Tension percolation

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Abstract

We introduce a new class of bootstrap percolation models where the local rules are of a geometric nature as opposed to simple counts of standard bootstrap percolation. Our geometric bootstrap percolation comes from rigidity theory and convex geometry. We outline two percolation models: a Poisson model and a lattice model. Our Poisson model describes how defects—holes is one of the possible interpretations of these defects—imposed on a tensed membrane result in a redistribution or loss of tension in this membrane; the lattice model is motivated by applications of Hooke spring networks to problems in material sciences. An analysis of the Poisson model is given by Menshikov, Rybnikov, and Volkov (1999). In the discrete set-up we consider regular and generic triangular lattices on the plane where each bond is removed with probability $1-p$. The problem of the existence of tension on such lattice is solved by reducing it to a bootstrap percolation model where the set of local rules follows from the geometry of stresses. We show that both regular and perturbed lattices cannot support tension for any $p<1$. Moreover, the complete relaxation of tension—as defined in Section 4—occurs in a finite time almost surely. Furthermore, we underline striking similarities in the properties of the Poisson and lattice models.

Keywords: Equilibrium tension, self-stress, spider web, triangular lattice, percolation on graphs, bootstrap percolation, graph rigidity

1 Introduction

Consider a planar tensed membrane in space clamped on its boundary. What happens when holes are created in this structure? When will it still support tension? When will there be floppy portions that bend and flex? When will the whole structure become floppy with tension vanishing throughout the membrane? Naturally this depends on how

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the holes are distributed and just what the structure of the membrane is. We present two classes of percolation models, both discrete, where tension can exist in a natural sense, and where the creation of holes can have the consequence of relieving the tension. One approach is a continuous bootstrap-like percolation of compact defects distributed with a Poisson Law. The other is a bootstrap percolation on a triangular lattice. In both of these models it is the geometric properties of the underlying structure (after the holes are created) that determines whether or not the tension exists. Thus, our paper introduces a new class of percolation models, geometric bootstrap percolation models.

We prove here that an infinite triangular (regular or perturbed) lattice, where each edge has been removed independently with probability $1 - p > 0$, cannot support an equilibrium tension almost surely (a.s.—throughout the text). There are strong parallels between this result and the result described in Menshikov, Rybnikov and Volkov (1999), our continuous model. In the continuous model the positions of numerous holes are distributed homogeneously in the plane according to a Poisson Law with $\lambda > 0$, and their shapes are independently identically distributed (i.i.d.—throughout the text) random functions on a circle independent of the Poisson Process. As with the lattice model, tension vanishes almost surely. But in this model, the criterion for having tension exist is that there is some triangulation of the complement of the holes such that an equilibrium tension exists in the 1-skeleton of that framework.

In our definition, stress (and, in particular, tension) is a real scalar quantity $\omega_{ij} = \omega_{ji}$ associated to each edge ij between vertex i and vertex j of a graph underlying a framework that triangulates the region in the plane. This stress is said to be an equilibrium stress if the vector sum $\sum_j \omega_{ij}(\mathbf{v_i} - \mathbf{v_j}) = 0$ for each vertex $\mathbf{v_i}$ (treated as a vector in Euclidean space) of the graph other than pinned vertices. There is no equilibrium condition for pinned vertices. If an edge (i, j) has $\omega_{ij} > 0$, it is said to be in tension. When a framework is connected and has all of its edges in tension it is easy to show that this framework is rigid. This is one of the main tools to show rigidity and one of the main reasons that the existence of an equilibrium stress with all of its members in tension is of interest here. But the stress, as it is defined here, is more accurately thought of as a stress coefficient, rather than what might be usually referred to as a stress in physics or engineering. Each ω_{ij} is not a force by itself. The vector quantity $\omega_{ij}(\mathbf{v_i}-\mathbf{v_j})$ is essentially the physical force involved. But the ω_{ij} are more easily dealt with mathematically, and they are what has been used in the mathematical literature.

When the percolation process of edge removal or hole creation is performed, for any particular graph (lattice model) or complement of the holes (Poisson model), the determination of whether there exists an equilibrium stress that is positive on all the edges of some graph can be difficult to determine. Fortunately, however, to calculate the critical tension threshold for the Poisson model it is enough to consider only the situation where convex holes intersect. If there is a region in the plane that is removed, creating a hole that is not convex, then the convex hull of a connected component has no tension in its interior. We call such an area *defective*. It turns out that with high probability, these holes coalesce into defective areas that and eventually cover the entire plane in the infinite case. The proof of this is one of the main points of the paper by Menshikov, Rybnikov and Volkov (1999).

In Section 1.1 we carefully define the notions of stress and rigidity. In Section 1.2 we show how the discrete approach based on these notions can be applied to help to understand the rigidity of regions in the plane. Section 2 treats the continuous membrane model and recent probabilistic results for it. The main probabilistic result of this paper states that the relaxation of tension on a triangular lattice (regular or slightly perturbed) where each edge has been removed independently with probability $1-p>0$ occurs in a finite time (discrete time for our bootstrap process is defined in Section 3) almost surely; this is proved in Section 3. Menshikov et al. (1999) showed that in the continuous case the relaxation of tension also occures in a finite time a.s. In Section 3 we analyze similarities between the processes of tension relaxation for the continuous and discrete cases. In addition, we conjecture that our methods used for triangular lattices can be applied to a broader class of planar graphs. In the last section we discuss connections between the problem of tension percolation for the infinite regular triangular lattice and the same problem for finite subgraphs of this lattice.

Both models assume that after the edge removal or hole creation the remaining medium remain static, i.e., it is not plastic. If the medium, lattice or membrane, has

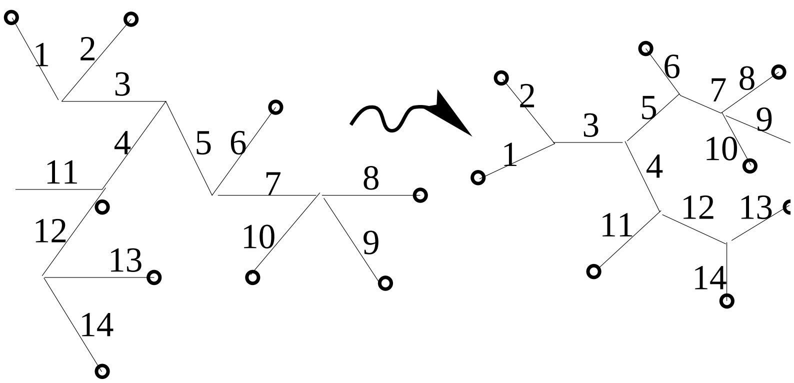

Figure 1: Perturbed fragment of a sub-graph of the triangular lattice; vertices marked with circles may be adjacent to other edges of the sub-graph

the property that it can deform and recreate another stressed configuration after the removal of the edges (lattice model) or the holes creation (continuous model), it could arrive at a new stressed configuration in equilibrium. If the medium has such plastic properties, both results should rather be interpreted not as immediate relaxation, but as an inevitable displacement restoring the ability of the system to support tension; in other words, to preserve strong stability the system has to rearrange itself. For example, in the lattice model a star of ϵ-type (see Figure 10 in Section 3) can reshape into the star that can support tension (see Figures 2 and 1). In the continuous model the displacement can be pictured in many different ways. Of course, if the probability distribution is such that the medium is not even connected, then the material will not even be able to rearrange itself after the edges have been cut or the holes have been created. Note, that unlike tension percolation, this connectivity percolation has a critical probability value below which the medium remains connected, almost surely, and above which it is disconnected, almost surely (Menshikov, Sidorenko 1987).

In other words, to preserve stability, the system has to rearrange itself. Figure 1 shows an example of this sort of phenomenon.

1.1 Frameworks: Rigidity and Stresses

A bar-and-joint framework is a graph (possibly, with countably many vertices) together with its realization in $\mathbb{R}^d$. *We consider only discrete frameworks: any compact subset of* $\mathbb{R}^d$ *may contain only a finite number of vertices of a framework.* Denote by $G(E, V, V_0; \mathbf{p})$

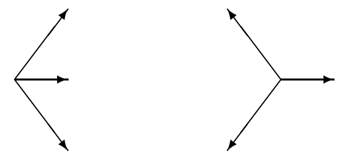

Figure 2: ϵ ϵ-star can reshape into a star supporting tension

a framework in $\mathbb{R}^d$ with the edge set E, and the vertex set V with pinned (fixed in $\mathbb{R}^d$) subset of vertices $V_0 \subset V$; here $\mathbf{p}$ is the set of all the coordinates of the vertices of the framework. We will denote the graph of the framework by $G(E, V, V_0)$, where V_0 is the set of vertices that must be pinned in a realization. Thus, in our notation $\mathbf{p}$ defines a realization of the graph $G(E, V, V_0)$ in $\mathbb{R}^d$. Vertices that are not pinned are called *free*. If $V_0 = \emptyset$, we will write simply $G(E, V; \mathbf{p})$. Notice, that in the mathematics of rigidity there is a tendency to use term *framework* instead of *network* preferred by physicists. Denote by $\mathbf{v}_i$ the vector of coordinates of vertex $v_i \in V$.

Definition 1.1 *An equilibrium stress (or self-stress) is an assignment of real numbers $\omega_{ij} = \omega_{ji}$ to the edges, a tension if the sign is positive, or a compression if the sign is negative, so that the equilibrium conditions*

$$\sum_{\{j \mid (ij) \in E\}} \omega_{ij}(\mathbf{v}_j - \mathbf{v}_i) = 0$$

hold at each vertex $\mathbf{v}_i \in V \setminus V_0$ (see **Fig. 3***).*

Definition 1.2 *A framework $G(E, V, V_0; \mathbf{p})$ that has an equilibrium stress, positive on all edges, is referred to as a spider web.*

In other words, $G(E, V, V_0; \mathbf{p})$ is a *spider web* if it supports an equilibrium tension. For example, an infinite regular triangular lattice is, obviously, a spider web (see Figure 4 for a piece of such grid). The following obvious observations immidiately follow from the definition of stress.

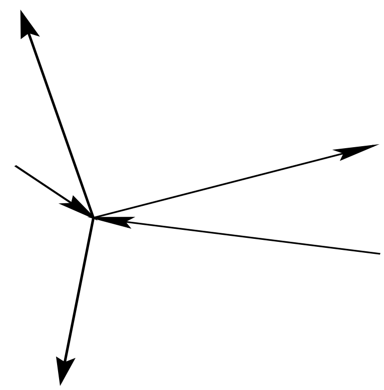

Figure 3: Equilibrium stress

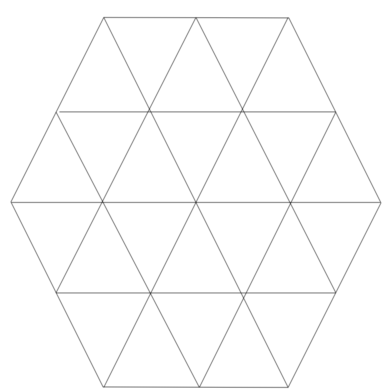

Figure 4: Triangular grid

Proposition 1.3 *If* $G = (E, V; \mathbf{p})$ *is a spider web in* $\mathbb{R}^d$*, then* G *has infinitely many edges and vertices and their convex hull is an affine subspace of* $\mathbb{R}^d$*.*

Proposition 1.4 *If* $G(E, V; \mathbf{p})$ *is a spider web, then for each vertex* $\mathbf{v}$ *of* G *the convex hull of the vertices adjacent to* $\mathbf{v}$ *contains* $\mathbf{v}$*.*

Denote the set of vertices adjacent to $\mathbf{v}$ by $A(\mathbf{v})$. The following proposition follows from the definition of spider web.

Proposition 1.5 *Let* $\mathbf{v}$ *be a vertex of a spider web* $G = (E, V; \mathbf{p})$ *in* $\mathbb{R}^d$*. Suppose there is a subset of vertices of* $A(\mathbf{v})$ *such that its convex hull affinely spans a hyperplane in* $\mathbb{R}^d$ *passing through* V*. Then, if* $A(\mathbf{v})$ *affinely spans* $\mathbb{R}^d$*, the convex hull of* $A(\mathbf{v})$ *intersects both open half-spaces determined by this hyperplane.*

Two frameworks in $\mathbb{R}^d$ are called *edge equivalent* if they have the same graph and the same lengths of all edges. Two edge equivalent frameworks in $\mathbb{R}^d$ are called *congruent* if all distances between corresponding pairs of vertices are the same. Notice, that for a finite framework the set of the vertex coordianates $\mathbf{p}$ can, obviously, be regarded as a point $\mathbf{p}(G)$ in the space of parameters $\mathbb{R}^{d|V|}$.

Definition 1.6 *A finite framework* $G(E, V, V_0; \mathbf{p})$ *in* $\mathbb{R}^d$ *is called* rigid *in* $\mathbb{R}^d$ *if there is a neighborhood* $N_{\mathbf{p}} \subset \mathbb{R}^{d|V|}$ *of* $\mathbf{p}$ *such that any other realization* $\mathbf{q}$ *of graph* $G(E, V, V_0)$ *satisfying the following conditions (1)-(3) is congruent to* G*.*

(1) $G(E, V, V_0; \mathbf{q})$ is edge equivalent to $G(E, V, V_0; \mathbf{p})$,

(2) $\mathbf{q} \in N_{\mathbf{p}}$,

(3) the pinned vertices of G' *coincide with* the pinned vertices of G.

If a framework G satisfies the above definition with $N_{\mathbf{p}} = \mathbb{R}^{d|V|}$, it is called *globally rigid* in $\mathbb{R}^{d|V|}$. Note that a globally rigid framework is automatically rigid. A framework that is not rigid is called *flexible.* It is important to specify the dimension of the space where our framework G is considered. A graph can be rigid in $\mathbb{R}^2$, but not rigid in $\mathbb{R}^3$: for example, the graph depicted in Figure 5 has motions that keep the boundary vertices on the plane, but move the vertices U, V and W, lying inside the tringle, from the plane (dashed lines show that the extensions of the edges do not have a common point: this is a sufficient condition for this graph to be flexible in the space). There are a few ways to define rigidity for infinite graphs, but the existing tools of rigidity theory allow one to work only with those definitions where the rigidity of an infinite graph is understood as the rigidity of its finite subgraphs. It is natural to refer to this type of rigidity as *finite* rigidity (see Connelly (1990)). Since in this paper we deal only with finite types of rigidity we shall omitt the word *finite* throughout the rest of the paper.

In the following definition all frameworks are assumed to have no pinned vertices.

Definition 1.7 *An infinite framework $G(E, V; \mathbf{p})$ in $\mathbb{R}^d$ is referred to as* rigid *if any finite sub-framework of $G(E, V; \mathbf{p})$ is contained in a rigid finite sub-framework of $G(E, V, \mathbf{p})$.*

The above definition of rigidity was adopted by Holroyd (1998,1999) in his studies of generic rigidity percolation on lattices (see also Grimmett (1999)). Since the definition of rigidity for finite graphs can be applied to infinite graphs with all but finitely many vertices pinned, the following definition of pseudorigidity is consistent.

Definition 1.8 *An infinite framework $G(E, V; \mathbf{p})$ in $\mathbb{R}^d$ is referred to as (globally)* pseudorigid *if for any finite subset V' of V the framework obtained from $G(E, V; \mathbf{p})$ by pinning all of the vertices in $V \backslash V'$ is (globally) rigid.*

To avoid confusion, let us notice that in some papers (e.g. Connelly (1990)) the above property is called finite rigidity. Rigidity in $\mathbb{R}^d$ obviously, implies pseudorigidity in $\mathbb{R}^d$. However, rigidity in $\mathbb{R}^2$ does not imply pseudorigidity in $\mathbb{R}^3$. For example, an infinite graph triangulating $\mathbb{R}^2$ is always rigid in the plane, however, if it has a subgraph shown in Figure 5, it is not pseudorigid in the space. Let us illustrate the differences between rigidity and pseudorigidity: the regular triangular lattice is rigid in $\mathbb{R}^2$ and pseudorigid in $\mathbb{R}^3$, but not rigid in $\mathbb{R}^3$, whereas the square lattice is pseudorigid in $\mathbb{R}^3$, but not rigid in $\mathbb{R}^d (r \geq 2)$. The pseudorigidity of the square lattice follows from the basic properties of spider webs (see Connelly (1982)).

Our motivation for introducing tension percolation models was to study the properties of random graphs that guarantee the rigidity not only in $\mathbb{R}^2$, but also in $\mathbb{R}^3$. One of such properties is the existence of an equilibrium tension (Connelly (1982); Connelly,

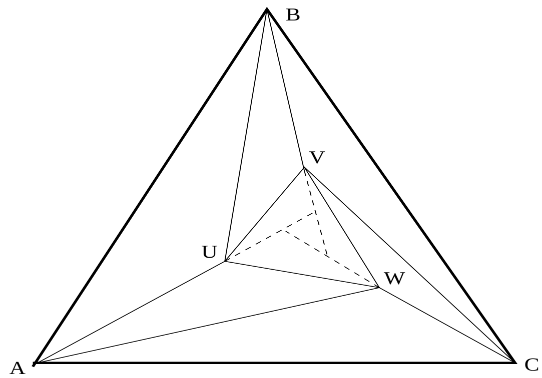

Figure 5: Rigid in the plane, but not in the space

Whiteley (1996)). An infinite framework can be rigid in $\mathbb{R}^2$ but not even pseudorigid in $\mathbb{R}^3$. For example, let ABC be a triangle in the regular triangular lattice. Now, add a triangle UVW and edges BU, AU, and CW to the lattice, as it is shown in Figure 5. The resulting infinite graph will still be rigid in the plane, but not in the 3-space, since the added vertices can be lifted from the plane without changing the lengths of the edges. The computer simulation program of Jacobs and Thorpe (1995, 1996) constructs large rigid clusters (finite, indeed) by pseudorandom edge removal from the triangular lattice; it is interesting that most of these clusters are rigid in $\mathbb{R}^2$, but flexible in $\mathbb{R}^3$ with the boundary pinned. Our main result explains, to some extent, why these clusters should not be rigid in $\mathbb{R}^3$ with the boundary pinned: a spider web is always pseudorigid (Proposition 1.11), but for a triangular lattice $\mathbb{T}$ any non-neglectable edge removal has the consequence that no infinite subset of $\mathbb{T}$ is a spider web (Theorem 3.1).

The rigidity and elasticity properties of a glass are related to how amenable the glass is to continuous deformations requiring little energy. From a physical point of view it is not enough to declare that the distance constraints force the structure to have only one configuration, since the bonds in a physical network do not behave as ideal bars in a framework. There should be a way of describing the behavior of the system as it is perturbed. That is why physicists often consider the energy function defined on the edges of a network of Hooke springs: each spring has some optimal length at which its energy is minimal, stretching or shortening a spring increases the energy of this connection. A *tensegrity* framework is a generalization of this model where besides Hooke springs there are members whose energy increases with the distance, and members whose energy decreases with the distance. In context of energy considerations it is often useful to work with the notion of tensegrity framework (Roth, Whiteley (1981), Connelly, Whiteley (1996)).

In a tensegrity framework all edges are partitioned into three types, cables E_+, struts E_-, and bars E_0, i.e. $E = E_0 \cup E_+ \cup E_-$. Together, struts, cables, and bars are called members. If a cable is stretched, the energy in the cable increases; if a strut is shortened, the energy in it increases too. Any change in the length of a bar forces the energy to increase. Therefore, networks of Hooke springs are bar tensegrities from a mathematical point of view.

Let $G(E_0, E_+, E_-; V, V_0; \mathbf{p})$ be some tensegrity framework in $\mathbb{R}^d$. The energy $\mathfrak{H}_{ij}$ of member (ij) considered as the function of its squared length l_{ij}^2

– is monotone increasing if (ij) is a cable,

– is monotone decreasing if (ij) is a strut,

– has a strict local minimum at l_{ij}^0 called the equilibrium length of (ij), if (ij) is a bar.

It is natural to define the energy function $\mathfrak{H}$ of a finite tensegrity framework (finite network of Hooke springs) as the sum of the energy functions of its members. Thus,

$$(1)\quad \mathfrak{H} = \frac{1}{2} \sum_{(ij)\in E} \mathfrak{H}_{ij}(|\mathbf{v}_j - \mathbf{v}_i|^2) = \frac{1}{2} \sum_{(ij)\in E} \mathfrak{H}_{ij}(l_{ij}^2).$$

When all members are bars the simplest way to define the energy function is as follows

$$(2)\quad \mathfrak{H} = \frac{1}{2} \sum_{(ij)\in E} a_{ij}(l_{ij} - l_{ij}^0)^2,$$

where the sum is over all ordered pairs of vertices of the framework, l_{ij} is the length of the bond between i and j, l_{ij}^0 is the equilibrium bond length, and $a_{ij} > 0$ is the spring constant of the bond between vertices v_i and v_j. Here $\mathfrak{H}_{ij}(x) = x + (l_{ij}^0)^2 - 2\sqrt{x}l_{ij}^0$.

In the spirit of the definition of equilibrium stress we assume that a strut can support only compression, a cable can support only tension, and a bar can be under either type of stress, depending on whether its length is larger or smaller than l_{ij}^0. For more detailed information on tensegrities see the works of Roth and Whiteley (1981), Connelly and Whiteley (1996), and Connelly (1993).

Definition 1.9 *A finite tensegrity framework $G(V, E, V_0; \mathbf{p})$ in $\mathbb{R}^d$ with pinned vertices $V_0 \subset V$ is called prestress stable if*

(1) The first derivatives of $\mathfrak{H}_{ij}(x)$ evaluated at $x = |\mathbf{v}_i - \mathbf{v}_j|^2$ constitute an equilibrium stress on G.

(2) the second differential of $\mathfrak{H}(|\mathbf{v}_i - \mathbf{v}_j)|^2)$—regarded as the function of the coordiantes of point $\mathbf{p} \in \mathbb{R}^{d|V|}$—is a positive semidefinite quadratic form whose kernel restricted to infinitesimal motions leaving V_0 unmoved consists of trivial infinitesimal motions of the framework.

As in the case of rigidity this definition can be applied to infinite frameworks with only finitely many free vertices—and we use this in the following definition.

Definition 1.10 *An infinite framework $G(E, V; \mathbf{p})$ in $\mathbb{R}^d$ is called* prestress stable *if for any finite subset V' of V there is an interpretation of the edges with at least one vertex in V' as either cables or struts such that the* tensegrity *framework obtained from $G(E, V; \mathbf{p})$ by pinning all of the vertices in $V \setminus V'$ is* prestress stable *in $\mathbb{R}^d$.*

The concept of prestress stability comes from engineering and, basically, accounts for local minima of the energy function. This concept is defined in Connelly (1993) and Connelly and Whiteley (1993, 1996). If $\mathfrak{H}_{ij}$ are twice continuously differentiable, a prestress stable framework realizes a local minimum of the global energy function $\mathfrak{H}$ (Connelly, Whiteley (1996)). Note that if $\mathfrak{H}_{ij}$ are defined by formula 2, they are twice continuously differentiable on $(0, \infty)$.

1.2 Tension

The existence of a tension (a positive equilibrium stress) on a framework in the plane implies some important rigidity properties for this framework considered living in the three-space. This may have some interesting consequences for modeling physical properties of materials with networks of Hooke springs and geometry of convex surfaces. The rigidity properties of infinite graphs (lattices) drew the attention of physicists since thr early 80's. It turns out that real glasses are well represented by random central-force networks of Hooke springs (Thorpe (1983)). The sucsess of these methods resulted in good characterization of elastic properties of glasses like $Ge_xAs_ySe_{1-y}$ (Thorpe (1983)). In their experiments the variation of the parameters x and y is directly linked to variation of the probability of edge removal in the independent rigidity percolation model on the triangular lattice. The rigidity analysis of random networks has also been used for modeling physical properties of proteins, polymers and semiconductors (Thorpe and Duxbury (1999)).

Let G be an infinite framework rigid in $\mathbb{R}^2$. The example discussed in Secion 1 clearly demonstrates that G need not be rigid in $\mathbb{R}^3$. The pictures produced by Jacobs and Thorpe's program (1995, 1996) also give examples of $\mathbb{R}^2$-rigid, but not $\mathbb{R}^3$-rigid random graphs. Meanwhile, the spider web property implies the pseudorigidity in $\mathbb{R}^d$ for any $d > 1$.

Proposition 1.11 *Let $G(E, V, V_0; \mathbf{p})$ be a (possibly infinite) spider web in $\mathbb{R}^2$ with pinned vertices $V_0 \subset V$. Then*

1) $G(E, V, V_0; \mathbf{p})$ is globally pseudorigid in $\mathbb{R}^d$ ($d > 1$);

2) $G(E, V, V_0; \mathbf{p})$ is prestress stable in $\mathbb{R}^d$ ($d > 1$).

The first part can be derived from Connelly (1982), where it is proved for finite frameworks (see also Connelly (1993) and Connelly, Whiteley (1996)). This proof directly applies to infinite frameworks, since the pseudorigidity has been defined via finite subgraphs of G. The second part is non-trivial; the proof will appear elsewhere (mentioned in Connelly 1998).

2 Poisson Model

2.1 Tension in a membrane

Let M be a tensed membrane (film) clamped on its boundary. A small convex hole made in the membrane results in the redistribution of tension in the rest of the membrane. Clearly, if we have a non-convex hole (which can also be interpreted as the union of a number of convex overlapping holes) tension ought to vanish on the convex hull of this set (see **Fig. 6**).

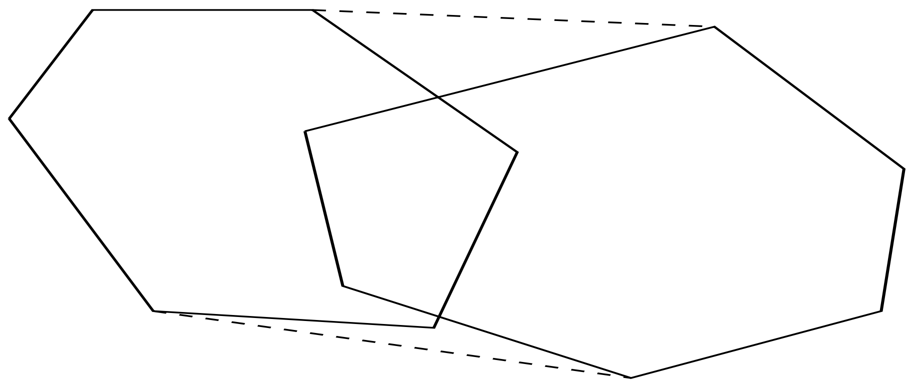

Figure 6: Two overlapping holes: tension must disappear on the convex hull of them

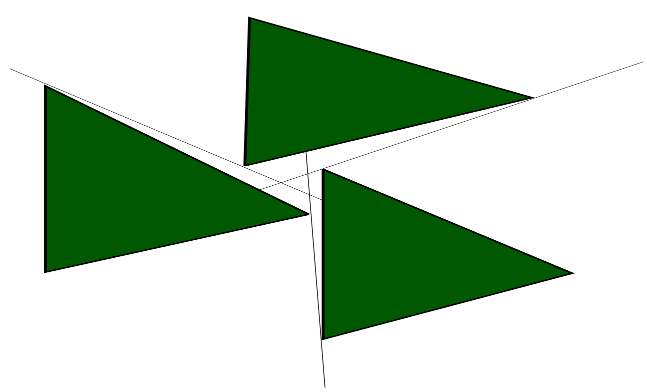

Figure 7: Three non-overlapping holes: tension must disappear on the convex hull of them

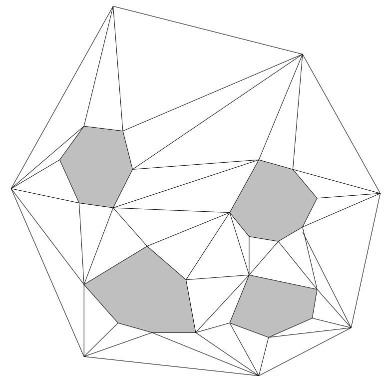

Figure 8: A triangulation

It is, however, less intuitive that tension may vanish at some subset of the complement of a collection of convex non-overlapping holes. For example, the convex hull of three holes shown on **Fig. 7** cannot support tension; this can even be verified with a sheet of some elastic material and scissors. Therefore, if the area where tension vanishes is interpreted as defective, all three polygons on **Fig. 7** ought to coalesce into one big defect. A mathematical explanation of this *coalescence effect of a "pinwheel configuration"* is given in Menshikov et al. (1999). Roughly speaking, the non-existence of tension on the convex hull of the three holes is due to the visible "swirl" in the area where the triangles "almost" meet.

By a (convex) tiling of a closed planar set with piecewise-linear or no boundary we mean a *locally finite* partition of this set into subsets of three types: open convex polygons called 2-cells, open segments called edges or 1-cells, and points called vertices or 0-cells. The 1-skeleton of a tiling is a framework whose vertex set is the vertex set of the tiling, and whose edge set is the tiling's edge set. A triangular tiling where any two triangles whose closures intersect can only make contact either at a common vertex or at a full common edge is called a *triangulation* (see **Fig. 8**)

Definition 2.1 *Let M be a set with a polygonal or no boundary in $\mathbb{R}^2$ (M might be all of $\mathbb{R}^2$), and let $\mathcal{H}$ be a collection of open polygons in M, such that the number of polygons intersecting any compact subset of $\mathbb{R}^2$ is finite. We call the elements of $\mathcal{H}$* holes *and denote by H the pointwise union of the holes.*

Definition 2.2 *In the context of the above definition we say that $M \setminus H$ supports tension* *if $M\backslash H$ admits a partition with the edge set E and vertex set V such that the framework $(E, V, V \cap \partial M)$ is a spider web. Let S be a closed subset of $M\backslash H$. We say that tension is lost on S if there is no closed subset A of $M\backslash H$ such that A supports tension and contains S.*

Evidently, in this definition a general convex tiling can be replaced by a triangulation without any loss of generality. A direct generalization of this definition to the case of general dimension is possible, but not quite natural, since not all spider webs in dimensions higher than 2 can be interpreted as 1-skeletons of polyhedral tilings (see Connelly and Whiteley (1996)). In the planar case the situation is simplified by the fact that any spider web with self-intersections can be turned into the 1-skeleton of a polygonal tiling by adding points of self-intersections to the vertex set of the framework, and modifying the edge set accordingly: the cone of positive stresses of the 1-skeleton of the new partition contains the cone of positive stresses of the original skeleton. A more natural definition for the general dimension would be one in which we require the existence of a three-dimensional spider web in the complement of the holes such that each vertex of each hole is incident to at least one edge of the web.

Let us now make some observations about holes. First, if a hole is non-convex, then there is no triangulation of the complement such that its 1-skeleton (vertices on the boundary of M are pinned, indeed) supports a non-zero equilibrium tension. For instance, the equilibrium of forces at vertex $\mathbf{v}$ in **Fig. 9** is impossible, if all edges incident to this vertex are under tension. Therefore, if two holes overlap, and their union is not convex, such as in **Fig. 6**, tension vanishes on all of their convex hull. Intuitevely, the vanishing of tension is rather obvious—the pieces ABC and EFG are "floppy" in the space. This is called the *coalescence effect of overlapping holes.* We summarize this observation in the following proposition.

Proposition 2.3 *Let H be a an open polygonal subset of $\mathbb{R}^2$. If $\mathbb{R}^2\backslash H$ supports tension, all connected components of H are convex polygons.*

Notice, that the above proposition is not valid for dimensions greater than two. Obviously, a saddle point of a 3-dimensional hole can serve as a vertex of a spider web realized

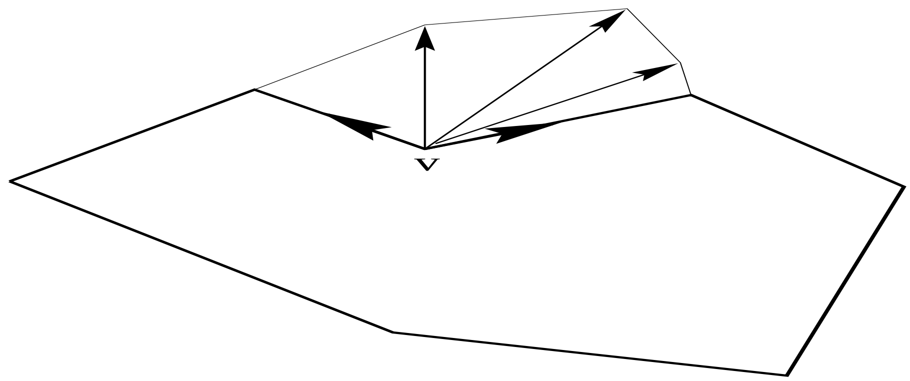

Figure 9: Non-convex hole

in the complement of the hole. Nevertheless, by Proposition 1.5 a set supporting tension in $\mathbb{R}^d$ cannot have points of strict convexity.

Proposition 2.4 *(Menshikov et al. (1999)) Let M be a convex subset of $\mathbb{R}^2$ with a polygonal or no boundary (M might be all of $\mathbb{R}^2$). For a finite set of polygonal holes $\mathcal{H}$ there is a supporting tension subset S_{max} of $M \backslash H$ such that any subset of $M \backslash H$ supporting tension is contained in S_{max}.*

Thus, when the number of holes is finite, $M \backslash H$ can be partitioned into two polygonal subsets, the unique *maximal* (with respect to inclusion) subset supporting tension and its complement where tension vanishes. The case of infinite system of holes is more complicated. Even under additional restrictions on the system of holes, for example, if the vertices of the holes form a discrete point system where the distance between every two point is no less than some r, and there is no empty circle of radius greater than some R, or, that the sizes of the holes are uniformly bounded both from above and below, it is not obvious that the union of all subsets of holes supporting tension can be represented as the complement of a discrete set of non-overlapping polygons.

Conjecture 2.5 *Let $\mathcal{H}$ be an infinite discrete system of polygons in $\mathbb{R}^2$. Then the union of all subsets of $\mathbb{R}^2 \backslash H$ supporting tension can be represented as the complement of a discrete set of edge-disjoint convex polygons.*

Let us summarize the implications of the existence of tension in the complement of the holes. They directly follow from (non-trivial) Theorem 1.11 the first part of which can be derived from the results of Connelly (1982) and Connelly, Whiteley (1996).

Proposition 2.6 *Let $\mathcal{H}$ be a discrete collection of convex open polygons in $\mathbb{R}^2$ possibly overlapping. If $M \backslash (H \cup \partial M)$ supports tension, then the 1-skeleton of any triangulation of $\mathbb{R}^2 \backslash H$ is globally pseudorigid and prestress stable in $\mathbb{R}^3$.*

There are also interesting connections between our model and convex geometry that, in its original form, are due to Maxwell (1864, 1869-1872) and Cremona (1872); they are outlined in Menshikov et al. (1999).

2.2 Bootstrap Percolation of Convex Defects

Menshikov et al. (1999) assume that holes are associated with the nodes of a Poisson point process on $\mathbb{R}^2$. They show that if the "centers" of the holes are distributed in $\mathbb{R}^2$ according to a Poisson law and their shapes are i.i.d., tension disappears on all of $\mathbb{R}^2$ a.s. In fact, this result follows from a more general theorem of the authors on the behavior of iterated convex hulls of connected subsets of $\mathbb{R}^d$, where the initial configuration of subsets is distributed according to a Poisson law, and the shapes of the elements of the original configuration are independent of this Poisson distribution. For the latter problem they establish the existence of a critical threshold in terms of the number of iterated convex hull operations required for covering all of $\mathbb{R}^d$. Below we give a short account of their results.

Definition 2.7 *A hole (G-hole) centered at $p \in \mathbb{R}^d$ is a region*

$$H(p, f) = \{p + f(\frac{\mathbf{x}}{\|\mathbf{x}\|})\mathbf{x} \ \Big| \ \|\mathbf{x}\| \leq 1\}$$

where G is a continuous positive function defined on a unit $(d-1)$-sphere.

Therefore, each hole is a continuous function on a unit circle. Consider a $d-$dimensional Poisson point process with rate λ. Let $Y = Y(\omega)$ be the collection of nodes of some realization ω of the process. Each node $y \in Y(\omega)$ is the center of a hole $H(y, f_y)$, where function G_y is positive and continuous. Let μ be a probability measure on some subset of positive continuous functions on the unit $(d-1)$-sphere. Suppose that for each y the function G_y is chosen from a distribution μ independently of the other functions and the configuration ω. Therefore, the holes $H(y, f_y)$ are i.i.d..

Definition 2.8 *Let $\mathcal{H}$ be a set of holes. Elements of $\mathcal{H}$ are called defects of 0^{th} generation.*

Definition 2.9 *A connectivity component (understood topologically) of defects of the k^{th} generation is referred to as a $k-$cluster.*

Definition 2.10 *A defect of the $(k+1)^{th}$ generation is the convex hull of a $k-$cluster.*

Lemma 2.11 *(Menshikov et al. (1999)) Let our membrane M be all of $\mathbb{R}^2$. Then tension vanishes on a defect of any generation.*

The following theorem from Menshikov at al., 1999 is the main result for the independent Poisson model of tension percolation.

Theorem 2.12 *For any distribution μ and any $\lambda > 0$ there exists a non-random non-negative integer $N = N(\mu, \lambda)$ such that N-cluster coincides with $\mathbb{R}^d$ a.s.*

In this paper we establish a similar result for tension percolation on a triangular lattice. There are strong parallels between the continuous and the lattice models. The probability $1-p$ of independent edge deletion plays the role of the Poisson density λ. The number of applications of local rules (see Theorem 4.2) required to eliminate all the infinite connected components in the triangular lattice $\mathbb{T}_p$ is, in a way, similar to the number $N = N(\mu, \lambda)$ from the above theorem. $M(p)$ and $N = N(\mu, \lambda)$ are both referred to as the *destruction* time.

3 Triangular Lattice Models

We consider a regular or slightly perturbed triangular lattice $\mathbb{T}$ on the plane where each edge is removed independently with probability $1-p$, $p > 0$. Is there a critical value $p_c < 1$, such that for $p > p_c$ there is an infinite spider web subgraph a.s.? We show that

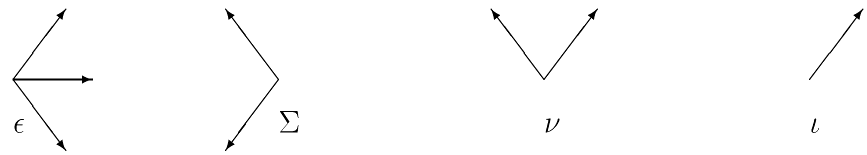

Figure 10: Local removal rules

for any $p < 1$ there no spider web subgraph a.s. Thus, no non-trivial p_c exists. Our percolation model is related to so-called "bootstrap percolation" introduced on trees by Chalupa, Leath, and Reich (1979) and, later, on d-dimensional lattices by Kogut and Leath (1981). In these models, points are independently occupied with a low density and the resulting configuration is taken as the initial state for dynamics based on some collection of local rules, in which the occupation status of a point is updated according to the configuration of its neighbors. Van Enter (1987) conducted a rigorous analysis of theses models (see also Aizenman and Lebowitz (1988)). For a review of bootstrap percolation models see Adler (1991). For the latest results on bootstrap percolation see Dehghanpour and Schonmann (1997).

Consider the affine plane $\mathbb{R}^2$ and two vectors $\vec{\mathbf{e}}_\mathbf{1}$ and $\vec{\mathbf{e}}_\mathbf{2}$ with coordinates $(1, 0)$ and $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ respectfully . Also, set $\vec{\mathbf{e}}_\mathbf{3} = \vec{\mathbf{e}}_\mathbf{2} - \vec{\mathbf{e}}_\mathbf{1}$. The regular triangular lattice $\mathbb{T}$ is a framework whose vertex set is the collection of all points with coordinates $V(\mathbb{T}) = \{i\vec{\mathbf{e}}_\mathbf{1} + j\vec{\mathbf{e}}_\mathbf{2},\ (i, j) \in \mathbb{Z}^2\}$, and whose edge set $E(\mathbb{T})$ consists of all edges between vertices $\mathbf{a}, \mathbf{b} \in V(\mathbb{T})$ such that $\mathbf{a} - \mathbf{b} = \vec{\mathbf{e}}_\mathbf{k}$ or $\mathbf{a} - \mathbf{b} = -\vec{\mathbf{e}}_\mathbf{k}$ for $k = 1$, 2 or 3. Let us denote an edge between a and b by $(\mathbf{a}, \mathbf{b})$.

Suppose some edge have been removed from $\mathbb{T}$. Denote the resulting lattice by $\mathbb{T}'$. By Proposition 1.4 edges in configurations congruent to those depicted in Figure 10 cannot support tension. We call configurations in Figure 10 ϵ-, Σ-, ν-, and ι -configurations respectfully, and refer to any such configuration as *relaxed*. By Proposition 1.5 edge $(\mathbf{va})$ and edges $(\mathbf{wa})$ and $(\mathbf{wb})$ in Figure 11 cannot support tension. We call such edges *legs* in λ- and π-configurations. We refer to λ- and π-configurations as *partially relaxed.* Therefore, if $\mathbb{T}'$ contains a spider web as a subgraph, this spider web does not have edges

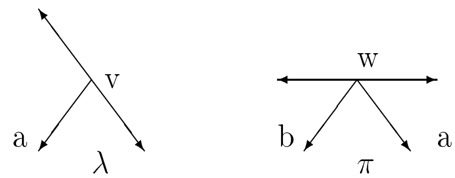

Figure 11: Local partial removal rules

in configuration depicted in Figure 10 and edges that are legs in λ- or π-configurations.

Assume we have an infinite parallel Turing machine that can operate on the stars of the vertices of an infinite (but locally finite) grid; the machine works on all stars similtaneously. Once the machine sees a star where edges form one of the configurations congruent to those depicted in Figure 11 or Figure 10 (ϵ, Σ, ν, ι, λ, π), it removes all the edges that cannot support tension. The machine proceeds for as long as there are edges that can be removed using the local rules given by Figure 11 and Figure 10.

In Section 4 we show that if the initial lattice $\mathbb{T}_p$ was obtained from $\mathbb{T}$ as the result of the independent edge removal with probability $1-p$, the parallel machine operating on the grid requires only a finite number of steps to turn $\mathbb{T}_p$ into a graph with no infinite connected components.

Our main result is

Theorem 3.1 *For any $p < 1$ the lattice $\mathbb{T}_p$ obtained from the regular or perturbed triangular lattice $\mathbb{T}$ as the result of the independent edge removal with probability $1-p > 0$ cannot support tension almost surely.*

However, first we want to prove

Lemma 3.2 *With a* positive *probability $\mathbb{T}_p$ cannot support tension.*

Proof of the Lemma. By Proposition 1.4 an edge incident to a vertex whose star is congruent to one of the stars depicted in Figure 10 cannot support tension. Therefore, the lattice $\mathbb{T}_p$ can support tension if and only if the lattice $\mathbb{T}_p(1)$ obtained from $\mathbb{T}_p$ by removing all edges in such relaxed configurations can support tension. We call these edges *implicitly* removed, as opposed to *initially* removed edges, that is, $E(\mathbb{T}\backslash\mathbb{T}_p)$. Similarly, we construct the lattice $\mathbb{T}_p(2)$ by removing all edges from $\mathbb{T}_p(1)$ in configurations congruent to the ones in Figure 10. In the same manner we define lattices $\mathbb{T}_p(3)$, $\mathbb{T}_p(4)$, ..., etc. Notice, that if $\mathbb{T}_p(n+1) \equiv \mathbb{T}_p(n)$ for some n, then $\mathbb{T}_p(n+k) \equiv \mathbb{T}_p(n)$ for any positive integer k.

Let k be a positive integer and $\mathcal{H}(k)$ be a regular hexagon centered at the origin with a side of length k, i.e., the hexagon with the vertices $k\vec{\mathbf{e}}_1$, $k\vec{\mathbf{e}}_2$, $k\vec{\mathbf{e}}_3$, $(-k)\vec{\mathbf{e}}_1$, $(-k)\vec{\mathbf{e}}_2$ and $(-k)\vec{\mathbf{e}}_3$. Let $F(k)$ denote the event "all interior edges of $\mathcal{H}(k)$ have been, possibly implicitly, removed from $\mathbb{T}_p(k)$ for some k". It is obvious from geometric observations that for any $k_0 < k$

$$(3) \quad \mathsf{P}\left(F(k+1) \mid \bigcap_{i=k_0}^{k} F(i)\right) = \mathsf{P}(F(k+1) \mid F(k)).$$

Let us show that

$$\mathsf{P}\left(\bigcap_{i=k_0+1}^{\infty} F(i) \mid F(k_0)\right) = \prod_{k=k_0}^{\infty} P(F(k+1) \mid F(k)),$$

Indeed, for $k > k_0$

$$\mathsf{P}\left(\bigcap_{i=k_0+1}^{k} F(i) \mid F(k_0)\right) = \frac{\mathsf{P}\left(F(k) \mid \bigcap_{i=k_0}^{k-1} F(i)\right)\mathsf{P}\left(\bigcap_{i=k_0}^{k-1} F(i)\right)}{\mathsf{P}(F(k_0))}$$

$$= \mathsf{P}(F(k) \mid F(k-1))\mathsf{P}\left(\bigcap_{i=k_0+1}^{k-1} F(i) \mid F(k_0)\right) = \cdots = \prod_{i=k_0}^{k} \mathsf{P}(F(i+1) \mid F(i))$$

Letting $k \to \infty$ proves (3).

We are about to show that for large k the probability of the event $G(k+1) \mid G(k)$ is greater than $1-\gamma_k$, for some sequence $\{\gamma_k\}$, such that $\sum_{k=k_0}^{\infty} \gamma_k < \infty$. This would yield

$$\mathsf{P}\left(\bigcap_{i=k_0+1}^{\infty} F(i) \mid F(k_0)\right) \geq \prod_{k=k_0}^{\infty} (1-\gamma_k) > 0,$$

which, in turn, would prove the Lemma, since $P(F(k_0)) > 0$ for any fixed k_0 and positive p.

Indeed, the probability that on each of the six sides of $\mathcal{H}(k)$ at least one edge has been initially removed is

$$\left(1-(1-p)^k\right)^6 = \left(1-e^{-\alpha k}\right)^6 > 1-6e^{-\alpha k}$$

where $\alpha = -\log(1-p) > 0$. Now, pick k_0 so large that $1 - 6e^{-\alpha k}$ is positive as soon as $k \geq k_0$. Set $\gamma_k = e^{-\alpha k}$. Then $\sum \gamma_k$ is, indeed, finite. Meanwhile, as one can conclude upon studying Fig. 12, whenever there are no edges inside $\mathcal{H}(k)$, and at least one edge is

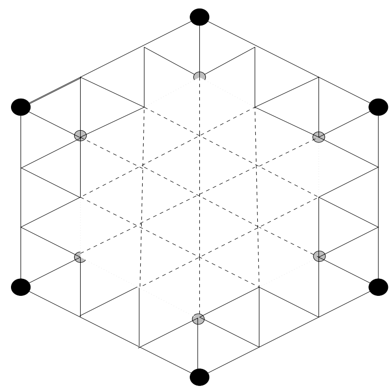

Figure 12: Typical propagation of a regular hexagon. Solid lines are remaining edges, dotted lines are removed ones. Dark circles are vertices of $\mathcal{H}(k)$, and grey circles are vertices of $\mathcal{H}(k-1)$

removed on each side of it, an incremental application of the removal rules will eventually, (in a number of steps not exceeding k), delete all edges inside $\mathcal{H}(k+1)$. Therefore, with a positive probability the event $F(k)$ implies that all the edges of our lattice are eventually removed. $\square$

Notice that in our model an empty hexagon propagating to infinity plays the role of a "critical droplet", sometimes called "Straley void". Before returning to our main theorem we would like to make a few important observations. Below, we will refer to the process described in the above proof as "hexagon propagation". We will make use of the following definition.

Definition 3.3 *We say that the sequence of planar lattices $L(n)$* eventually disappears *and write $L(n) \to \emptyset$, if for any fixed bounded subset A of the plane there exists $N > 0$ such that $L(n) \bigcap A = \emptyset$ for all $n \geq N$.*

Therefore, the above Lemma immediately implies

Corollary 3.4 *With a positive probability, $\mathbb{T}_p(n) \to \emptyset$. Moreover, conditioned on the event $R_k =$"all edges are initially removed in $\mathcal{H}(k)$",*

$$\mathsf{P}(\mathbb{T}_p(n) \to \emptyset \mid R_k) \to 1$$

as $k \to \infty$.

We would like to make another observation about the proof of Lemma 3.2. Suppose the interior of $\mathcal{H}(k)$ is empty. Evidently, to remove all edges from $\mathcal{H}(k+1)$ using local removal rules described above we need that at least one edge is absent (initially removed) on each side of $\mathcal{H}(k)$. Suppose we are not allowed to look for such initially removed edges in the planar cones (angles) defined by inequalities $|\varphi| \le 30^o$ and $|\varphi - 180^o| \le 30^o$, in the polar coordinate system (ρ, φ) (see Fig. 13). It is not hard to check that the arguments

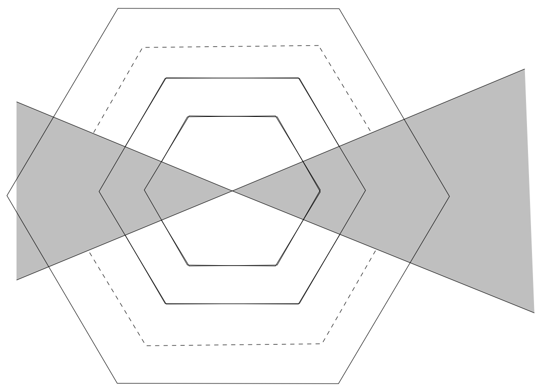

Figure 13: Hexagon propagation avoiding two angles

of the proof of Lemma 3.2 can be carried through virtually unchanged. Thus we have

Corollary 3.5 *Independently of the initial configuration inside the above mentioned cones*

$$\mathsf{P}(\mathbb{T}_p(n) \to \emptyset \mid R(k)) \to 1$$

as $k \to \infty$.

Let us return to our main statement.

Proof of Theorem 3.1. Fix $\epsilon > 0$. By Corollary 3.5 there is N such that if each edge in $\mathcal{H}(N)$ has been removed, the probability that $\mathbb{T}_p(n) \to \emptyset$ is greater than $1 - \epsilon/2$, regardless of the configuration inside the two cones. Let $q = q(N)$ be the probability that all edges inside $\mathcal{H}(N)$ have been initially removed. Obviously, $q > 0$ for any positive p. There is a positive integer M such that

$$1 - (1-q)^M > 1 - \epsilon/2.$$

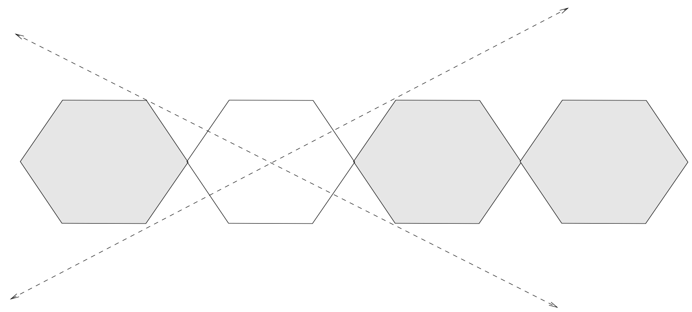

Figure 14: M hexagons; with probability $1-\epsilon/2$ all edges are removed in at least one of them

Consider M non-overlapping hexagons $\mathcal{H}_i(N)$, $i = 0,\dots,M-1$ of size N along the horizontal axis with the centers at $\mathbf{0}$, $N\vec{\mathbf{e}}_\mathbf{1}$, $2N\vec{\mathbf{e}}_\mathbf{1},\dots$, $(M-1)N\vec{\mathbf{e}}_\mathbf{1}$ (see Fig. 14). Notice, that each of the hexagons lies fully inside $\pm 30^o$ angles for all the others; therefore, due to the symmetry and space homogeneity, there is $N>0$ such that each $\mathcal{H}_i(N)$ propagates to infinity in the way described in the proof of Lemma 3.2 (with an angular restriction of Corollary 3.5) with probability at least $1-\epsilon/2$ *independently* of the initial configuration inside the others. Thus, the probability that inside of at least one of the M hexagons all the edges have been initially removed, *and* it will propagate to infinity is greater than

$$(1-\epsilon/2)^2 \geq 1-\epsilon.$$

Now, recall the definition of tension. A framework supports tension if there is a subgraph of this framework that can bear an equilibrium tension. The local rules cull only those edges that cannot support tension. The arguments above show that eventually *all* edges are bound to be removed with probability at least $1-\epsilon$. Since $\epsilon > 0$ is arbitrary, Theorem 3.1 holds. □

Studies of rigidity percolation (Jacobs et. al. (1995,1996,1997), Holroyd 1998) show that the behavior of a regular triangular lattice may differ from the behavior of a generic triangular lattice. A generic lattice in a strong sense is a realization of a graph in $\mathbb{R}^d$ where the dimension of the space of stresses of any finite subgraph of the lattice is minimal. All theorems and lemmas in this section hold not only for a regular triangular lattice $\mathbb{T}$, but also for any generic triangular lattice obtained from $\mathbb{T}$ by a sufficiently small perturbation, for we essentially need only three removal rules: the ν-rule, the Σ-rule, the

ϵ-rule, and the ι-rule, which are "robust" to such perturbations. (see Lemma 3.2 and Figure 13). Of course, our tension percolation problem for a perturbed triangular lattice makes sense only if there are perturbations of the regular lattice preserving the property of the lattice to support an equilibrium tension. It follows from the results of Baranyi and Dolbilin (1988) or Connelly (1988) on the uniform stability of sphere packings that there is $\epsilon > 0$ such that any ϵ-perturbation of the regular triangular lattice supports an equilibrium tension (see also Bezdek, Bezdek and Connelly (1998)).

We suspect that all our results hold for a larger class of generic triangular lattices, although our method cannot be applied staightforwardly to the case of an arbitrary generic triangular lattice, because a perturbation can turn a relaxed configuration into a non-relaxed configuration (see Fig. 2).

A general tension percolation problem can be stated as follows. Let G be an infinite framework in $\mathbb{R}^d$ with discrete vertex set. Remove each edge with probability $1-p$ independently of the other edges, and denote the resulting graph by G^p. What is the infinum of p's such that G^p supports tension a.s.? We call this number the critical probability of tension percolation. We have a general conjecture about tension percolation on planar graphs. To formulate this conjecture we need to introduce the notion of directional spectrum of a framework. By the direction of a line on the plane we understand the angle this line forms with, say, the horisontal axis. If G is a framework on the plane, the set of directions defined by the edges of G is called the *directional spectrum* of G. The edge set of a framework G is said to have the (l, L) property if the edge lengths of G are bounded from below by some $l > 0$ and from above by some $L > 0$.

Conjecture 3.6 *Let $G = (E, V, V_0)$ be an infinite framework on the plane realized without self-intersections. Suppose the directional spectrum of G is finite, and E has the (l, L) property. Then the critical probability of tension percolation is 1.*

The notion of an (r, R) point set is widely used in discrete geometry and mathematical crystallography. A point set V is called an (r, R)-system, or a Delaunay system, if

1) for any point $\mathbf{v} \in V$ the ball of radius r centered at $\mathbf{v}$ does not contain any other vertices of V, and

2) any ball of radius R contains at least one point of V.

Notice, that for a graph with a finite directional spectrum the (l, L)-property of the edge set is equivalent to the (r, R)-property of the vertex set.

4 Finite Time of Relaxation

Assume that it takes one unit of time for an infinite parallel Turing machine to remove all the configurations of edges congruent to those depicted in Figure 10. Thus, the lattice $\mathbb{T}_p$ is transformed to $\mathbb{T}_p(n)$ by time n. Let us call this process the *relaxation of tension on* $\mathbb{T}_p$, and say that tension has been completely lost if there is no infinite connected component of non-removed edges on the lattice. We shall show that the complete relaxation of tension occurs in a finite time a.s. We shall also show that there is a non-random time $N \geq 1$ such that $\mathbb{T}_p(N)$ has no infinite connected components a.s., but $\mathbb{T}_p(N-1)$ has an infinite component a.s. (by convention, we let $\mathbb{T}_p(0) = \mathbb{T}$, the triangular lattice we all edges).

Lemma 4.1 *The event $T(p, N) :=$ "$\mathbb{T}_p(n)$ has an infinite connected subgraph" is a tail event.*

Proof. We need to show that this event does not depend on the state of any finite subset of $\mathbb{T}$. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two subgraphs of our triangular lattice $\mathbb{T}$ such that $\mathbb{T}_1$ can be obtained from $\mathbb{T}_2$ by adding and/or removing only a finite number of edges. Let E be those edges of $\mathbb{T}_1$ that are absent in $\mathbb{T}_2$. Denote by $\mathbb{T}_1(n)$ and $\mathbb{T}_2(n)$ the results of n iterated applications of the local rules to $\mathbb{T}_1$ and $\mathbb{T}_2$. Suppose $\mathbb{T}_1(n)$ has an infinite connected component C. If $\mathbb{T}_2(n)$ has no infinite connected component, $\mathbb{T}_2(n)$ differs from $\mathbb{T}_1(n)$ at infinitely many places. An edge e of C can be absent from $\mathbb{T}_2(n)$, only if there is an edge path on $\mathbb{T}$ of length no greater than n connecting a vertex of e to one of the vertices of E. Thus, only those edges of C can be missing from $\mathbb{T}_2$ that lie at distance no greater than n from the vertex set of E. The number of such edges is finite. Therefore $\mathbb{T}_2$ contains an infinite connected component of C which is, in turn, is an infinite connected subgraph of $\mathbb{T}$. $\square$

Theorem 4.2 *There is a non-random number N_{cr} such that $\mathbb{T}_p(N_{cr})$ is a union of finite disjoint graphs a.s., but $\mathbb{T}_p(N_{cr}-1)$ has an infinite connected component a.s. (and $\mathbb{T}_p(0) \equiv \mathbb{T}$ as before).*

Proof. From the above lemma and Kolmogorov's $0-1$ law it follows that $\mathsf{P}(T(p,n))$ is either zero or one. Moreover, this probability is non-increasing as n grows, and $\mathsf{P}(T(p,0)) = 1$. Therefore, either there is N_{cr} such that $\mathsf{P}(T(p,N_{cr}-1)) = 1$ and $\mathsf{P}(T(p,N_{cr})) = 0$, or $\mathsf{P}(T(p,n)) = 1$ for all n. To rule out the second possibility, it suffices to show that there is positive integer N such that $\mathbb{T}_p(N)$ has no infinite connected component a.s.

The idea of the proof is based on Theorem 3.1. Let N and M are the same as in the proof of Theorem 3.1. Pick $\epsilon < \frac{1}{10}$ and N and M corresponding to this ϵ. Consider a partition of the plane into the boxes with the side length $S = 13(M-1)N > (4\sqrt{3}+6)(M-1)N$. Assume one of the boxes – call it B_0 – is centered at $\mathbf{0}$. In this box consider hexagons with the side N centered at $\mathbf{0}$, $N\vec{\mathbf{e}}_\mathbf{1}$, $2N\vec{\mathbf{e}}_\mathbf{1}, \ldots,$ $(M-1)N\vec{\mathbf{e}}_\mathbf{1}$.

We call box B_0 *open* if (1) one of these hexagons has all the edges removed, (2) using the procedure of implied edge removal as described by Lemma 3.2 and avoiding $\pm 30^o$ cones, it will grow till its upper and lower sides coincide with those of the box B_0 and (3) one of the edges on its upper side with the X-coordinate between 0 and $(M-1)N$ has been initially removed (see Fig. 15 and 14).

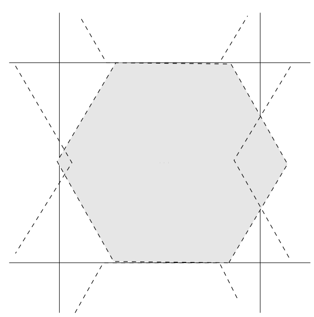

Figure 15: Hexagon propagation inside a box

Following the line of arguments in Theorem 3.1, we can conclude that the probability that B_0 is open can be made greater than 0.9 (however, we might need to have N quite large). The same is true about the other boxes of the tiling $\{B_0 + i\vec{\mathbf{e}}_\mathbf{1} + \mathbf{j} \times (\vec{\mathbf{1},\mathbf{0}}),\ (\mathbf{i},\mathbf{j}) \in$

$\mathbb{Z}^2\}$. Moreover, both vertical and horizontal neighbors are open independently, since they "look for" different initially removed edges (this is because we ignore the interior of the cones described between Corollaries 3.4 and 3.5). Therefore, all the boxes are open independently of each other. Besides, if two neighboring (at a side) boxes are open, their inside areas where edges are removed are connected.

Now, let us couple the boxes with the vertices in the site percolation model on $\mathbb{Z}^2$ where each site is open with probability 0.9 and closed otherwise. There is a unique open cluster of open sites and no infinite cluster of closed sites (e.g. see Aizenman et al. (1987) or Grimmett (1989)). Therefore, each cluster of closed sites is surrounded by a finite contour of open site. Geometrically, for our triangular lattice, it implies that each connected component of non-removed edges is surrounded by a contour of removed ones, and therefore each such component is finite. Thus, after N (or even less) iterations $\mathbb{T}_p(N)$ has no infinite connected component a.s. □

5 Tension on Finite Subgraphs of a Triangular Lattice

While discussing tension on finite graphs, we assume that some of the vertices are pinned. For example, if a finite graph can be regarded as the 1-skeleton of a tiling of a convex polygon (e.g like in Figures 8 and 4), we normally assume that all the boundary vertices of the polygon are pinned. There are a few reasons to study tension percolation on finite subgraphs of a regular infinite graph. First, a finite spider web has desirable properties (see Section 1.2). Second, it is reasonable to suggest that in some cases a very large finite piece of a triangular lattice decribes the behavior of a physical system better than an infinite triangular lattice. Third, to study tension percolation on 3-dimensional lattices it is important to understand quantitatively the effect of edge removal on the ability of a finite subset of a 2-lattice to support tension. While the method developed in the previous sections explains how a triangular lattice loses the ability to support tension as a result of any non-neglectable edge removal, it barely helps to estimate the probability of the existence of tension on a subset of a triangular lattice where each edge has been removed with probability $1-p$ as a function of the size of the subset.

Let us denote by $\mathcal{H}_n$ a hexagonal chunk of a regular triangular lattice with each side of length n. We define the distribution function $\mathbb{F}_n(p)$ (where $0 \le p \le 1$, $pn \in \mathbb{Z}$) as the ratio of the number of supporting tension subgraphs of $\mathcal{H}_n \backslash \partial\mathcal{H}_n$ on pn edges (with $\partial\mathcal{H}_n$ pinned) and the total number of subgraphs of $\mathcal{H}_n \backslash \partial\mathcal{H}_n$ on pn edges. $\mathbb{F}_n(p)$ can be, indeed, interpreted as the probability that after the independent deletion of pn edges $\mathcal{H}_n$ still supports tension (with the boundary pinned). Obviously, for each n it is a decreasing function of p. Numerical experiments also show that for each p $\mathbb{F}_n(p)$ decreases as $n \to \infty$. For large n function $\mathbb{F}_n(p)$ should look like a non-decreasing function of p with one inflexion point, although it is very hard to formally prove that $\mathbb{F}_n(p)$ converges to such a function. It is known that for the connectivity and rigidity percolation problems the analogous distribution function has such a shape. In connectivity percolation it is the proportion of the subgraphs on pn edges having a component connecting two opposite sides of a rectangle (hexagon) of size n. In the case of rigidity percolation it is the proportion of the subgraphs on pn edges having a rigid component connecting two opposite sides of a rectangle (hexagon)of size n. Jacobs, Thorpe, and Duxbury's simulation results suggest that in the case of rigidity percolation the distribution function converges to an increasing function with one inflexion point. We believe that the limiting behavior of $\mathbb{F}_n(p)$, as $n \to \infty$, is described by the distribution function for the probability model described in Section 3. Let $\mathsf{P}(p)$ be the probability that $\mathbb{T}_p$ supports tension, or in other words, that it has a subgraph which is a spider web. We conjecture that $\mathbb{F}_n(p) \to \mathsf{P}(p)$, as $n \to \infty$.

Before having proved that the critical probability for the 2-dimensional problem is one, we had conducted numerical experiments for the hexagonal fragments of the *regular* triangular lattice of sizes $n = 10$–75. The purpose of the experiments was not only to see the behavior of the value of the critical threshold, but also to produce pictures for subsequent visual analysis. We also compared two algorithms finding the maximum spider web in a graph. One of these algorithms is an integer LP algorithm which solves an optimization problem for the stress matrix of the graph. The other one is a combinatorial approximation algorithm that removes edges from the graph according to the local rules (see above). When it cannot find a removable configuration of edges it declares the remaining subgraph a “spider web”, which may not be true. The advantage of this algorithm is its linear running time.

The threshold value of p was estimated through Monte-Carlo trials of the following kind. Remove independently a fraction of edges, and check if the remaining subgraph $\mathbb{T}'$ supports tension. If not, then remove a smaller fraction of edges and start from the beginning. Otherwise remove an edge at random from $\mathbb{T}'$, and check if the resulting graph still has a spider web subgraph; do it until it does not have a spider web component. The concentration of the remaining edges is the threshold value of the conducted trial. For each concentration the average of the threshold values of different trials (we used 50-100 trials) was taken as its threshold value. In fact, we verified the existence of a spider web subgraph only for small values of n. For larger sizes (> 10) we confined ourself to applications of local rules (see above). Therefore, we got the estimates of the threshold value from below; however, for sizes 10–12 we did not encounter any situations were local rules were not able to establish the absence of a spider web subgraph.

The exact algorithm needs an integer linear programming (LP) routine over integers, since the matrix of stresses for a piece of a regular triangular grid has only 0, 1, and 2 entries, most of which are 0. Of course, at the implementation stage it is possible to replace an integer LP feasibility routine by a floating number LP routine, but it did not work well for LP implementations that we used.

Physicists are convinced that in connectivity and rigidity percolation the value of the critical threshold for a finite system (which is defined differently for different problems) approximately follows the power law, namely $Y(n) \asymp P_c - A(\frac{1}{n})^B$; here n is the linear size of the system, $Y(n)$ is the value of the critical threshold for a piece of linear size n (we used a regular hexagonal piece – $\mathcal{H}_n$ in our notation), P_c is the value of critical probability for the infinite lattice, and A, B are positive constants (see Stauffer (1989), Jacobs (1995)). However, if the crytical probability is 1, $Y(n)$ is more likely to behave as $P_c - \frac{C}{ln N}$, $C > 0$ (see Aizenman, Lebowitz (1988)), or as $P_c - \frac{C}{ln\, ln\, N}$, $C > 0$ (see Cirillo-Cerf (1999)). Below we give the estimates of C in cases the finite-size correction follows the inverse logarithmic and inverse double logarithmic law respectfully. ($P_c = 1$

by Theorem 4.2).

	Value	*Std.Error*	*Res.Std.Error*
C	0.5482	0.0094	0.01510

Table 1: Parameters of the model $Y \asymp 1 - \frac{C}{ln N}$

	Value	*Std.Error*	*Res.Std.Error*
C	0.1980	0.0027	0.01218

Table 2: Parameters of the model $Y \asymp 1 - \frac{C}{ln\, ln\, N}$

5.1 3-Dimensional Lattices

Let us sketch the connections between tension percolation on a 3-dimensional triangular lattice and spider web properties of finite subgraphs of a 2-dimensional triangular lattice. To introduce a three-dimensional analog of $\mathbb{T}$ we need to enlist the notion of point lattice. Recall, that a point lattice is the set of all points in the affine space $\mathbb{R}^d$ that can be represented as integer linear combinations of the vectors of a fixed coordinate frame. A face-centered cubic lattice of points—*fcc* lattice—is constructed by adding the centers of all the facets of a tiling by cubes to the set of vertices of this tiling, lattice $\mathbb{Z}^3$. This lattice is a natural 3-dimensional generalization of the hexagonal point lattice (the vertex set of what is known in percolation theory as the regular triangular lattice), since it is generated by the edge set of a regular 3-simplex. Denote by $\mathbb{T}^3$ the graph whose vertex set is the *fcc* lattice, and whose edge set consists of all unordered pairs of vertices $(\mathbf{a}, \mathbf{b})$ such that $\mathbf{a} - \mathbf{b}$ is the shortest vector of our *fcc* lattice. Remove each edge independently with probability $1 - p$ and denote the resulting lattice by $\mathbb{T}^3_p$. For what values of p does the modified lattice support tension with a positive probability? Suppose we want to approach this problem in the same way we approached the 2-dimensional problem. Here, instead of a propagating hexagon we have a propagating 3-polytope (see Lemma 3.2). Notice, that regular triangular and square lattice are the only types of 2-sublattices of $\mathbb{T}^3$. We call a polytope a lattice polytope if all its faces lie on periodic subgraphs of $\mathbb{T}^3$; note, that our definition of a lattice polytope differs from the standard definition of a lattice polytope used in the theory of lattice points. Thus, the facets of a lattice 3-polytope can be of only two sorts: lying on a square sublattice and lying on a triangular sublattice. The geometry of a facet as well as the geometry of a lattice polytope is not important, since there are only a finite number of lattice polytopes in $\mathbb{T}^3$ up to homothety. From this remark it becomes clear that, in principal, tension percolation on $\mathbb{T}^3$ is no different from tension percolation on any periodic graph in $\mathbb{R}^3$ with triangular planar subgraphs. Now, let us compare the hexagon propagation and the polytope propagation. If a side of the propagating hexagon misses an edge, an entire side has to go; however, one missing edge on a facet of a triangular type is not enough to conclude that the rest (with fixed boundary) is not able to bear tension. Let n be the length of the longest edge of a facet of $\mathbb{T}^3$. Denote by $P_p(n)$ the probability that after the independent edge removal from a facet of size n with probability $1-p$ the resulting graph with fixed boundary can support tension in internal edges. If $\sum_n P_p(n)$ converges, the arguments of Lemma 3.2 and Theorem 3.1 work, and $\mathbb{T}^3_p$, $p < 1$ cannot support tension a.s.

If for $\mathbb{T}_2$ the critical concentration approches the critical probability according to the inverse logarithmic law (see above), the power series $\sum_n P_p(n)$ introduced above diverges, and Lemma 3.2 will not work for $\mathbb{T}^3$. In other words, in this case $\mathbb{T}^3$ does not have the property that once the tear starts (see Lemma 3.2), it propagates to infinity with a positive probability. Of course, even if Lemma 3.2 does not work in dimension 3, the critical probability of tension percolation for $\mathbb{T}^3$ may well be 1. Our considerations here are somewhat related to Schonmann's work (1992). We end up with the following problem.

Problem 5.1 *Is it true that the critical probability of tension percolation for* $\mathbb{T}^3$ *is less than 1?*

We tend to think that the answer to the above problem is yes.

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