g_m)] = 1_{\{n=m\}}n!\langle f_n, g_m\rangle_{l^2(\mathbb{N}^*)^{\otimes n}}$ (see Relation (13) in [8]) does not hold in general, e.g. for the binomial process we have $E[J_n(1^{\circ n}_{[1,N]})^2] = (n!)^2\binom{N}{n} < n!\|1^{\circ n}_{[1,N]}\|^2_{l^2(\mathbb{N}^*)^{\circ n}}$, cf. Sect. 4. $\square$

From the expression

$$f^{\circ n} = \sum_{d=1}^{d=n} \sum_{\substack{1\leq i_1<\cdots<i_d\\ n_1+\cdots+n_d=n}} \frac{n!}{n_1!\cdots n_d!} f^{n_1}(i_1)\cdots f^{n_d}(i_d) e_{i_1}^{\circ n_1}\circ\cdots\circ e_{i_d}^{\circ n_d}, \qquad f\in l^2(\mathbb{N}^*). \tag{5}$$

We have

$$J_n(1^{\circ n}_{[1,N]}) = \sum_{d=1}^{d=n} \sum_{\substack{1\leq i_1<\cdots<i_d\leq N\\ n_1+\cdots+n_d=n}} \frac{n!}{n_1!\cdots n_d!} \prod_{k=1}^{k=d} \phi^{n_k}(S_{i_k} \mid S_{i_k-1}),$$

and a stochastic exponential $\mathcal{E}^\circ_N(z)$ can be constructed as

$$\mathcal{E}^\circ_N(z) = \sum_{n=0}^{n=N} z^n J_n(1^{\circ n}_{[1,N]}) = \sum_{n=0}^{n=N} z^n \sum_{d=1}^{d=n} \sum_{\substack{1\leq i_1<\cdots<i_d\leq N\\ n_1+\cdots+n_d=n}} \frac{n!}{n_1!\cdots n_d!} \prod_{k=1}^{k=d} \phi^{n_k}(S_{i_k} \mid S_{i_k-1}), \quad z\in\mathbb{R}. \tag{6}$$

# 3 Orthogonal expansions for i.i.d. processes

From now on we consider processes with i.i.d. increments, i.e. $(S_n - S_{n-1})_{n\geq 1} = (X_n)_{n\geq 1}$ is a family of i.i.d. random variables. In this case the function $\phi^n(x \mid y)$ depends only on the difference $x-y$, so that we denote $\phi^n(x \mid y) = \phi^n(x-y)$. We have

$$J_n(1^{\circ n}_{[1,N]}) = \sum_{d=1}^{d=n} \sum_{\substack{1\leq i_1<\cdots<i_d\leq N\\ n_1+\cdots+n_d=n}} \frac{n!}{n_1!\cdots n_d!} \phi^{n_1}(X_{i_1})\cdots\phi^{n_d}(X_{i_d}).$$

**Definition 2** *Given $f\in l^2(\mathbb{N}^*)$ we denote by $f^{\odot n}$ the symmetrization in $n$ variables of*

$$(k_1,\ldots,k_n) \mapsto 1_{\{k_1\neq\cdots\neq k_n\}} f(k_1)\cdots f(k_n),$$

*and call $l^2(\mathbb{N}^*)^{\odot n} \subset l^2(\mathbb{N}^*)^{\otimes n}$ the completed linear span generated by $\{f^{\odot n} : f\in l^2(\mathbb{N}^*)\}$.*

The space $l^2(\mathbb{N}^*)^{\odot n}$ consists in fact of all the symmetric functions in $l^2(\mathbb{N}^*)^{\circ n}$ that vanish on every diagonal in $(\mathbb{N}^*)^n$. We have

$$\langle e_{i_1}^{\odot n_1}\odot\cdots\odot e_{i_d}^{\odot n_d}, e_{i_1}^{\odot n_1}\odot\cdots\odot e_{i_d}^{\odot n_d}\rangle_{l^2(\mathbb{N}^*)^{\otimes n}} = 1_{\{n_1=\cdots=n_d=1\}}1_{\{i_1\neq\cdots\neq i_d\}}, \tag{7}$$

and $l^2(\mathbb{N}^*)^{\odot n}$ is also the $n$-th chaos of the toy Fock space, cf. Meyer [9], p. 14. For all symmetric function $f_n\in l^2(\mathbb{N}^*)^{\odot n}$ we have

$$J_n(f_n) = \sum_{1\leq i_1<\cdots<i_n} f_n(i_1,\ldots,i_n)\phi^1(X_{i_1})\cdots\phi^1(X_{i_n}).$$

**Proposition 2** *The functional $J_n(f_n)$ is orthogonal to $J_m(g_m)$ in $L^2(\Omega)$ if $n \neq m$, and $J_n : l^2(\mathbb{N}^*)^{\odot n} \longrightarrow L^2(\Omega)$ extends as a linear continuous operator with*

$$E[J_n(f_n)^2] = \|f_n\|^2_{l^2(\mathbb{N}^*)^{\otimes n}}, \quad f_n \in l^2(\mathbb{N}^*)^{\odot n},\ n \in \mathbb{N}. \tag{8}$$

**Proof:** The orthogonality property follows from Prop. 1. By independence we have if $\{i_1, \ldots, i_n\} \neq \{j_1, \ldots j_n\}$,

$$E[J_n(e_{i_1} \odot \cdots \odot e_{i_n}) J_n(e_{j_1} \odot \cdots \odot e_{j_n})] = 0, \quad n \geq 0,$$

and

$$E[(J_n(e_{i_1} \odot \cdots \odot e_{i_n}))^2] = 1_{\{i_1 \neq \cdots \neq i_n\}}.$$

With $f_n \in l^2(\mathbb{N}^*)^{\odot n}$ we have

$$E[J_n(f_n)^2] = \sum_{1 \leq i_1 < \cdots < i_n} f_n(i_1, \ldots, i_n)^2 = \langle f_n, f_n \rangle_{l^2(\mathbb{N}^*)^{\otimes n}}. \quad \square$$

The corresponding exponential martingale $\mathcal{E}^{\odot}_N(z)$ is constructed as

$$\mathcal{E}^{\odot}_N(z) = \sum_{n=0}^{n=N} z^n J_n(1^{\odot n}_{[1,N]}) = \prod_{i=1}^{i=N} (1 + z\phi^1(X_i)) = \prod_{i=1}^{i=N} (1 + z(\alpha + \beta X_i)), \quad z \in \mathbb{R}. \tag{9}$$

We now completes the result of [5] by showing that the integral $J_n(1^{\odot n}_{[1,N]})$ is a polynomial in $S_N$ if and only if $(S_n)_{n \in \mathbb{N}}$ is the binomial process. We have

$$J_1(1_{[1,N]}) = \sum_{i=1}^{i=N} J_1(e_i) = \sum_{i=1}^{i=N} \phi^1(X_i) = \alpha N + \beta S_N.$$

**Proposition 3** *Each of the following statements holds if and only if $(S_n)_{n \geq 0}$ is a binomial process:*

*i) The exponentials $\mathcal{E}^{\circ}_N(z)$ and $\mathcal{E}^{\odot}_N(z)$ coincide, i.e.*

$$\mathcal{E}^{\circ}_N(z) = \sum_{n=0}^{n=N} z^n J_n(1^{\circ n}_{[1,N]}) = \sum_{n=0}^{n=N} z^n J_n(1^{\odot n}_{[1,N]}) = \mathcal{E}^{\odot}_N(z), \quad z \in \mathbb{R}.$$

*ii) The integrals $J_n(f^{\odot n})$ and $J_n(f^{\circ n})$ coincide, $f \in l^2(\mathbb{N}^*)$, $n \in \mathbb{N}$.*

*iii) The integral $J_2(1^{\odot 2}_{[1,N]})$ can be expressed as a second degree polynomial in $S_N$ for all $N \geq 1$.*

**Proof:** We note that $(i)$ is equivalent to $(ii)$. Taking $n = N = 2$, $(ii)$ implies $\phi^2 = 0$, hence the distribution of $X_k$ is supported by two points only and $(S_n)_{n\geq 0}$ is the binomial process. Concerning $(iii)$ we have $1_{[1,N]}^{\odot 2} = \sum_{1\leq i\neq j\leq N} e_i \circ e_j$, and

$$J_2(1_{[1,N]}^{\odot 2}) = \sum_{1\leq i\neq j\leq N} \phi^1(X_i)\phi^1(X_j) = (\alpha N + \beta S_N)^2 - \sum_{1\leq i\leq N} \phi^1(X_i)^2.$$

If $J_2(1_{[1,N]}^{\odot 2})$ is a second degree polynomial in $S_N$, then $\sum_{i=1}^{i=N} \phi^1(X_i)^2 = c_N S_N^2 + d_N S_N + e_N$ is polynomial of degree at most two in $S_N$, hence

$$\phi^1(X_N)^2 = c_N S_N^2 + d_N S_N + e_N - c_{N-1} S_{N-1}^2 - d_{N-1} S_{N-1} - e_{N-1}$$

Hence, i.e. with $S_N = X_N + S_{N-1}$, we have for $N \geq 1$:

$$(\alpha + \beta X_N)^2 = c_N (X_N + S_{N-1})^2 + d_N (X_N + S_{N-1}) + e_N - c_{N-1} S_{N-1}^2 - d_{N-1} S_{N-1} - e_{N-1}$$

or

$$\begin{aligned} & X_N^2(\beta^2 - c_N) + X_N(2\alpha\beta - 2c_N S_{N-1} - d_N) \\ & \quad - c_N S_{N-1}^2 - d_N S_{N-1} - e_N + c_{N-1} S_{N-1}^2 + d_{N-1} S_{N-1} + e_{N-1} + \alpha^2 = 0. \end{aligned}$$

If $X_N$ is allowed to take at least three distinct values, then $c_N = \beta^2$ and $2\alpha\beta - 2c_N S_{N-1} - d_N = 0$, $N \geq 1$, which is impossible. As $X_N$ can only attain 2 values, the process $S_n$ is a binomial process. The fact that $J_n(1_{[1,N]}^{\circ n})$ is a polynomial in $S_N$ if $(S_n)_{n\in\mathbb{N}}$ is the binomial process will be proved in Sect. 4. $\square$

# 4 Krawtchouk polynomials and the binomial process

In the following we assume that $(S_n)_{n\in\mathbb{N}}$ is the binomial process with parameter $p$, i.e. $P(x, y) = q1_{\{y=x\}} + p1_{\{y=x+1\}}$, $x, y \in \mathbb{N}$. In other terms, $S_0 = 0$ and $(X_i)_{i\geq 1} = (S_i - S_{i-1})_{i\geq 1}$ is a family of i.i.d. Bernoulli random variables with parameter $0 < p = \Pr(X_i = 1) < 1$, and $S_N$ has a binomial distribution $\mathrm{Bin}(N, p)$, given by the probabilities $\binom{N}{i} p^i q^{N-i}$, $i = 0, 1, \ldots, N$. The monic orthogonal polynomials with respect to the binomial distribution are the monic Krawtchouk polynomials and are determined by the generating function (Koekoek and Swarttouw [7] , 1998):

$$Y(x, N, z) = \sum_{n=0}^{N} \tilde{K}_n(x; N, p) \frac{z^n}{n!} = (1 + qz)^x (1 - pz)^{N-x},$$

where $N \in \mathbb{N}$, $0 < p < 1$ and $p + q = 1$, with $\tilde{K}_n(x; N, p) = 0$, $x \in \mathbb{R}$, $n > N$. Explicitly, this implies

$$\tilde{K}_n(x; N, p) = p^n (-N)_n \sum_{i=0}^{i=n} \frac{(-n)_i (-x)_i}{(-N)_i} \frac{(1/p)^i}{i!}, \qquad x, n \in \mathbb{N}, \tag{10}$$

and in particular $\tilde{K}_1(x;N,p) = x - Np$, where $(a)_k = a(a+1)\dots(a+k+1)$ denotes the Pochhammer symbol, with $(a)_0 = 1$, $a \in \mathbb{R}$. The Krawtchouk polynomials are orthogonal with respect to the binomial distribution $\mathrm{Bin}(N,p)$:

$$\sum_{x=0}^{x=N} \binom{N}{x} p^x q^{N-x} \tilde{K}_n(x;N,p)\tilde{K}_m(x;N,p) = (-1)^n n!(-N)_n (pq)^n 1_{\{n=m\}}, \quad 0 \le n,m \le N, \tag{11}$$

see [7] (1988). We have $\phi^1(x \mid y) = (pq)^{-1/2}\tilde{K}_1(x-y;1,p) = (pq)^{-1/2}(x-y-p)$ and $\phi^n(x \mid y) = 0$, $x,y \in \mathbb{R}$, $n > 1$.

**Proposition 4** *With $(S_n)_{n\in\mathbb{N}}$ the binomial process we have*

$$J_n(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ n_d}) = (pq)^{-n/2} \prod_{k=1}^{k=d} \tilde{K}_{n_k}(S_{N_k} - S_{M_k}; N_k - M_k, p),$$

*$n_1 + \cdots + n_d = n$, $0 \le M_i < N_i \le M_{i+1}$, $i = 1,\dots,d-1$, and $M_d < N_d$.*

**Proof:** From (5) we have

$$1_{[M+1,N]}^{\circ n} = \sum_{d=1}^{d=n} \sum_{\substack{M<i_1<\cdots<i_d\le N \\ n_1+\cdots+n_d=n}} \frac{n!}{n_1!\cdots n_d!} e_{i_1}^{\circ n_1} \circ \cdots \circ e_{i_d}^{\circ n_d},$$

hence

$$\begin{aligned} J_n(1_{[M+1,N]}^{\circ n}) &= 1_{\{0\le n\le N-M\}} n! \sum_{M<i_1<\cdots<i_n\le N} \prod_{k=1}^{k=n} \phi^1(X_{i_k}) \\ &= 1_{\{0\le n\le N-M\}} (pq)^{-n/2} n! \sum_{M<i_1<\cdots<i_n\le N} \prod_{k=1}^{k=n} (X_{i_k} - p). \end{aligned}$$

This also shows that $J_n(1_{[M+1,N]}^{\circ n})$ is a (polynomial) functional of $S_N - S_M$ since it depends only on the number of jumps of $(S_n)_{n\ge1}$ on $\{M+1,\dots,N\}$, and not on jump times. Moreover, $J_n(1_{[M+1,N]}^{\circ n})$ satisfies the same orthogonality property as the Krawtchouk polynomials. Since

$$E[J_n(1_{[M+1,N]}^{\circ n})^2] = (n!)^2 \binom{N-M}{n} = n!(-1)^n(M-N)_n$$

and from the orthogonality relation (11): $E[(\tilde{K}_n(S_N - S_M; N-M, p))^2] = n!(-1)^n(M-N)_n(pq)^n$, we obtain

$$J_n(1_{[M+1,N]}^{\circ n}) = (pq)^{-n/2}\tilde{K}_n(S_N - S_M; N-M, p).$$

Finally, from the definition of $J_n$ we have since $\phi^1(X_{i_k})$ is independent of $S_{i_k-1}$:

$$\begin{aligned}
J_n(1_{[M_1+1,N_1]}^{\circ n_1}\circ\cdots\circ 1_{[M_d+1,N_d]}^{\circ n_d}) &= \prod_{l=1}^{l=d} n_l! \sum_{M_l<i_1<\cdots<i_{n_l}\leq N_l} \prod_{k=1}^{k=n_l}\phi^1(X_{i_k}) \\
&= \prod_{l=1}^{l=d} J_{n_l}(1_{[M_l+1,N_l]}^{\circ n_l}) \\
&= (pq)^{-n/2}\prod_{l=1}^{l=d}\tilde{K}_{n_l}(S_{N_l}-S_{M_l};N_l-M_l,p). \quad \square
\end{aligned}$$

Note that as $N$ goes to infinity, $N^{-n}E[J_n(1_{[1,N]}^{\circ n})^2]$ converges to $n!$, which is the usual value of the square norm of the multiple stochastic integral over $[0,1]^n$ with respect to a continuous time normal martingale. We also obtained the relation

$$J_n(1_{[1,N]}^{\circ n}) = \tilde{K}_n(S_N;N,p) = n!\sum_{1\leq i_1<\cdots<i_n\leq N}\prod_{k=1}^{k=n}(X_{i_k}-p) = \sum_{1\leq i_1\neq\cdots\neq i_n\leq N}\prod_{k=1}^{k=n}(X_{i_k}-p),$$

see §V-9-3 of [4] for the symmetric case $p=q=1/2$.

# 5 Iterated stochastic summation with respect to the binomial process

In the usual continuous time stochastic integration with respect to a martingale $(M_t)_{t\in\mathbb{R}_+}$, the multiple stochastic integral $I_n(f_n)$ of a symmetric function of $n$ real variables is $n!$ times the iterated integral of $f_n$ over the simplex $\{0\leq t_1\leq\cdots\leq t_n\}$:

$$\frac{1}{n!}I_n(f_n) = \int_0^\infty\int_0^{t_n}\cdots\int_0^{t_2} f_n(t_1,\ldots,t_n)dM_{t_1}\cdots dM_{t_n}.$$

Given $f_n\in L^2(\mathbb{R}_+^n)$ one lets $I_n(f_n)=I_n(\tilde{f}_n)$, where $\tilde{f}_n$ denotes the symmetrization of $f_n$ in $n$ variables. Given $f_{n+1}\in L^2(\mathbb{R}_+)^{\circ n}\otimes L^2(\mathbb{R}_+)$ this implies

$$\int_0^\infty I_n(f_{n+1}(\cdot,t)1_{[0,t]^n}(\cdot)1_{\Delta_n}(\cdot))dM_t = I_{n+1}(f_{n+1}1_{\Delta_{n+1}}),$$

where $f_{n+1}(\cdot,t)1_{[0,t]^n}(\cdot)1_{\Delta_n}(\cdot)$ is the function of $n$ variables defined as

$$(t_1,\ldots,t_n)\mapsto f_{n+1}(t_1,\ldots,t_n,t)1_{[0,t]}(t_1)\cdots 1_{[0,t]}(t_n)1_{\Delta_n}(t_1,\ldots,t_n), \tag{12}$$

and $I_n(f_n)=n!I_n(f_n1_{\Delta_n})$, $f_n\in L^2(\mathbb{R}_+)^{\circ n}$. We will show that analogously, the functional $\frac{1}{n!}J_n(f_n)$ is an iterated multiple stochastic integral in discrete time with respect to the compensated binomial process $(S_n-np)_{n\in\mathbb{N}}$. We set $J_n(f_n)=J_n(\tilde{f}_n)$ if $f_n\in l^2(\mathbb{N}^n)$ is not symmetric, and let $\Delta_n=\{0\leq k_1<\cdots<k_n\}$ denote the simplex in $\mathbb{N}^n$, and let $Y_k=(X_k-p)/\sqrt{pq}$, $k\geq 1$, denote the normalised (centered with variance one) increment of $(S_n)_{n\in\mathbb{N}^*}$.

**Theorem 1** *We have for $f_{n+1} \in l^2(\mathbb{N}^*)^{\circ n} \otimes l^2(\mathbb{N}^*)$:*

$$\sum_{k=1}^{k=\infty} Y_k J_n(f_{n+1}(\cdot,k)1_{[1,k-1]^n}(\cdot)1_{\Delta_n}(\cdot)) = J_{n+1}(f_{n+1}1_{\Delta_{n+1}}), \tag{13}$$

*where $f_{n+1}(\cdot,k)1_{[1,k-1]^n}(\cdot)1_{\Delta_n}(\cdot)$, $k \geq n+1$, is defined as in (12).*

**Proof:** First we note that $J_n(f_{n+1}(\cdot,k)1_{[1,k-1]^n}(\cdot)) = 0$ if $n > k-1$, so that the summation (13) actually starts at $k = n+1$. We start by proving that

$$\sum_{k=M+1}^{k=N} (X_k - p)\tilde{K}_n(S_{k-1} - S_M; k-1-M, p) = \frac{\tilde{K}_{n+1}(S_N - S_M; N-M, p)}{n+1}, \tag{14}$$

with $\tilde{K}_n(x; N, p)$ the monic Krawtchouk polynomial of degree $n$. Using the generating function it is sufficient to prove

$$\sum_{k=M+1}^{k=N} (X_k - p)Y(S_{k-1} - S_M, k-1-M, z) = \frac{Y(S_N - S_M, N-M, z) - 1}{z},$$

This follows immediately from the fact that

$$\begin{aligned}
&\frac{Y(S_k - S_M, k-M, z) - Y(S_{k-1} - S_M, k-1-M, z)}{z} \\
&= \frac{Y(S_{k-1} - S_M, k-1-M, z)}{z}\left(\frac{(1+qz)^{X_k}}{(1-pz)^{X_k - 1}} - 1\right) \\
&= Y(S_{k-1} - S_M, k-1-M, z)(X_k - p).
\end{aligned}$$

Another way of proving (14) is to directly use the representation formula (10). From the relation

$$J_n(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ n_d}) = \prod_{k=1}^{k=d} J_{n_k}(1_{[M_k+1,N_k]}^{\circ n_k}),$$

we deduce that (13) holds for

$$f_{n+1} = 1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ n_d} \otimes 1_{[M_l+1,N_l]}.$$

For this it suffices to consider $l = d$ and to note that

$$\begin{aligned}
&\sum_{k=1}^{k=\infty} \frac{X_k - p}{\sqrt{pq}} J_n(f_{n+1}(\cdot,k)1_{[1,k-1]^n}(\cdot)1_{\Delta_n}(\cdot)) \\
&= \sum_{k=1}^{k=\infty} \frac{1}{n!}\frac{X_k - p}{\sqrt{pq}} J_n(f_{n+1}(\cdot,k)1_{[1,k-1]^n}(\cdot)) \\
&= \frac{1}{n!}\sum_{k=M_d+1}^{k=N_d} \frac{X_k - p}{\sqrt{pq}} J_n(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,k-1]}^{\circ n_d}) \\
&= \frac{1}{n!}(pq)^{-(n+1)/2} \prod_{k=1}^{k=d-1} \tilde{K}_{n_k}(S_{N_k} - S_{M_k}; N_k - M_k, p)
\end{aligned}$$

$$
\begin{aligned}
& \times \sum_{k=M_d+1}^{k=N_d} \tilde{K}_{n_d}(S_{k-1}-S_{M_d}; k-1-M_d, p)(X_k-p) \\
= \; & \frac{(pq)^{-(n+1)/2}}{n!(n_d+1)} \tilde{K}_{n_d+1}(S_{N_d}-S_{M_d}; N_d-M_d, p) \prod_{k=1}^{k=d-1} \tilde{K}_{n_k}(S_{N_k}-S_{M_k}; N_k-M_k, p) \\
= \; & \frac{1}{n!(n_d+1)} J_{n+1}(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ(n_d+1)}).
\end{aligned}
$$

Now, for $0 \leq k_1 < \cdots < k_{n+1}$ we have

$$
\begin{aligned}
& \left(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ(n_d+1)}\right)(k_1, \ldots, k_{n+1}) \\
& \quad = \frac{n!(n_d+1)}{(n+1)!} \left(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ n_d} \otimes 1_{[M_d+1,N_d]}\right)(k_1, \ldots, k_{n+1}) \\
& \quad = \frac{n!(n_d+1)}{(n+1)!} f_{n+1}(k_1, \ldots, k_{n+1}) \\
& \quad = \frac{n!(n_d+1)}{(n+1)!} f_{n+1}(k_1, \ldots, k_{n+1}) 1_{\Delta_{n+1}}(k_1, \ldots, k_{n+1}).
\end{aligned}
$$

This shows that $1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ(n_d+1)}$ is $n!(n_d+1)$ times the symmetrization of $f_{n+1} 1_{\Delta_{n+1}}$. Hence

$$
\sum_{k=1}^{k=\infty} \frac{1}{n!} \frac{X_k - p}{\sqrt{pq}} J_n(f_{n+1}(\cdot, k) 1_{[1,k-1]^n}(\cdot)) = J_{n+1}(f_{n+1} 1_{\Delta_{n+1}}).
$$

Finally from (4), by linearity and density, Relation (13) holds for all $f_{n+1} \in l^2(\mathbb{N}^*)^{\circ n} \otimes l^2(\mathbb{N}^*)$. $\square$

The interpretation of this result is that the Krawtchouk polynomials are the stochastic counterparts of the usual powers $(S_N - Np)^n = (\tilde{K}_1(S_N; N, p))^n$, $n \geq 0$ for the compensated binomial process $\{S_n - np, n \in \mathbb{N}\}$. Also we found that the role of the classical exponential function, now is taken by $Y(S_n, n, 1) = (1+q)^{S_n} q^{n-S_n}$ because of the relation

$$
\sum_{i=1}^{i=n} \left(\frac{1+q}{q}\right)^{S_{i-1}} q^{i-1}(X_i - p) = \left(\frac{1+q}{q}\right)^{S_n} q^n - 1.
$$

Furthermore there is a striking similarity with integration with respect to Brownian motion and Hermite polynomials on the one hand and the Poisson process and the Charlier polynomials on the other hand.

# 6 Gradient, divergence and Clark formula

In this section we introduce gradient and divergence operators, and obtain a Clark formula for the functionals of $(S_n)_{n\geq 1}$. We use the convention $1_{[N,M]} = 0$ if $M < N$. Let $\mathcal{P}$ denote the set of polynomials in $X_1, X_2, X_3, \ldots$, and let $\mathcal{U}$ denote the space of discrete-time processes $(u(k))_{k\geq 1}$, with finite support in $k \geq 1$ and such that $u(k) \in \mathcal{P}$, $k \geq 1$. The space $\mathcal{P}$ is clearly dense in $L^2(\Omega, P)$, hence the process $(S_n)_{n\geq 1}$ has the chaos representation property, i.e. any $F \in L^2(\Omega, P)$ can be represented as a series of multiple stochastic integrals:

$$F = \sum_{n=0}^{\infty} J_n(f_n), \quad f_k \in l^2(\mathbb{N}^*)^{\circ k}, \; k \in \mathbb{N}^*,$$

with $J_0(f_0) = E[F]$.

**Definition 3** *We define the gradient operator $D : L^2(\Omega) \longrightarrow L^2(\Omega \times \mathbb{N}^*)$ on $\mathcal{P}$ as*

$$D_k F = \sum_{i=1}^{i=d} n_i 1_{[M_i+n_i,N_i]}(k) J_{n-1}(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_i+1,N_i]}^{\circ n_i - 1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ n_d})$$

*with $F = J_n(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ n_d})$.*

We have in particular

$$D_k J_n(1_{[M+1,N]}^{\circ n}) = n 1_{[M+n,N]}(k) J_{n-1}(1_{[M+1,N]}^{\circ(n-1)}), \quad k \in \mathbb{N}^*.$$

**Definition 4** *The divergence operator $\delta : L^2(\Omega \times \mathbb{N}^*) \longrightarrow L^2(\Omega)$ is defined on $\mathcal{U}$ as*

$$\delta(J_n(f_{n+1}(*, \cdot))) = J_{n+1}(\tilde{f}_{n+1}) = J_{n+1}(f_{n+1}),$$

*$f_{n+1} \in l^2(\mathbb{N}^*)^n \otimes l^2(\mathbb{N}^*)$.*

We have in particular

$$\delta(u) = J_{n+1}(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_l+1,N_l]}^{\circ n_l + 1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ n_d})$$

with $u(k) = J_n(1_{[M_1+1,N_1]}^{\circ n_1} \circ \cdots \circ 1_{[M_d+1,N_d]}^{\circ n_d}) 1_{[M_l+1,N_l]}(k)$. Let $\mathcal{F}_k$ denote the $\sigma$-field generated by $X_1, \ldots, X_k$.

**Proposition 5** *Let $(u(k))_{k\geq 1}$ be a predictable square-integrable process, i.e. $u(k)$ is $\mathcal{F}_{k-1}$-measurable, $k \geq 1$, and $E[\|u\|_{l^2(\mathbb{N}^*)}^2] < \infty$. Then $\delta(u)$ coincides with the discrete time stochastic integral with respect to $(S_n)_{n\geq 1}$:*

$$\delta(u) = \sum_{k=1}^{\infty} Y_k u(k),$$

*with $E[\delta(u)^2] = E[\|u\|_{l^2(\mathbb{N}^*)}^2]$.*

**Proof.** Given $f_{n+1} \in l^2(\mathbb{N})^{\circ n} \otimes l^2(\mathbb{N}^*)$ and $u(k) = J_n(f_{n+1}(\cdot, k))$, $k \geq 1$, the predictability condition means that $f_{n+1}(\cdot, k) = f_{n+1}(\cdot, k) 1_{[1,k-1]^n}(\cdot)$, hence the symmetrization of $f_{n+1}$ is $n!$ times the symmetrization of $f_{n+1} 1_{\Delta_{n+1}}$. Thus from Th. 1 we have,

$$\begin{aligned}
\delta(J_n(f_{n+1}(*, \cdot))) &= J_{n+1}(f_{n+1}) = n! J_{n+1}(f_{n+1} 1_{\Delta_{n+1}}) \\
&= n! \sum_{k=1}^{\infty} Y_k J_n(f_{n+1}(\cdot) 1_{[1,k-1]^n}(\cdot) 1_{\Delta_n}(\cdot)) \\
&= \sum_{k=1}^{\infty} Y_k J_n(f_{n+1}(\cdot) 1_{[1,k-1]^n}(\cdot)) = \sum_{k=1}^{\infty} Y_k u(k). \quad \square
\end{aligned}$$

The truncation by the function $1_{[n,N]}(k)$ in

$$D_k J_n(1_{[1,N]}^{\circ n}) = n 1_{[n,N]}(k) J_{n-1}(1_{[1,N]}^{\circ(n-1)}),$$

is not present in continuous time. The following proposition shows that it disappears after taking the predictable projection of the gradient process.

**Proposition 6** *Let $f_n \in l^2(\mathbb{N})^{\circ n}$. We have*

$$E[D_k J_n(f_n) \mid \mathcal{F}_{k-1}] = n J_{n-1}(f_n(\cdot, k) 1_{[1,k-1]^{n-1}}(\cdot)), \quad k \in \mathbb{N}^*.$$

**Proof.** It is sufficient to note that if $M < k \leq N$,

$$\begin{aligned}
E[D_k J_n(1_{[M+1,N]}^{\circ n}) \mid \mathcal{F}_{k-1}] &= n 1_{[M+n,N]}(k) E[J_{n-1}(1_{[M+1,N]}^{\circ(n-1)}) \mid \mathcal{F}_{k-1}] \\
&= n 1_{[M+n,N]}(k) J_{n-1}(1_{[M+1,k-1]}^{\circ(n-1)}) \\
&= n J_{n-1}(1_{[M+1,k-1]}^{\circ(n-1)}),
\end{aligned}$$

since $J_{n-1}(1_{[M+1,N\wedge(k-1)]}^{\circ(n-1)}) = 0$ if $n > k - M$. If $k > N$ or $k \leq M$ we have $0 = E[D_k J_n(1_{[M+1,N]}^{\circ n}) \mid \mathcal{F}_{k-1}] = n 1_{[M+1,N]}(k) J_{n-1}(1_{[M+1,N]}^{\circ(n-1)})$. $\square$

The following proposition shows that $D : L^2(\Omega) \longrightarrow L^2(\Omega \times \mathbb{N}^*)$ and $\delta : L^2(\Omega \times \mathbb{N}^*) \longrightarrow L^2(\Omega)$ are mutually adjoint.

**Proposition 7** *We have*

$$E[(DF, u)_{l^2(\mathbb{N}^*)}] = E[F \delta(u)], \quad u \in \mathcal{U}, \ F \in \mathcal{P},$$

*and $D : L^2(\Omega) \longrightarrow L^2(\Omega \times \mathbb{N}^*)$, $\delta : L^2(\Omega \times \mathbb{N}^*) \longrightarrow L^2(\Omega)$ closable.*

**Proof.** It suffices to consider $F = J_n(1_{[M+1,N]}^{\circ n})$ and $u(k) = J_m(1_{[M'+1,N']}^{\circ m}) 1_{[M'+1,N']}(k)$, with $M' \leq M < N' \leq N$. We have if $n = m + 1$:

$$\begin{aligned}
E[(DF, u)_{l^2(\mathbb{N}^*)}] &= n \sum_{k=1}^{\infty} 1_{[M+n,N']}(k) E[J_{n-1}(1_{[M+1,N]}^{\circ(n-1)}) J_m(1_{[M'+1,N']}^{\circ m})] \\
&= n(N' - M - m) m! \frac{(N' - M)!}{(N' - M - m)!} = (m+1)! \frac{(N' - M)!}{(N' - M - m - 1)!} \\
&= E[J_n(1_{[M+1,N]}^{\circ n}) J_{m+1}(1_{[M'+1,N']}^{\circ(m+1)})] = E[F \delta(u)].
\end{aligned}$$

If $n \neq m+1$ then $0 = E[(DF, u)_{l^2(\mathbb{N}^*)}] = E[F\delta(u)]$. The closability of $D$ and $\delta$ is a consequence of the duality formula and of the density of $\mathcal{P}$ and $\mathcal{U}$ in $L^2(\Omega)$ and $L^2(\Omega\times\mathbb{N}^*)$ respectively. $\square$

The normalised increment $Y_i$ of $(S_n)_{n\in\mathbb{N}^*}$ satisfies the structure equation

$$Y_i^2 = 1 + \varphi Y_i, \quad i \geq 1, \quad \text{with} \quad \varphi = \frac{q-p}{\sqrt{pq}},$$

see §II-2-1 of [9] (1993), and [2] (1990). This implies in particular the following elementary product formula for single stochastic integrals:

$$J_1(f)J_1(g) = J_2(f\circ g) + (f,g)_{l^2(\mathbb{N}^*)} + \varphi J_1(fg), \tag{15}$$

for $f, g \in l^2(\mathbb{N}^*)$ such that $fg \in l^2(\mathbb{N}^*)$. We also have

$$\begin{aligned} & J_n(1_{[M+1,N]}^{\circ n})J_1(1_{[M+1,N]}) \\ & = \quad J_{n+1}(1_{[M+1,N]}^{\circ(n+1)}) + n(N-M-n+1)J_{n-1}(1_{[M+1,N]}^{\circ(n-1)}) + \varphi n J_n(1_{[M+1,N]}^{\circ n}), \end{aligned}$$

from the three term recurrence relation for Krawtchouk polynomials, see e.g. [7]:

$$\begin{aligned} & \tilde{K}_1(S_N - S_M; N-M, p)\tilde{K}_n(S_N - S_M; N-M, p) = \tilde{K}_{n+1}(S_N - S_M; N-M, p) \\ & \quad + npq(N-M-n+1)\tilde{K}_{n-1}(S_N - S_M; N-M, p) + n(q-p)\tilde{K}_n(S_N - S_M; N-M, p). \end{aligned}$$

The operator $D$ and $\delta$ do not satisfy the same product rules as in the continuous time case (cf. Prop. 1.3 of [11]), since we have in general:

$$D_k(FG) \neq FD_kG + GD_kF + \varphi D_kFD_kG, \quad k \in \mathbb{N}^*, \tag{16}$$

and

$$F\delta(u) \neq \delta(Fu) + (DF, u)_{l^2(\mathbb{N}^*)} + \delta(\varphi u DF), \quad F \in \mathcal{P},\ u \in \mathcal{U}. \tag{17}$$

The latter inequality expresses the fact that there is no explicit formula for the product $\tilde{K}_n(x; N, p)\tilde{K}_m(x; N, p)$, $n, m > 1$. In fact $D_k$ is not the operator $a_k^-$ that acts by removal of an eventual jump at time $k$ as

$$a_k^- \tilde{K}_n(S_N; p, N) = \tilde{K}_n(S_N - 1_{\{X_k=1\}}; p, N),$$

cf. §II-2-2 of [9], since we have $D_k\tilde{K}_n(S_N; p, N) = n1_{[n,N]}(k)\tilde{K}_{n-1}(S_N; p, N)$. The next result is the predictable representation of the functionals of $(S_n)_{n\geq 1}$.

**Proposition 8** *We have the Clark formula*

$$F = E[F] + \sum_{k=1}^{\infty} E[D_kF \mid \mathcal{F}_{k-1}]Y_k = E[F] + \delta(E[D.F \mid \mathcal{F}_{.-1}]), \quad F \in L^2(\Omega).$$

**Proof.** For $F = J_n(f_n)$ we have

$$E[D_k J_n(f_n) \mid \mathcal{F}_{k-1}] = nJ_{n-1}(f_n(\cdot,k)1_{[1,k-1]^{n-1}}) = n!J_{n-1}(f_n(\cdot,k)1_{[1,k-1]^{n-1}}(\cdot)1_{\Delta_{n-1}}(\cdot)).$$

We apply Th. 1:

$$\begin{aligned} F &= J_n(f_n) = n!J_n(f_n 1_{\Delta_n}) = n!\sum_{k=1}^{\infty} J_{n-1}(f_n(\cdot,k)1_{[1,k-1]^{n-1}}(\cdot)1_{\Delta_{n-1}}(\cdot))Y_k \\ &= n\sum_{k=1}^{\infty} J_{n-1}(f_n(\cdot,k)1_{[1,k-1]^{n-1}}(\cdot))Y_k = \sum_{k=1}^{\infty} E[D_k J_n(f_n) \mid \mathcal{F}_{k-1}]Y_k. \end{aligned}$$

Next we apply Prop. 5 to the predictable process $u = (E[D_k F \mid \mathcal{F}_{k-1}])_{k\geq 1}$:

$$F = \sum_{k=1}^{\infty} E[D_k J_n(f_n) \mid \mathcal{F}_{k-1}]Y_k = \delta(E[D_\cdot F \mid \mathcal{F}_{\cdot-1}]), \quad F \in \mathcal{P}.$$

This identity also shows that $F \mapsto E[D_\cdot F \mid \mathcal{F}_{\cdot-1}]$ has norm bounded by one as an operator from $L^2(\Omega)$ into $L^2(\Omega\times\mathbb{N}^*)$:

$$\|E[D_\cdot F \mid \mathcal{F}_{\cdot-1}]\|^2_{L^2(\Omega\times\mathbb{N}^*)} = \|F - E[F]\|^2_{L^2(\Omega)} \leq \|F - E[F]\|^2_{L^2(\Omega)} + E[F]^2 \leq \|F\|^2_{L^2(\Omega)},$$

hence the Clark formula extends to $F \in L^2(\Omega)$. □

A generalisation consists in replacing the constant $\varphi$ in the structure equation $Y_k^2 = 1+\varphi Y_k$ by a deterministic function $\varphi : \mathbb{N}^* \longrightarrow \mathbb{R}$ and considering the solution of the structure equation

$$Z_i^2 = 1 + \varphi_i Z_i, \quad i \in \mathbb{N}^*,$$

i.e.

$$Z_i = \sqrt{\varphi_i^2+4}\left(X_i + \frac{-1+\varphi_i/\sqrt{\varphi_i^2+4}}{2}\right), \quad i \geq 1.$$

The process $(Z_1 + \cdots + Z_n)_{n\geq 1}$ will be a martingale under $P$ if and only if

$$\frac{-1+\varphi_i/\sqrt{\varphi_i^2+4}}{2} = -p,$$

i.e. $\varphi = (q-p)/\sqrt{qp}$. In fact it is a Markov chain with transition matrix

$$P(i,i) = \frac{1}{2} + \frac{\varphi_i}{2\sqrt{\varphi_i^2+4}} = q_i, \qquad P(i,i+1) = \frac{1}{2} - \frac{\varphi_i}{2\sqrt{\varphi_i^2+4}} = p_i, \quad i \geq 1.$$

**Remark:** The results of Sects. 5 and 6 can be generalised by replacing $(Y_1+\cdots+Y_n)_{n\geq 1}$ by the process $(Z_1+\cdots+Z_n)_{n\geq 1}$ solution of $Z_i^2 = 1+\varphi_i Z_i$, $i \in \mathbb{N}^*$, except for the fact that $J_n(1^{\circ n}_{[1,N]})$ is not a (polynomial) functional of $S_N$ if $\varphi_i$ is dependent on $i$ since $J_n(1^{\circ n}_{[1,N]})$ depends not only on the number of jumps from 1 to $N$ but also on their positions.

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## References

[1] Ph. Biane. Chaotic representations for finite Markov chains. *Stochastics and Stochastics Reports*, 30:61–68, 1989.

[2] M. Emery. On the Azéma martingales. In *Séminaire de Probabilités XXIII*, volume 1372 of *Lecture Notes in Mathematics*, pages 66–87. Springer Verlag, 1990.

[3] D.D. Engel. *The Multiple Stochastic Integral*, volume 265 of *Memoirs of the American Mathematical Society.* AMS, Providence, 1982.

[4] P. Feinsilver and R. Schott. *Algebraic Structures and Operator Calculus, Vol. I: Representations and Probability Theory*, volume 241 of *Mathematics and Its Applications.* Kluwer Academic Publishers, Dordrecht, 1993.

[5] Ph. Feinsilver. Some classes of orthogonal polynomials associated with martingales. *Proc. Amer. Math. Soc.*, 98(2):298–302, 1986.

[6] Ph. Feinsilver. Bernoulli fields. In *Quantum probability and applications, IV (Rome, 1987)*, pages 158–181. Springer, Berlin, 1989.

[7] R. Koekoek and R.F. Swarttouw. The askey-scheme of hypergeometric orthogonal polynomials and its $q$-analogue. Delft University of Technology, Report 98-17, 1998.

[8] J.P. Kroeker. Wiener analysis of functionals of a Markov chain: Application to neural transformations of random signals. *Biol. Cybernetics*, 36:243–248, 1980.

[9] P.A. Meyer. *Quantum Probability for Probabilists*, volume 1538 of *Lecture Notes in Mathematics.* Springer-Verlag, 1993.

[10] H. Ogura. Orthogonal functionals of the Poisson process. *IEEE Transactions on Information Theory*, IT-18(4):474–481, 1972.

[11] N. Privault. Independence of a class of multiple stochastic integrals. In *Seminar on Stochastic Analysis, Random Fields and Applications*, pages 249–259, Ascona, 1996.

[12] N. Privault, J.L. Solé, and J. Vives. Chaotic Kabanov formula for the Azéma martingales. Preprint No 367, Centre de Recerca Matemàtica, Barcelona, 1997, to appear in Bernoulli.

[13] W. Schoutens. Stochastic processes in the askey scheme. Doctoral Thesis. K.U. Leuven, Leuven, 1999.

[14] W. Schoutens and J.L. Teugels. Lévy processes, polynomials and martingales. *Commun. Statist. and Stochastic Models*, 14:335–349, 1998.