The Travel Time in Carousel Systems under the Nearest Item Heuristic

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Abstract

A carousel is an automated warehousing system consisting of a large number of drawers rotating in a closed loop. In this paper we study the travel time needed to pick a list of items when the carousel operates under the Nearest Item heuristic. We find a closed form expression for all moments and the distribution of the travel time. We also analyze the asymptotic behavior of the travel time when the number of items tends to infinity. All results follow from probabilistic arguments based on properties of uniform order statistics.

1 Introduction

A carousel is an automated warehousing system consisting of a large number of drawers rotating in a closed loop in either direction. Such systems are mostly used for storage and retrieval of small and medium sized goods, which are requested moderately often. The picker has a fixed position in front of the carousel, which rotates the required items to the picker. For a recent review of literature on carousels, as part of a general overview of planning and control of warehousing systems, we refer to Van den Berg [2].

An important performance characteristic is the total time needed to pick a list of items. This consists of the pure pick time plus the travel time. Only the latter depends on the pick strategy. In this paper we consider the Nearest Item (NI) heuristic, where the next item to be picked is always the nearest one. We will study the statistical properties of the travel time under this heuristic.

We model the carousel as a circle of length 1. For ease of presentation, we act as if the picker travels to the items, instead of the other way around. The picker travels at unit speed and has to pick $n(> 0)$ items under the NI heuristic. Their positions are uniformly distributed on the circle. Using probabilistic arguments based on properties of uniform order statistics we derive closed form expressions for all moments and the distribution of the travel time needed to pick n items. We also investigate the asymptotic behavior of the travel time as n tends to infinity.

The performance of the NI heuristic has also been investigated by Bartoldi and Platzman [1]. They prove that the travel time under the NI heuristic is never greater than one rotation of the carousel. Litvak *et al.* [5] improve this upper bound and show that the new upper bound is tight. Using an analytical approach, they find the mean and variance of the remaining travel time under the NI heuristic, i.e., the travel time, when there is an empty space at one side of the picker's position. They also introduce a probabilistic approach to determine the mean total travel time. In fact, in the present paper we elaborate this probabilistic approach and we show that it enables us to completely analyze the travel time.

The paper is organized as follows. In Section 2 we prove that the travel time under the NI heuristic can be represented as a sum of independent normalized exponential random variables. In Section 3 we use this representation to obtain all moments of the travel time. Further, in Section 4 we derive a closed form expression for the distribution of the travel time. Finally, in Section 5 we give an exhaustive analysis of the limiting behavior of the travel time distribution.

2 The travel time as a sum of exponentials

Let the random variable U_0 be the picker's starting point and the random variable U_i, where $i = 1, 2, \ldots, n$, be the position of the ith item. We suppose that the U_i's, $i = 0, 1, \ldots, n$, are independent and uniformly distributed on the interval $[0, 1)$. Let $U_{0:n+1}, U_{1:n+1}, \ldots, U_{n:n+1}$ denote the order statistics of the random variables $U_0, \ldots, U_n$ on $[0, 1)$. These order statistics partition the circle into $n+1$ spacings with lengths

$$D_1 = U_{1:n+1} - U_{0:n+1}, \quad \ldots, \quad D_n = U_{n:n+1} - U_{n-1:n+1}, \quad D_{n+1} = 1 - U_{n:n+1} + U_{0:n+1}.$$

To find the distribution of the travel time under the NI heuristic we use the following very useful property of these spacings. If $Y_1, \ldots, Y_{n+1}$ are independent exponentials with the same mean, then the random vectors $(D_1, \ldots, D_{n+1})$ and $(Y_1/\sum_{i=1}^{n+1} Y_i, \ldots, Y_{n+1}/\sum_{i=1}^{n+1} Y_i)$ are indentically distributed (cf. Pyke [6, 7], or Sec. 13.1 in Karlin and Taylor [4]). Hence the spacings are normalized exponentials.

Under the NI heuristic the picker does not have to know all spacings at once. He first considers the two spacings adjacent to his starting position and then travels to the nearest item. Next he also looks at the other spacing adjacent to that item and compares the distance to the item located at the endpoint of that spacing and the distance to the first item in the other direction, which is the sum of the spacings previously considered. Then he travels again to the nearest item, and so on. Furthermore, we may act as if the picker faces non-normalized exponential spacings, and afterwards divide the travel time by the sum of all spacings. Then it is clear that each new spacing faced by the picker is independent of the ones already observed. Now let X_i, where $i = 1, \ldots, n+1$, denote the i-th non-normalized exponential spacing faced by the picker. So the spacings are numbered

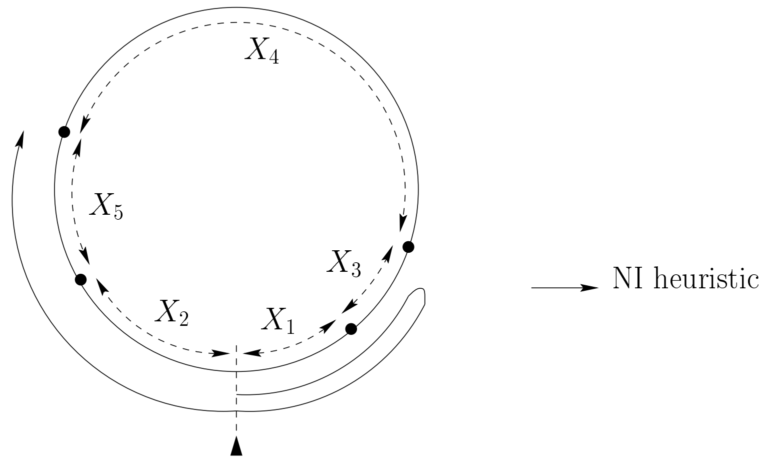

Figure 1: The NI route of the picker facing 5 exponential spacings.

as observed by the picker operating under the NI heuristic (see Fig. 1). Denote

$$S_i = \sum_{j=1}^{i} X_j, \quad i = 1, 2, \ldots$$

and let the random variable T_n denote the travel time needed to pick n items under the NI heuristic. Then T_n can be expressed as

$$T_n = \sum_{i=1}^{n} \frac{\min(S_i, X_{i+1})}{S_{n+1}} . \tag{1}$$

In this section we introduce a quite simple random variable, which has the same distribution as the right-hand side of (1). Further, we shall use the common notation

$$X \stackrel{d}{=} Y$$

to indicate that the random variables X and Y have the same distribution. Now we are going to prove the following theorem, which is crucial to the rest of the paper.

Theorem 2.1 *Let $X_1, X_2, \ldots$ be independent exponentials with mean μ. Then it holds for all $n = 1, 2, \ldots$ that*

$$\sum_{i=1}^{n} \frac{\min(S_i, X_{i+1})}{S_{n+1}} \stackrel{d}{=} \sum_{i=1}^{n} \left(1 - \frac{1}{2^i}\right) \frac{X_i}{S_{n+1}} . \tag{2}$$

Proof. Let $Y_1, Y_2, \ldots$ be independent exponentials with mean μ. We will use the Y_i's to subsequently consider all minima in the left-hand side of (2). For the first minimum, i.e., $\min(S_1, X_2) = \min(X_1, X_2)$ there are two cases. If $X_1 < X_2$, then we can put $X_1 = \frac{1}{2}Y_1$. The overshoot of X_2 is again an independent exponential with mean μ, so we can put

$X_2 = \frac{1}{2}Y_1 + Y_2$. The same arguments can be used when $X_2 < X_1$. Then we can put $X_2 = \frac{1}{2}Y_1$ and $X_1 = \frac{1}{2}Y_1 + Y_2$. Hence, in both cases we have

$$\min(S_1, X_2) = \frac{1}{2}Y_1.$$

Further, since we have not made extra assumptions about $X_3, X_4, \ldots, X_{n+1}$, we can say that $X_l = Y_l$, $l = 3, 4, \ldots, n+1$. Also, note that for any $i = 2, 3, \ldots n+1$ we have

$$\begin{aligned} S_i &= \frac{1}{2}Y_1 + \frac{1}{2}Y_1 + Y_2 + Y_3 + \cdots + Y_i \\ &= Y_1 + Y_2 + \cdots + Y_i\,. \end{aligned}$$

From the arguments above it follows that

$$\sum_{i=1}^{n} \frac{\min(S_i, X_{i+1})}{S_{n+1}} \stackrel{d}{=} \frac{1}{2}\,\frac{Y_1}{Y_1 + \cdots + Y_{n+1}} + \sum_{i=2}^{n} \frac{\min(\sum_{j=1}^{i} Y_j, Y_{i+1})}{Y_1 + \cdots + Y_{n+1}}\,. \tag{3}$$

Since $X_1, X_2, \ldots, X_{n+1}$ and $Y_1, Y_2, \ldots, Y_{n+1}$ are just two sets of independent exponentials with the same mean, expression (3) may, of course, be rewritten as

$$\sum_{i=1}^{n} \frac{\min(S_i, X_{i+1})}{S_{n+1}} \stackrel{d}{=} \frac{1}{2}\,\frac{X_1}{S_{n+1}} + \sum_{i=2}^{n} \frac{\min(S_i, X_{i+1})}{S_{n+1}}\,. \tag{4}$$

Now, let us assume that for some $i = 2, 3, \ldots, n$ it holds that

$$\sum_{i=1}^{n} \frac{\min(S_i, X_{i+1})}{S_{n+1}} \stackrel{d}{=} \sum_{j=1}^{i-1} \left(1 - \frac{1}{2^{i-j}}\right) \frac{X_j}{S_{n+1}} + \sum_{j=i}^{n} \frac{\min(S_j, X_{j+1})}{S_{n+1}}. \tag{5}$$

By virtue of (4), we know that it indeed holds for $i = 2$. Below we show, by expanding $\min(S_i, X_{i+1})$, that if equality (5) is valid for i, then it is also valid for $i+1$.

Given the event

$$E_{i+1,k} = [S_{k-1} < X_{i+1} < S_k]\,,$$

for some $k = 1, \ldots, i$, the random variables $X_1, \ldots, X_{n+1}$ can be coupled as

$$\begin{aligned} &X_l = \frac{1}{2}\,Y_l, \quad l = 1, \ldots, k-1; \qquad && X_k = \frac{1}{2}\,Y_k + Y_{k+1}; \\ &X_l = Y_{l+1}, \quad l = k+1, \ldots, i; \qquad && X_{i+1} = \sum_{l=1}^{k} \frac{1}{2}\,Y_l; \\ &X_l = Y_l, \quad l = i+2, \ldots, n+1, && \end{aligned}$$

where $Y_1, Y_2, \ldots$ are independent exponentials with mean μ. This follows by observing that, given $E_{i+1,k}$, the random variable X_1 is the minimum of X_1 and X_{i+1}, and thus it is exponential with mean $\mu/2$. Since the overshoot of X_{i+1} is again exponential with mean μ we can repeat the argument for X_2 and so on. Eventually $X_{i+1} - S_{k-1}$ is less than

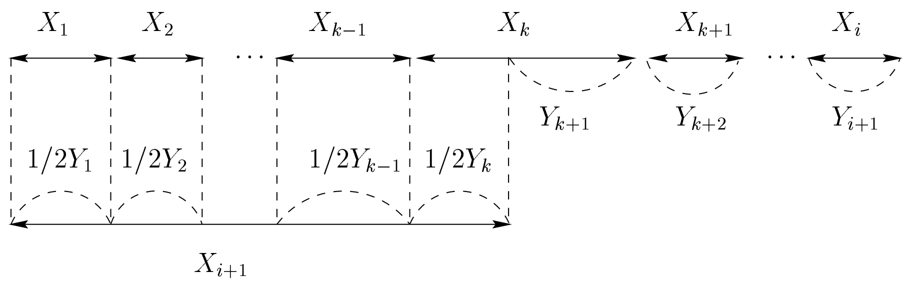

Figure 2: Coupling of the random variables $X_1, \ldots, X_{n+1}$ under event $E_{i+1,k}$.

X_k, so it is exponential with mean $\mu/2$. The random variable X_k is then the sum of two exponentials, one with mean $\mu/2$ and the other part (i.e., the overshoot) with mean μ (see also Fig. 2). Since the event $E_{i+1,k}$ does not provide any information on the other random variables, they remain exponential with mean μ.

Now, given the event $E_{i+1,k}$, it follows that

$$\min(S_i, X_{i+1}) = X_{i+1} = \sum_{l=1}^{k} \frac{1}{2} Y_l,$$

and for any $j = i+1, \ldots, n+1$ we have

$$\begin{aligned} S_j &= \frac{1}{2} Y_1 + \cdots + \frac{1}{2} Y_k + Y_{k+1} + \cdots + Y_{i+1} + \sum_{l=1}^{k} \frac{1}{2} Y_l + Y_{i+2} + \cdots + Y_j \\ &= Y_1 + \cdots + Y_j. \end{aligned}$$

Hence, given $E_{i+1,k}$, we can replace the X_j's by Y_j's in the right-hand side of (5), yielding

$$\sum_{j=1}^{i} \left(1 - \frac{1}{2^{i+1-j}}\right) \frac{Y_j}{Y_1 + \cdots + Y_{n+1}} + \sum_{j=i+1}^{n} \frac{\min(\sum_{l=1}^{j} Y_l, Y_{j+1})}{Y_1 + \cdots + Y_{n+1}}. \tag{6}$$

Note that expression (6) does not depend on k.

Along the same lines, it can be verified that, given the event $[X_{i+1} > S_i]$, the right-hand side of (5) has again the same distribution as (6). Now it immediately follows from the law of full probability that

$$\sum_{i=1}^{n} \frac{\min(S_i, X_{i+1})}{S_{n+1}} \stackrel{d}{=} \sum_{j=1}^{i} \left(1 - \frac{1}{2^{i+1-j}}\right) \frac{X_j}{S_{n+1}} + \sum_{j=i+1}^{n} \frac{\min(S_j, X_{j+1})}{S_{n+1}},$$

where the Y_i's in (6) are replaced again by X_i's. Thus, by subsequently expanding $\min(S_1, X_2)$, $\min(S_2, X_3)$, ..., $\min(S_n, X_{n+1})$ we finally obtain:

$$\sum_{i=1}^{n} \frac{\min(S_i, X_{i+1})}{S_{n+1}} \stackrel{d}{=} \sum_{j=1}^{n} \left(1 - \frac{1}{2^{n+1-j}}\right) \frac{X_j}{S_{n+1}},$$

which is exactly (2). □

3 Moments of the travel time

In this short section we shall use representation (2) to directly calculate the moments of the travel time T_n. From (2) we obtain for the k-th moment of the travel time,

$$\begin{aligned} \mathrm{E}\left[T_n^k\right] &= \mathrm{E}\left[\left(\sum_{i=1}^{n}\left(1-\frac{1}{2^i}\right)\frac{X_i}{S_{n+1}}\right)^k\right] \\ &= \sum_{\substack{k_1,k_2,\ldots,k_n\geq 0\\ k_1+k_2+\ldots+k_n=k}} \frac{k!}{k_1!k_2!\ldots k_n!}\left(\prod_{j=1}^{n}\left(1-\frac{1}{2^j}\right)^{k_j}\right)\mathrm{E}\left(\frac{X_1^{k_1}X_2^{k_2}\ldots X_n^{k_n}}{S_{n+1}^k}\right). \end{aligned} \tag{7}$$

To determine the expectation

$$\mathrm{E}\left(\frac{X_1^{k_1}X_2^{k_2}\ldots X_n^{k_n}}{S_{n+1}^k}\right),$$

first recall that the random variables $X_i/S_{n+1} = D_i$, $i = 1, 2, \ldots, n+1$ are uniform spacings. Under the condition that $D_3 = d_3, \ldots, D_{n+1} = d_{n+1}$ the random variable D_1 is uniform on the interval $[0, 1 - d_3 - \cdots - d_{n+1}]$ (cf. Sec. 13.1 in Karlin and Taylor [4]). Hence, by conditioning and partial integration we obtain

$$\mathrm{E}\left(\frac{D_1^{k_1}}{k_1!}\frac{D_2^{k_2}}{k_2!}\frac{D_3^{k_3}}{k_3!}\cdots\frac{D_{n+1}^{k_{n+1}}}{k_{n+1}!}\right) = \mathrm{E}\left(\frac{D_1^{k_1+1}}{(k_1+1)!}\frac{D_2^{k_2-1}}{(k_2-1)!}\frac{D_3^{k_3}}{k_3!}\cdots\frac{D_{n+1}^{k_{n+1}}}{k_{n+1}!}\right).$$

By symmetry and repeatedly applying this equality we find

$$\mathrm{E}\left(\frac{D_1^{k_1}}{k_1!}\frac{D_2^{k_2}}{k_2!}\cdots\frac{D_{n+1}^{k_{n+1}}}{k_{n+1}!}\right) = \mathrm{E}\left(\frac{D_1^{k_1+\cdots+k_{n+1}}}{(k_1+\cdots+k_{n+1})!}\right).$$

Using $\mathrm{E}(D_1^k/k!) = n!/(n+k)!$ and substituting $D_i = X_i/S_{n+1}$ yields

$$\mathrm{E}\left(\frac{X_1^{k_1}X_2^{k_2}\ldots X_{n+1}^{k_{n+1}}}{S_{n+1}^{k_1+k_2+\cdots+k_{n+1}}}\right) = \frac{k_1!k_2!\ldots k_{n+1}!n!}{(n+k_1+k_2+\cdots+k_{n+1})!},$$

which is valid for any collection $k_1, k_2, \ldots, k_{n+1}$ of nonnegative integers. Hence, (7) becomes

$$\mathrm{E}\left[T_n^k\right] = \binom{n+k}{k}^{-1} \sum_{\substack{k_1,k_2,\ldots,k_n\geq 0\\ k_1+k_2+\cdots+k_n=k}} \prod_{j=1}^{n}\left(1-\frac{1}{2^j}\right)^{k_j}.$$

For example, for $k = 1$ we have:

$$\mathrm{E}(T_n) = \frac{n}{n+1} - \frac{1}{n+1}\left(1-\frac{1}{2^n}\right). \tag{8}$$

This formula has already been derived in Litvak *et al.* [5].

4 The distribution of the travel time

Now we are going to determine the distribution of T_n. Due to Theorem 2.1 we have:

$$\Pr(T_n < x) = \Pr\left(\sum_{i=1}^{n}\left(1-\frac{1}{2^i}\right)X_i < xS_{n+1}\right) \tag{9}$$

So we need to find an expression for the right-hand side of (9). First note that it immediately follows from (9) that

$$\Pr(T_n < x) = 1, \quad x \geq 1-\frac{1}{2^n}, \tag{10}$$

which automatically yields the upper bound of Litvak *et al.* [5] (see Corollary 3.4). Also, we will have different expressions for the distribution function in the intervals $(0,1/2]$, $(1/2,3/4]$, …, $(1-1/2^{n-1}, 1-1/2^n]$. Namely, for $1-1/2^{k-1} < x \leq 1-1/2^k$, where $k=1,2,\ldots,n$, we have

$$\begin{aligned}&\Pr\left(\sum_{i=1}^{n}\left(1-\frac{1}{2^i}\right)X_i < xS_{n+1}\right)\\ &\quad = \Pr\left(\sum_{j=k}^{n}\left(1-\frac{1}{2^j}-x\right)X_j < \sum_{j=1}^{k-1}\left(x-1+\frac{1}{2^j}\right)X_j + xX_{n+1}\right).\end{aligned} \tag{11}$$

Equation (11) can be rewritten as

$$\begin{aligned}\Pr(T_n < x) \quad &= \quad \Pr\left(\sum_{l=1}^{n+1-k}\left(1-\frac{1}{2^{k-1+l}}-x\right)X_l < \sum_{j=1}^{k}\left(x-\left(1-\frac{1}{2^{j-1}}\right)\right)Y_j\right),\\ &\quad 1-1/2^{k-1} < x \leq 1-1/2^k,\end{aligned} \tag{12}$$

where $X_1, X_2, \ldots, Y_1, Y_2, \ldots$ are again independent exponentials with the same mean. To obtain a closed-form expression for $\Pr(T_n < x)$ we need the following lemma.

Lemma 4.1 *Let $X_1, X_2, \ldots, Y_1, Y_2, \ldots$ be independent exponentials with the same mean, and let $a_1, a_2, \ldots, b_1, b_2, \ldots$ be positive numbers. Then for any $M, N > 0$ we have*

$$\begin{aligned}&\Pr\left(\sum_{j=1}^{M} a_jX_j > \sum_{l=1}^{N} b_lY_l\right)\\ &\quad = \sum_{m=1}^{M}\ \sum_{0=k_0\leq k_1\leq\ldots\leq k_{m-1}<k_m=N}\left(\prod_{j=1}^{m}\prod_{l=k_{j-1}+1}^{k_j}\frac{a_j}{a_j+b_l}\right)\left(\prod_{j=1}^{m-1}\frac{b_{k_j+1}}{a_j+b_{k_j+1}}\right).\end{aligned} \tag{13}$$

Proof. The proof is based on the memoryless property of the exponential distribution. Let us say that a_jX_j 'beats' b_lY_l, if a_jX_j is greater than b_lY_l. Now consider a_1X_1 'competing' with b_1Y_1. If a_1X_1 wins (which happens with probability $a_1/(a_1+b_1)$), then the overshoot of a_1X_1 has again the same distribution as a_1X_1, and it will compete with b_2Y_2. If, on the other hand, b_1Y_1 wins (with probability $b_1/(a_1+b_1)$), then the overshoot of b_1Y_1 is distributed as b_1Y_1, and it will compete with a_2X_2. Formally,

$$\begin{aligned}\Pr\left(\sum_{j=1}^{M} a_jX_j > \sum_{l=1}^{N} b_lY_l\right) &= \frac{a_1}{a_1+b_1}\Pr\left(\sum_{j=1}^{M} a_jX_j > \sum_{l=1}^{N} b_lY_l \,|\, a_1X_1 > b_1Y_1\right)\\ &+\frac{b_1}{a_1+b_1}\Pr\left(\sum_{j=1}^{M} a_jX_j > \sum_{l=1}^{N} b_lY_l \,|\, a_1X_1 < b_1Y_1\right)\\ &= \frac{a_1}{a_1+b_1}\Pr\left(\sum_{j=1}^{M} a_jX_j > \sum_{l=2}^{N} b_lY_l\right)+\frac{b_1}{a_1+b_1}\Pr\left(\sum_{j=2}^{M} a_jX_j > \sum_{l=1}^{N} b_lY_l\right).\end{aligned}$$

Now we can repeat the arguments to reduce the two sums in the right-hand side. This results in the following process: a_1X_1 beats the first k_1 terms of $\sum_{l=1}^{N} b_lY_l$, where k_1 may be zero. If $k_1 < N$, then it means that $b_{k_1+1}Y_{k_1+1}$ beats a_1X_1 and proceeds to compete with a_2X_2. Then, a_2X_2 beats the following k_2-k_1 terms of $\sum_{l=k_1+1}^{N} b_lY_l$, where k_2-k_1 may again be zero. If $k_2 < N$, then it means that $b_{k_2+1}Y_{k_2+1}$ beats a_2X_2 and further competes with a_3X_3, and so on. Now the 'X-player' wins if eventually, a_mX_m beats b_NY_N for some $m \le M$. Note that $k_m - k_{m-1} > 0$, because a_mX_m clearly wins at least once. Formally applying the law of full probability, as shown above for a_1X_1 and b_1Y_1, we obtain formula (13). □

Now we can derive closed-form expressions for the distribution function of T_n in the sequence of intervals $(0,1/2], (1/2,3/4], \ldots, (1-1/2^{n-1}, 1-1/2^n]$. The expression in $(0,1/2]$ is the simplest one:

$$\Pr(T_n < x) = 2\cdot\frac{4}{3}\cdots\frac{2^n}{2^n-1}\,x^n, \qquad 0 < x \le 1/2.$$

In the next interval we get one additional term:

$$\Pr(T_n < x) = 2\cdot\frac{4}{3}\cdots\frac{2^n}{2^n-1}\,x^n - 2\cdot\frac{4}{3}\cdots\frac{2^{n-1}}{2^{n-1}-1}\,(2x-1)^n, \qquad 1/2 < x \le 3/4.$$

In fact an extra term appears in each of the following intervals. This is formulated in the theorem below.

Theorem 4.2 *For all $n = 1,2,\ldots$ it holds that*

$$\begin{aligned}\Pr(T_n < x) &= \sum_{k=0}^{n-1}(-1)^k\left(\prod_{i=1}^{k}\frac{1}{2^k-1}\right)c_{n-k}\,\left(2^kx-2^k+1\right)^n\,\mathbf{1}_{\{x>1-\frac{1}{2^k}\}}, \\ &\quad 0 < x \le 1-1/2^n,\end{aligned} \tag{14}$$

where the coefficients c_m are defined as

$$c_m = 2\cdot\frac{4}{3}\cdots\frac{2^m}{2^m-1}, \qquad m \ge 1. \tag{15}$$

Proof. We first combine (12) and (13) to obtain

$$\begin{aligned}
&\Pr(T_n < x)\\
&= \sum_{m=1}^{k} \sum_{0=k_0\le\ldots\le k_{m-1}<k_m=n+1-k} \left(\prod_{j=1}^{m} \left(2^{j-1}x - (2^{j-1}-1)\right)^{k_j-k_{j-1}} \prod_{l=k_{j-1}+1}^{k_j} \frac{2^{k+l-j}}{2^{k+l-j}-1} \right)\\
&\times \prod_{j=1}^{m-1} \left(1 - \frac{2^{k+k_j-j+1}}{2^{k+k_j-j+1}-1} \left(2^{j-1}x - (2^{j-1}-1)\right)\right), \; 1-1/2^{k-1} \le x < 1-1/2^k. \quad (16)
\end{aligned}$$

Putting $k = 1$ in (16) gives

$$\Pr(T_n < x) \;=\; c_n\, x^n, \qquad 0 < x \le 1/2,$$

which coincides with (14). Let us show that for $1/2 < x \le 3/4$ the formulas (16) and (14) are again the same. For $k = 2$ formula (16) gives

$$\begin{aligned}
\Pr(T_n < x) \;&=\; \frac{4}{3}\cdot\frac{8}{7}\cdots\frac{2^n}{2^n-1}\, x^{n-1}\\
&+\; \frac{4}{3}\cdot\frac{8}{7}\cdots\frac{2^{n-1}}{2^{n-1}-1}\, x^{n-2} \left(1 - \frac{2^n}{2^n-1}\, x\right) \frac{2^{n-1}}{2^{n-1}-1}\,(2x-1)\\
&+\; \frac{4}{3}\cdot\frac{8}{7}\cdots\frac{2^{n-2}}{2^{n-2}-1}\, x^{n-3} \left(1 - \frac{2^{n-1}}{2^{n-1}-1}\, x\right) \frac{2^{n-2}}{2^{n-2}-1}\cdot\frac{2^{n-1}}{2^{n-1}-1}\,(2x-1)^2\\
&+\; \cdots + \left(1 - \frac{4}{3}\, x\right) c_{n-1}\,(2x-1)^{n-1}. \qquad (17)
\end{aligned}$$

The first term in the right-hand side of (17) can be rewritten as

$$\frac{4}{3}\cdot\frac{8}{7}\cdots\frac{2^n}{2^n-1}\, x^{n-1} = c_n\, x^n - \frac{4}{3}\cdot\frac{8}{7}\cdots\frac{2^n}{2^n-1}\, x^{n-1}\,(2x-1).$$

Now adding it to the second term gives

$$\begin{aligned}
&c_n\, x^n - \frac{4}{3}\cdot\frac{8}{7}\cdots\frac{2^{n-1}}{2^{n-1}-1}\, x^{n-2}\,(2x-1) \left\{ \frac{2^n}{2^n-1}\, x - \left(1 - \frac{2^n}{2^n-1}\, x\right) \frac{2^{n-1}}{2^{n-1}-1} \right\}\\
&\quad = c_n\, x^n - \frac{4}{3}\cdot\frac{8}{7}\cdots\frac{2^{n-1}}{2^{n-1}-1}\, x^{n-2}\, \frac{2^{n-1}}{2^{n-1}-1}\,(2x-1)^2,
\end{aligned}$$

since

$$\frac{2^l}{2^l-1}\, x - \left(1 - \frac{2^l}{2^l-1}\, x\right) \frac{2^{l-1}}{2^{l-1}-1} = \frac{2^{l-1}}{2^{l-1}-1}\,(2x-1), \qquad l > 1. \qquad (18)$$

Subsequently adding the terms in the right-hand side of (17) and using (18) we finally obtain:

$$\Pr(T_n < x) = c_n\, x^n - c_{n-1}\,(2x-1)^n, \qquad 1/2 < x \le 3/4.$$

Similarly we can rewrite (16) for arbitrary $k = 1, 2, \dots, n$. To do this we use the equalities

$$\frac{2^m}{2^m-1}\cdot\frac{1}{2^l-1}\left(2^jx-2^j+1\right)-\frac{2^{m-l}}{2^{m-l}-1}\left(1-\frac{2^m}{2^m-1}\left(2^jx-2^j+1\right)\right)$$
$$=\frac{2^{m-l}}{2^{m-l}-1}\cdot\frac{1}{2^l-1}\left(2^{j+l}x-2^{j+l}+1\right),\qquad m>l\geq 1,\quad j\geq 0,\tag{19}$$

and

$$c_m\left(2^jx-2^j+1\right)^l-c_{m-1}\left(2^{j+1}x-2^{j+1}+1\right)^l+\frac{1}{3}\,c_{m-2}\left(2^{j+2}x-2^{j+2}+1\right)^l$$
$$-\cdots+(-1)^m\left(\prod_{i=1}^{m}\frac{1}{2^m-1}\right)c_0\left(2^{j+m}x-2^{j+m}+1\right)^l=1,\tag{20}$$
$$m\geq l\geq 0,\quad j\geq 0,$$

where $c_0 = 1$ by convention. Equality (19) is a generalization of (18) and it can be checked directly. Equality (20) can be established in the following way. Let us consider the function

$$f(s,x)=\frac{1}{1-s}\cdot\frac{2}{2-s}\cdots\frac{2^m}{2^m-s}\left(s\,2^jx-s\,2^j+1\right)^l.$$

For $l \leq m$ it has the following expansion in rational fractions of s:

$$\begin{aligned}f(x,s)\;&=\;c_m\left(2^jx-2^j+1\right)^l\frac{1}{1-s}-c_{m-1}\left(2\cdot 2^jx-2\cdot 2^j+1\right)^l\frac{2}{2-s}\\&+\;\frac{1}{3}\,c_{m-2}\left(4\cdot 2^jx-4\cdot 2^j+1\right)^l\frac{4}{4-s}\\&-\;\cdots+(-1)^m\left(\prod_{i=1}^{m}\frac{1}{2^m-1}\right)c_0\left(2^m\cdot 2^jx-2^m\cdot 2^j+1\right)^l\frac{2^m}{2^m-s},\\&\quad m\geq l\geq 0,\quad j\geq 0.\end{aligned}$$

Putting $s = 0$ we get (20).

Combining the terms of (16), starting with the ones containing powers of $(2^{k-1}x - 2^{k-1} + 1)$, we subsequently apply (20) and then (19). This finally leads to

$$\begin{aligned}\Pr(T_n<x)\;&=\;c_n\,x^n-c_{n-1}\,(2x-1)^n+\frac{1}{3}\,c_{n-2}\,(4x-3)^n\\&-\;\cdots+(-1)^{k-1}\frac{1}{3}\cdot\frac{1}{7}\cdots\frac{1}{2^{k-1}-1}\cdot c_{n-k+1}\,(2^{k-1}x-(2^{k-1}-1))^n,\\&\quad 1-1/2^{k-1}<x\leq 1-1/2^k,\end{aligned}$$

which coincides with (14). □

Formula (20) holds in particular for $0 < x \leq 1$, $j = 0$ and $l = m = n$. This means that (10) and (14) can be combined as follows:

$$\Pr(T_n<x)\;=\;\sum_{k=0}^{n}(-1)^k\left(\prod_{i=1}^{k}\frac{1}{2^k-1}\right)c_{n-k}\left(2^kx-2^k+1\right)^n\,\mathbf{1}_{\{x>1-\frac{1}{2^k}\}},\qquad 0<x\leq 1.\tag{21}$$

For $x > 1 - 1/2^n$ the right-hand side of (21) just adds up to 1, according to (20). Also, note that due to (20) the distribution of T_n can be written in another form:

$$\Pr(T_n < x) = 1 - \sum_{k=0}^{n}(-1)^k \left(\prod_{i=1}^{k} \frac{1}{2^k - 1}\right) c_{n-k} \left(2^k x - 2^k + 1\right)^n \mathbf{1}_{\{x<1-\frac{1}{2^k}\}}, \quad 0 < x \leq 1.$$

It gives a simpler expression for $1 - 1/2^{k-1} < x \leq 1 - 1/2^k$, when k is greater than $n/2$. In Fig. 3 we show the distribution of the travel time for several values of n.

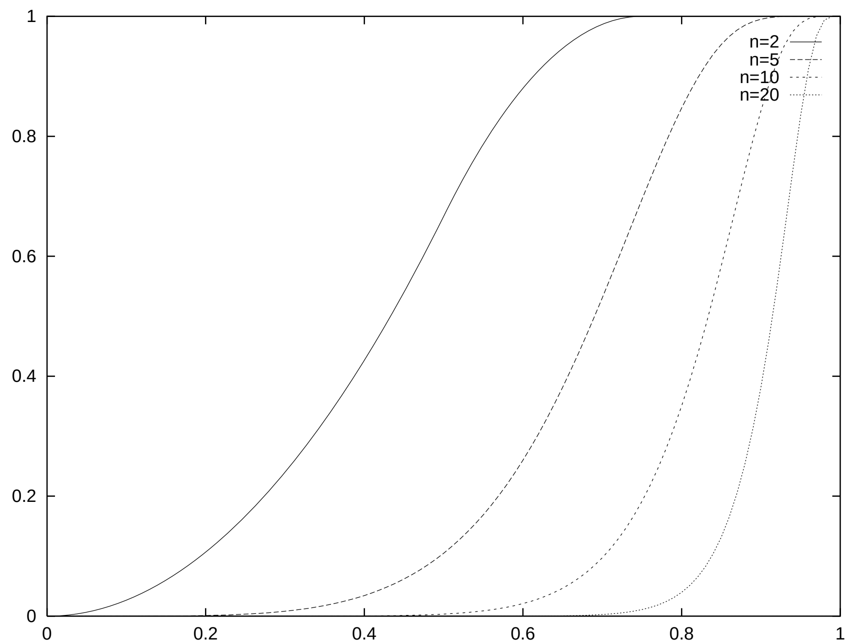

Figure 3: The distribution of T_n for $n = 2, 5, 10, 20$.

5 Asymptotic results

In this section we analyze the distribution of the travel time under the NI heuristic, when the number of items n tends to infinity. In fact, we consider $1 - T_n$, which is the difference between the travel time under the NI heuristic and a complete rotation of the carousel. It is clear that $1 - T_n$ converges in distribution to zero as $n \to \infty$. However, since (8) gives

$$\mathrm{E}\,(1 - T_n) = \frac{2}{n+1}\left(1 - \frac{1}{2^{n+1}}\right),$$

one may expect that $(n+1)(1 - T_n)$ has a proper limiting distribution.

We will use the common notation $Z_n \xrightarrow{d} Z$, if the sequence $Z_1, Z_2, \ldots$ converges in distribution to Z. The limiting distribution of $(n+1)(1 - T_n)$ is presented in the following theorem.

Theorem 5.1 *Let X_1, X_2, ... be independent exponentials with mean 1. Then*

$$(n+1)\,(1-T_n) \xrightarrow{d} \sum_{i=1}^{\infty} \frac{1}{2^{i-1}}\, X_i, \tag{22}$$

and the limiting distribution is given by

$$\Pr\left(\sum_{i=1}^{\infty} \frac{1}{2^{i-1}}\, X_i < x\right) = c_\infty \sum_{i=0}^{\infty} (-1)^i \left(\prod_{k=1}^{i} \frac{1}{2^k-1}\right) \Big(1-\exp(-2^i x)\Big), \quad x > 0, \tag{23}$$

where c_∞ is defined as (cf. (15))

$$c_\infty = \prod_{j=1}^{\infty} \frac{2^j}{2^j-1}\,.$$

Proof: Denote

$$\xi_n = \sum_{i=1}^{n} \frac{1}{2^{i-1}}\, X_i\,, \qquad \xi = \sum_{i=1}^{\infty} \frac{1}{2^{i-1}}\, X_i\,.$$

According the Monotone Convergence Theorem we have $\mathrm{E}(\xi) = \lim_{n\to\infty} \mathrm{E}(\xi_n) = 2$, which in particular implies $\Pr(\xi < \infty) = 1$.

To prove (22) we only need to rewrite (2) as

$$(n+1)(1-T_n) \stackrel{d}{=} \frac{(n+1)\xi_{n+1}}{S_{n+1}}.$$

By definition the sequence $\{\xi_n\}$ converges a.s. to ξ. Further, according to the strong law of large numbers, the sequence $\{S_n/n\}$ converges a.s to 1. Thus, the sequence $\{n\xi_n/S_n\}$ converges a.s. to ξ, which immediately gives (22).

The distribution of ξ can be determined via inversion of its Laplace-Stieltjes transform $\alpha(s)$, which is given by

$$\alpha(s) = \mathrm{E}(e^{-s\xi}) = \prod_{i=0}^{\infty} \frac{2^i}{s+2^i}\,.$$

It is readily verified that $\alpha(s)$ is a meromorphic function with simple poles $a_i = -2^i$, $i = 0, 1, \ldots$ The residues b_i at these poles are given by

$$b_i = c_\infty (-1)^i \left(\prod_{k=1}^{i} \frac{1}{2^k-1}\right) 2^i\,, \qquad i = 0, 1, \ldots$$

To invert $\alpha(s)$ we first expand this function in rational fractions of s, by following the approach in Whittaker and Watson [8], Sec. 7.4. This approach requires that $|\alpha(s)|$ is uniformly bounded on a sequence of circles C_j, with centre at 0 and radius R_j, not passing through any poles, and such that $R_j \to \infty$ as $j \to \infty$. In this case we can take $R_j = -(a_j + a_{j+1})/2 = 2^{j-1} + 2^j$ and it is straightforward to show that for all $s \in C_j$,

$$|\alpha(s)| \leq |\alpha(-R_j)| \leq 2 \cdot \prod_{k=0}^{\infty} \frac{2^{2+k}}{2^{2+k} - 3}\,.$$

Since this upper bound does not depend on j, the function $|\alpha(s)|$ is indeed uniformly bounded on the sequence of circles C_j. Now we can conclude from Sec. 7.4 in [8] that

$$\alpha(s) = \alpha(0) + \sum_{i=0}^{\infty} b_i \left[\frac{1}{s - a_i} + \frac{1}{a_i}\right]\,.$$

From Litvak *et al.* [5], Remark 10.3, it follows that

$$\sum_{i=0}^{\infty} \frac{b_i}{a_i} = c_\infty (-1)^{i+1} \sum_{i=0}^{\infty} \left(\prod_{k=1}^{i} \frac{1}{2^k - 1} \right) = -1 = -\alpha(0),$$

which implies that

$$\alpha(s) = \sum_{i=0}^{\infty} \frac{b_i}{s - a_i}\,.$$

Inversion of this expression yields (23). □

Further note that for any $k = 1, 2, \ldots$ we have

$$\begin{aligned}
&\mathrm{E}\left([(n+1)(1-T_n)]^k\right) \\
&= (n+1)^k \sum_{\substack{k_1, k_2, \ldots, k_{n+1} \geq 0 \\ k_1 + k_2 + \cdots + k_{n+1} = k}} \frac{k!}{k_1! k_2! \ldots k_{n+1}!} \mathrm{E}\left(\frac{X_1^{k_1} X_2^{k_2} \cdots X_{n+1}^{k_{n+1}}}{S_{n+1}}\right) \prod_{i=1}^{n+1} \left(\frac{1}{2^i}\right)^{k_i} \\
&= (n+1)^k \binom{n+k}{k}^{-1} \sum_{\substack{k_1, k_2, \ldots, k_{n+1} \geq 0 \\ k_1 + k_2 + \cdots + k_{n+1} = k}} \prod_{i=1}^{n+1} \left(\frac{1}{2^i}\right)^{k_i} \\
&\leq \frac{k!(n+1)^k}{(n+1)(n+2)\cdots(n+k)} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^n}\right)^k \leq k! 2^k.
\end{aligned}$$

Hence (see e.g. Chung [3], Sec. 4.5) for any $k = 1, 2, \ldots$ it holds that

$$\lim_{n \to \infty} \mathrm{E}\left([(n+1)(1-T_n)]^k\right) = \mathrm{E}\left(\xi^k\right) < \infty.$$

In Fig. 4 we demonstrate the rate at which the distribution of $(n+1)(1-T_n)$ converges to its limiting distribution.

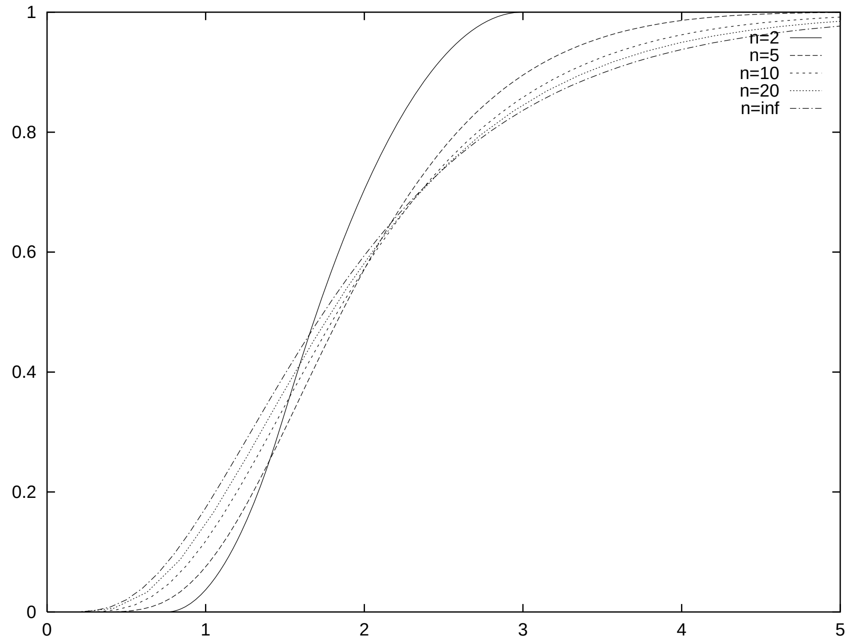

Figure 4: The distribution of $(n+1)(1-T_n)$ for $n = 2, 5, 10, 20$ and the limiting distribution as $n \to \infty$.

To find $\mathrm{E}\left(\xi^k\right)$ we use (23) and then change the order of integration and summation. This yields:

$$\mathrm{E}\left(\xi^k\right) = c_\infty \sum_{i=0}^{\infty} (-1)^i \left(\prod_{j=1}^{i} \frac{1}{2^j - 1} \right) \frac{k!}{2^{ki}} \,. \tag{24}$$

Changing the order of integration and summation is allowed, since the sum above is absolutely convergent. Expression (24) can be simplified by using the equality

$$\prod_{j=k+1}^{\infty} \left(1 - \frac{1}{2^j}\right) = \sum_{i=0}^{\infty} (-1)^i \left(\prod_{j=1}^{i} \frac{1}{2^j - 1} \right) \frac{1}{2^{ki}} \,,$$

which holds for $k = 0, 1, \ldots$. For $k = 0$ the proof of this equality is given in Litvak *et al.* [5], Remark 10.3. The case $k > 0$ can be proved along the same lines. Substituting this equality into (24) gives the simple expression:

$$\mathrm{E}\left(\xi^k\right) = k! \prod_{j=1}^{k} \frac{2^j}{2^j - 1} \,.$$

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