

ANNEALED SURVIVAL ASYMPTOTICS FOR BROWNIAN MOTION IN A SCALED POISSONIAN POTENTIAL

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Abstract

We consider d -dimensional Brownian motion evolving in a scaled Poissonian potential $\beta\varphi^{-2}(t)V$. $\beta > 0$ is a constant, φ is the scaling function, and V is obtained by translating a fixed non-negative compactly supported shape function to all the particles of a d -dimensional Poissonian point process. We are interested in the large t behavior of the annealed partition sum of Brownian motion up to time t under the influence of the natural Feynman-Kac weight associated to $\beta\varphi^{-2}(t)V$. We prove that for $d \geq 2$ there is a critical scale φ and a critical constant $\beta_c(d) > 0$ such that the annealed partition sum undergoes a phase transition if β crosses $\beta_c(d)$. In $d = 1$ this picture does not hold true, which can formally be interpreted that on the critical scale φ we have $\beta_c(1) = 0$.

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0 INTRODUCTION AND RESULTS

In the present article we study the behavior of d -dimensional Brownian motion under the influence of a scaled random soft potential, $d \geq 1$. The random soft potential is obtained by translating a fixed shape function W to all the points of a Poissonian cloud. Let \mathbb{P} stand for the law of a Poissonian point process $\omega = \sum_i \delta_{x_i} \in \Omega$ with fixed intensity $\nu = 1$ (Ω is the set of all simple pure locally finite point measures on \mathbb{R}^d). For $\omega \in \Omega$, $x \in \mathbb{R}^d$, the (unscaled) soft Poissonian potential is then defined as

$$V(x, \omega) \stackrel{\text{def}}{=} \int W(x - y) \omega(dy), \quad (0.1)$$

where we assume that the shape function $W \geq 0$ is measurable, bounded, compactly supported and $\int W(y)dy = 1$. For $x \in \mathbb{R}^d$, let P_x stand for the standard Wiener measure on

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$C(\mathbb{R}_+, \mathbb{R}^d)$ starting from x (its canonical process is denoted by Z). Then it is well known that the Feynman-Kac functional $u(x, t, \omega) = E_x \left[\exp \left\{ - \int_0^t V(Z_s, \omega) ds \right\} \right]$ represents the bounded solution in a classical sense when $V(\cdot, \omega)$ is regular and in a generalized sense else of the random potential parabolic equation:

$$\begin{cases} \partial_t u &= \frac{1}{2} \Delta u - V(\cdot, \omega)u, \\ u_{t=0} &= 1. \end{cases} \quad (0.2)$$

We know that the annealed large t behavior of that solution is

$$\mathbb{E}[u(0, t, \omega)] = \exp \left\{ -\tilde{c}(d, 1)t^{d/(d+2)}(1 + o(1)) \right\}, \quad \text{as } t \rightarrow \infty, \quad (0.3)$$

where $\tilde{c}(d, 1)$ is the constant defined in (0.10), below. This result goes back to Donsker-Varadhan [4], who used large deviation theory for occupation local times on Brownian motion on a torus. In a later version, Sznitman [12], Theorem 4.5.3, has proved the same result with the help of the method of enlargement of obstacles. Formula (0.3) is also true if one replaces the soft obstacles W by hard obstacles, which immediately kill the Brownian particles if they hit such an obstacle (traps) (see [12], Theorem 4.5.3). In the setting of rarified traps results have been obtained by Bolthausen [2], Sznitman [11], Bolthausen-den Hollander [3] and van den Berg-Bolthausen-den Hollander [1] (by scaling arguments the situation of rarified traps can be viewed to be equivalent to that of shrinking hard obstacles). Here we study a slightly different problem, instead of rarifying hard obstacles we scale the soft obstacles: For a scaling function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and $\beta > 0$, we examine the asymptotic behavior of

$$\mathbb{E} \otimes E_0 \left[\exp \left\{ -\beta\varphi(t)^{-2} \int_0^t V(Z_s, \omega) ds \right\} \right], \quad \text{as } t \rightarrow \infty. \quad (0.4)$$

This immediately motivates the study of the annealed problem for the principal Dirichlet eigenvalue of the random Schrödinger operator $H_{\beta\varphi(t)^{-2}V} \stackrel{\text{def}}{=} -\frac{1}{2}\Delta + \beta\varphi(t)^{-2}V$: let us give a heuristic argument for the leading order in (0.4) being determined by a principal eigenvalue:

$$E_0 \left[\exp \left\{ -\frac{\beta}{\varphi(t)^2} \int_0^t V(Z_s, \omega) ds \right\} \right] = e^{-(tH_{\beta\varphi(t)^{-2}V})} 1(0) \text{ “=” } \sum_i \phi_i(0) \langle \phi_i, 1 \rangle e^{-t\lambda_i}, \quad (0.5)$$

where the $\lambda_i \geq 0$ are the “eigenvalues” of $H_{\beta\varphi(t)^{-2}V}$ and ϕ_i the corresponding “eigenfunctions”. So let us consider the bottom $\lambda_{\beta\varphi(t)^{-2}V}(U)$ of the spectrum of $H_{\beta\varphi(t)^{-2}V}$ over a non-empty open subset U of \mathbb{R}^d ; more generally, for any measurable function $F : \mathbb{R}^d \rightarrow \mathbb{R}$ which is bounded from below the ground state energy with potential F is defined by

$$\lambda_F(U) \stackrel{\text{def}}{=} \inf \left\{ \frac{1}{2} \|\nabla \phi\|_2^2 + \int_U F \phi^2 dx : \phi \in C_c^\infty(U), \|\phi\|_2 = 1 \right\}. \quad (0.6)$$

For $l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $t > 0$ we define $\mathcal{T}_{l(t)} \stackrel{\text{def}}{=} (-l(t), l(t))^d$. The logarithmic moment generating function of a Poissonian point process is defined as follows ($\beta > 0$):

$$\Lambda_\phi(-\beta) \stackrel{\text{def}}{=} \log \mathbb{E} \left[\exp \left\{ -\beta \int_{\mathbb{R}^d} \phi^2 d\omega \right\} \right] = \int_{\mathbb{R}^d} (e^{-\beta\phi^2} - 1) dx, \quad (0.7)$$

for $\phi \in \Phi \stackrel{\text{def}}{=} \{\phi \in H^{1,2}(\mathbb{R}^d) : \phi \text{ is compactly supported, continuous, and } \|\phi\|_2 = 1\}$. Then

$$J(\beta) \stackrel{\text{def}}{=} \inf \left\{ \frac{1}{2} \|\nabla \phi\|_2^2 - \Lambda_\phi(-\beta) : \phi \in \Phi \right\}. \quad (0.8)$$

Finally we define the constant $\tilde{c}(d, 1)$: Let λ_d be the principal Dirichlet eigenvalue of $-\Delta/2$ on the d -dimensional unit ball $B_1(0)$. Then

$$\tilde{r}_d \stackrel{\text{def}}{=} \left(\frac{2\lambda_d}{d|B_1(0)|} \right)^{\frac{1}{d+2}}, \quad (0.9)$$

$$\tilde{c}(d, 1) \stackrel{\text{def}}{=} \inf_{r>0} \left(\frac{\lambda_d}{r^2} + r^d |B_1(0)| \right) = \frac{\lambda_d}{\tilde{r}_d^2} + \tilde{r}_d^d |B_1(0)|. \quad (0.10)$$

These quantities have already been introduced by Sznitman [12], formulas (4.5.30)–(4.5.32).

Our first main result is the following theorem (“ $a(t) \gg b(t)$ ” means that $a(t)/b(t) \rightarrow \infty$ as $t \rightarrow \infty$):

Theorem 0.1 *For $d \geq 1$, we choose $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $l(t) \gg (\varphi(t) \vee t^{1/(d+2)})$ and $\log l(t) \ll t (\varphi(t) \vee t^{1/(d+2)})^{-2}$. For $\beta > 0$ we have:*

a) *If $t^{1/(d+2)} \ll \varphi(t) \ll t^{1/2}$, then*

$$\lim_{t \rightarrow \infty} -\frac{\varphi(t)^2}{t} \log \mathbb{E} \left[\exp \left(-t \lambda_{\beta \varphi(t)^{-2} V}(\mathcal{T}_{l(t)}) \right) \right] = \beta. \quad (0.11)$$

b) *If $\varphi(t) = t^{1/(d+2)}$, then*

$$\lim_{t \rightarrow \infty} -t^{-d/(d+2)} \log \mathbb{E} \left[\exp \left(-t \lambda_{\beta \varphi(t)^{-2} V}(\mathcal{T}_{l(t)}) \right) \right] = J(\beta). \quad (0.12)$$

c) *If $\varphi(t) \ll t^{1/(d+2)}$, then*

$$\lim_{t \rightarrow \infty} -t^{-d/(d+2)} \log \mathbb{E} \left[\exp \left(-t \lambda_{\beta \varphi(t)^{-2} V}(\mathcal{T}_{l(t)}) \right) \right] = \tilde{c}(d, 1). \quad (0.13)$$

This result tells us that the critical scale, on which we may observe a phase transition, is $\varphi(t) = t^{1/(d+2)}$. It should be contrasted with the results obtained in the quenched case (see formulas (0.7), (0.15) and (0.16) in [10]). As the special case $\varphi(t) = 1$, (0.13) contains the result (0.3) by Donsker-Varadhan.

For the annealed partition sum we obtain a similar behavior:

Theorem 0.2 For $d \geq 1$ and $\beta > 0$ we have:

a) If $t^{1/(d+2)} \ll \varphi(t) \ll (t/\log t)^{1/2}$, then

$$\lim_{t \rightarrow \infty} -\frac{\varphi(t)^2}{t} \log \mathbb{E} \otimes E_0 \left[\exp \left(-\beta \varphi(t)^{-2} \int_0^t V(Z_s) ds \right) \right] = \beta. \quad (0.14)$$

b) If $\varphi(t) = t^{1/(d+2)}$, then

$$\lim_{t \rightarrow \infty} -t^{-d/(d+2)} \log \mathbb{E} \otimes E_0 \left[\exp \left(-\beta \varphi(t)^{-2} \int_0^t V(Z_s) ds \right) \right] = J(\beta). \quad (0.15)$$

c) If $\varphi(t) \ll t^{1/(d+2)}$, then

$$\lim_{t \rightarrow \infty} -t^{-d/(d+2)} \log \mathbb{E} \otimes E_0 \left[\exp \left(-\beta \varphi(t)^{-2} \int_0^t V(Z_s) ds \right) \right] = \tilde{c}(d, 1). \quad (0.16)$$

The upper bound in (0.16) is already contained in the proof of the upper bound in (0.3). Our next results prove that on the scale $\varphi(t) = t^{1/(d+2)}$ we have a phase transition in dimensions $d \geq 2$ but not in dimension $d = 1$.

Theorem 0.3 For $d \geq 2$, there is a critical point $\beta_c(d) > 0$ such that

$$J(\beta) = \beta \quad \text{for } 0 < \beta \leq \beta_c(d), \quad (0.17)$$

$$J(\beta) < \beta \quad \text{for } \beta > \beta_c(d). \quad (0.18)$$

However, this phase transition picture does not hold in dimension $d = 1$, as the following theorem shows:

Theorem 0.4 Assume $d = 1$. Then for all $\beta > 0$, $J(\beta) < \beta$. There are positive constants $\tilde{C}_1 \leq C_1$ and b_1 such that for all $\beta \in (0, b_1)$:

$$\beta - C_1 \beta^4 \leq J(\beta) \leq \beta - \tilde{C}_1 \beta^4. \quad (0.19)$$

As a consequence, $J(\beta)$ in dimension $d = 1$ is not proportional to β for small values of β : formally we may write $\beta_c(1) = 0$. One should compare Theorems 0.3 and 0.4 with Theorems 0.3 and 0.4 of [10]. The remarkable thing is that in the annealed case we observe the critical dimension $d = 2$ for having a phase transition, while the critical dimension in the quenched case equals $d = 4$.

The next theorem plays an analogous role for the annealed problem as Theorem (0.2) in [10] does in the quenched context:

Theorem 0.5 For any dimension $d \geq 1$ there are positive constants $C_2(d)$, $C_3(d)$, and $b_2(d)$ such that for all $\beta \geq b_2$ the following bounds hold:

$$\tilde{c}(d, 1) - C_2 \sqrt{\frac{\log \beta}{\beta}} \leq J(\beta) \leq \tilde{c}(d, 1) - \frac{C_3}{\sqrt{\beta}}. \quad (0.20)$$

This theorem shows that in the limit $\beta \rightarrow \infty$ one asymptotically approaches the Donsker-Varadhan picture for unscaled potentials; one may compare this with (0.16) and (0.3).

Let us explain how this article is organized: Formulas (0.11) and (0.12) are proved in Section 1, Lemmas 1.1 and 1.2. The claims (0.13) and (0.16) are also proved in Section 1. In Section 2 we show (0.14) and (0.15). The proof of the Theorems 0.3 - 0.4 is prepared in Section 3.1, but it is completed at the end of Section 3, whereas the proof of Theorem 0.5 is given in Section 3.2 (Lemmas 3.3 and 3.4).

1 ASYMPTOTIC BEHAVIOR OF THE PRINCIPAL DIRICHLET EIGENVALUE

In this section we prove Theorem 0.1 and the lower bound in (0.16). For $r > 0$, $y \in \mathbb{R}^d$, and a function ϕ the scaling operator S_y^r is defined by

$$(S_y^r \phi)(x) \stackrel{\text{def}}{=} r^{-d/2} \phi((x - y)/r). \quad (1.1)$$

Lemma 1.1 For all positive scaling functions $l(t) \gg \varphi(t)$ and $\beta > 0$:

$$\limsup_{t \rightarrow \infty} -\frac{\varphi(t)^2}{t} \log \mathbb{E} \left[\exp \left(-t \lambda_{\beta \varphi(t)^{-2} V}(\mathcal{T}_{l(t)}) \right) \right] \leq \beta. \quad (1.2)$$

For the special scaling function $\varphi(t) = t^{1/(d+2)}$ the following bound holds:

$$\limsup_{t \rightarrow \infty} -t^{-d/(d+2)} \log \mathbb{E} \left[\exp \left(-t \lambda_{\beta t^{-2/(d+2)} V}(\mathcal{T}_{l(t)}) \right) \right] \leq J(\beta). \quad (1.3)$$

Remark: For the special scaling function $\varphi(t) = t^{1/(d+2)}$, the inequality (1.3) is stronger than the general inequality (1.2) at least in some cases; see Theorem 0.3. The inequality (1.2) can be proven using Jensen's inequality. We do not proceed in this way, but treat instead (1.2) and (1.3) at the same time below.

Proof of Lemma 1.1. Choose $\phi \in \Phi$. Since ϕ is compactly supported and $l(t) \gg \varphi(t)$ we have for all sufficiently large t that the function $S_0^{\varphi(t)} \phi$ is supported in $\mathcal{T}_{l(t)}$. We

estimate for these large $t > 0$, using the notation $*$ for the convolution operator, and $W_r^-(x) \stackrel{\text{def}}{=} r^d W(-rx)$:

$$\begin{aligned}
& \log \mathbb{E} \left[\exp \left(-t \lambda_{\beta \varphi(t)^{-2} V}(\mathcal{I}_{l(t)}) \right) \right] \\
& \geq \log \mathbb{E} \left[\exp \left(-\frac{t}{2} \left\| \nabla (S_0^{\varphi(t)} \phi) \right\|_2^2 - \frac{\beta t}{\varphi(t)^2} \int_{\mathcal{I}_{l(t)}} (S_0^{\varphi(t)} \phi)^2 V dx \right) \right] \\
& = -\frac{t}{2\varphi(t)^2} \|\nabla \phi\|_2^2 + \log \mathbb{E} \left[\exp \left(-\frac{\beta t}{\varphi(t)^2} \int_{\mathbb{R}^d} W_1^- * (S_0^{\varphi(t)} \phi)^2 d\omega \right) \right] \\
& = -\frac{t}{2\varphi(t)^2} \|\nabla \phi\|_2^2 + \int_{\mathbb{R}^d} \left(\exp \left(-\frac{\beta t}{\varphi(t)^2} W_1^- * (S_0^{\varphi(t)} \phi)^2 \right) - 1 \right) dx \quad (1.4) \\
& = -\frac{t}{2\varphi(t)^2} \|\nabla \phi\|_2^2 + \varphi(t)^d \int_{\mathbb{R}^d} \left(\exp \left(-\frac{\beta t}{\varphi(t)^{d+2}} W_{\varphi(t)}^- * \phi^2 \right) - 1 \right) dx \\
& \geq -\frac{t}{2\varphi(t)^2} \|\nabla \phi\|_2^2 - \frac{\beta t}{\varphi(t)^2} \int_{\mathbb{R}^d} W_{\varphi(t)}^- * \phi^2 dx \\
& = -\frac{t}{\varphi(t)^2} \left(\frac{1}{2} \|\nabla \phi\|_2^2 + \beta \right);
\end{aligned}$$

we used $\|\phi\|_2^2 = 1$, $W \geq 0$, $\|W_r^-\|_1 = \|W\|_1 = 1$ in the last step. The estimate (1.4) implies

$$\limsup_{t \rightarrow \infty} -\frac{\varphi(t)^2}{t} \log \mathbb{E} \left[\exp \left(-t \lambda_{\beta \varphi(t)^{-2} V}(\mathcal{I}_{l(t)}) \right) \right] \leq \frac{1}{2} \|\nabla \phi\|_2^2 + \beta. \quad (1.5)$$

The gradient term $\|\nabla \phi\|_2^2$ can be made arbitrarily small; hence (1.5) implies the claim (1.2). To derive (1.3) in the case $\varphi(t) = t^{1/(d+2)}$, we proceed as follows: Using that ϕ is continuous, and $W \geq 0$, $\|W_r^-\|_1 = \|W\|_1 = 1$, we get $W_{\varphi(t)}^- * \phi^2 \xrightarrow{t \rightarrow \infty} \phi^2$ pointwise. Using the dominated convergence theorem one sees

$$\int_{\mathbb{R}^d} \left(\exp \left(-\beta W_{\varphi(t)}^- * \phi^2 \right) - 1 \right) dx \xrightarrow{t \rightarrow \infty} \Lambda_\phi(-\beta), \quad (1.6)$$

and hence, using the fifth line in (1.4):

$$\limsup_{t \rightarrow \infty} -t^{-d/(d+2)} \log \mathbb{E} \left[\exp \left(-t \lambda_{\beta \varphi(t)^{-2} V}(\mathcal{I}_{l(t)}) \right) \right] \leq \frac{1}{2} \|\nabla \phi\|_2^2 - \Lambda_\phi(-\beta). \quad (1.7)$$

Using the definition (0.8) of J , the claim (1.3) of Lemma 1.1 follows from (1.7). □

Lemma 1.2 *Assume that the scaling function φ satisfies either $t^{1/(d+2)} \ll \varphi(t) \ll t^{1/2}$ or $\varphi(t) = t^{1/(d+2)}$. Further assume that the scaling function l fulfills $\log l(t) \ll t/\varphi(t)^2$.*

Then for all $\beta > 0$:

$$\liminf_{t \rightarrow \infty} -\frac{\varphi(t)^2}{t} \log \mathbb{E} \left[\exp \left(-t \lambda_{\beta \varphi(t)^{-2} V}(\mathcal{I}_{l(t)}) \right) \right] \geq \begin{cases} J(\beta) & \text{for } \varphi(t) = t^{1/(d+2)}, \\ \beta & \text{for } t^{1/(d+2)} \ll \varphi(t) \ll t^{1/2}. \end{cases} \quad (1.8)$$

Proof of Lemma 1.2. We use some notations from [10], Section 2.1: for $M > 0$, $V^M \stackrel{\text{def}}{=} V \wedge M$, and for $\zeta > 0$, $j \in \zeta \mathbb{Z}^d$ we set $K_j(\zeta) \stackrel{\text{def}}{=} j + [0, \zeta)^d$ and $\omega^\zeta \stackrel{\text{def}}{=} \sum_{j \in \zeta \mathbb{Z}^d} \mathbf{1}_{\{\omega(K_j) \geq 1\}} \delta_j$; here δ_j means the Dirac measure located at j . We set $\tilde{V}^\zeta(x, \omega) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} W(x-y) \omega^\zeta(dy)$, this is the Bernoulli version of the Poissonian potential $V(\cdot, \omega)$. Finally for $\psi \in H^{1,2}(\mathbb{R}^d)$ and a measurable function F we abbreviate $\mathcal{E}_F(\psi) \stackrel{\text{def}}{=} \|\nabla \psi\|_2^2/2 + \int F \psi^2 dx$ (whenever the right-hand side is well-defined). The following notation deviates a little from the one chosen in [10], since we have now $l(t)$ instead of t as the length scale of the universe box: $Y_{R,t}^{\varphi,l} \stackrel{\text{def}}{=} \{y \in d^{-1/2} R \varphi(t) \mathbb{Z}^d : B_{R\varphi(t)}(y) \cap \mathcal{I}_{l(t)} \neq \emptyset\}$. Lemma 2.3 and Lemma 2.4 in [10], especially the estimate (2.40) there, show: For positive β , η and ζ there are $M > 0$, $R \geq 1$, a finite set $\Psi \subseteq \{\psi \in C_c^1(\overline{B_{R+1}}(0)) : \|\psi\|_2 = 1\}$ and $t_0 > 0$ such that for all $t > t_0$ and $\omega \in \Omega$:

$$\lambda_{\beta \varphi(t)^{-2} V}(\mathcal{I}_{l(t)}) \geq \min_{\substack{y \in Y_{R,t}^{\varphi,l} \\ \psi \in \Psi}} \mathcal{E}_{\beta \varphi(t)^{-2} V^M}(S_y^{\varphi(t)} \psi) - 3\varphi(t)^{-2} \eta, \quad (1.9)$$

and

$$\begin{aligned} & \varphi(t)^2 \max_{\substack{y \in Y_{R,t}^{\varphi,l} \\ \psi \in \Psi}} \left(\mathcal{E}_{\beta \varphi(t)^{-2} \tilde{V}^\zeta}(S_y^{\varphi(t)} \psi) - \mathcal{E}_{\beta \varphi(t)^{-2} V^M}(S_y^{\varphi(t)} \psi) \right) \\ & \leq 2\beta \max_{\psi \in \Psi} (\|\psi\|_\infty + \|\nabla \psi\|_\infty)^2 \sqrt{d} \zeta^{1-d} R^d |B_4(0)| \varphi(t)^{-1} \leq \eta. \end{aligned} \quad (1.10)$$

We emphasize the following fact: t_0 does not depend on $\omega \in \Omega$, since the first estimate in (1.10), which coincides with the last estimate in (2.40) in [10], is uniform in the Poissonian configuration ω . (1.9) and (1.10) yield

$$\lambda_{\beta \varphi(t)^{-2} V}(\mathcal{I}_{l(t)}) \geq \min_{\substack{y \in Y_{R,t}^{\varphi,l} \\ \psi \in \Psi}} \mathcal{E}_{\beta \varphi(t)^{-2} \tilde{V}^\zeta}(S_y^{\varphi(t)} \psi) - 4\varphi(t)^{-2} \eta. \quad (1.11)$$

Therefore (again for $t \geq t_0$):

$$\begin{aligned}
& \log \mathbb{E} \left[\exp \left(-t \lambda_{\beta \varphi(t)^{-2} V}(\mathcal{I}_l(t)) \right) \right] \\
& \leq \log \mathbb{E} \left[\max_{\substack{y \in Y_{R,t}^{\varphi,l} \\ \psi \in \Psi}} \exp \left(-t \mathcal{E}_{\beta \varphi(t)^{-2} \tilde{V}^\zeta}(S_y^{\varphi(t)} \psi) \right) \right] + 4t \varphi(t)^{-2} \eta \\
& \leq \log \sum_{\substack{y \in Y_{R,t}^{\varphi,l} \\ \psi \in \Psi}} \mathbb{E} \left[\exp \left(-t \mathcal{E}_{\beta \varphi(t)^{-2} \tilde{V}^\zeta}(S_y^{\varphi(t)} \psi) \right) \right] + 4t \varphi(t)^{-2} \eta \tag{1.12} \\
& \leq \sup_{\substack{y \in \mathbb{R}^d \\ \psi \in \Psi}} \log \mathbb{E} \left[\exp \left(-t \mathcal{E}_{\beta \varphi(t)^{-2} \tilde{V}^\zeta}(S_y^{\varphi(t)} \psi) \right) \right] + \log |Y_{R,t}^{\varphi,l}| + \log |\Psi| + 4t \varphi(t)^{-2} \eta.
\end{aligned}$$

To estimate the expectation in the last expression, we proceed analogous to the quenched case, see Lemma 2.5 in [10]: We define the discretized version $\nu^\zeta \stackrel{\text{def}}{=} \zeta^d \sum_{j \in \zeta \mathbb{Z}^d} \delta_j$ of the Lebesgue measure, abbreviate $m \stackrel{\text{def}}{=} \zeta^{-d} (1 - e^{-\zeta^d}) \xrightarrow{\zeta \rightarrow 0} 1$, and use the bound (2.45) in [10], which is the following estimate for the Laplace transform of a Bernoulli process (discretized Poissonian point process): $\log \mathbb{E} \left[\exp \left(\int f d\omega^\zeta \right) \right] \leq m \int (e^f - 1) d\nu^\zeta$.

We get for all $\psi \in \Psi$ (compare with (2.46) in [10]):

$$\begin{aligned}
& \frac{\varphi(t)^2}{t} \sup_{y \in \mathbb{R}^d} \log \mathbb{E} \left[\exp \left(-t \mathcal{E}_{\beta \varphi(t)^{-2} \tilde{V}^\zeta}(S_y^{\varphi(t)} \psi) \right) \right] \\
& = \sup_{y \in \mathbb{R}^d} -\frac{\varphi(t)^2}{2} \|\nabla S_y^{\varphi(t)} \psi\|_2^2 + \frac{\varphi(t)^2}{t} \log \mathbb{E} \left[\exp \left(-\frac{\beta t}{\varphi(t)^2} \int_{\mathbb{R}^d} (S_y^{\varphi(t)} \psi)^2 \tilde{V}^\zeta dx \right) \right] \\
& = -\frac{1}{2} \|\nabla \psi\|_2^2 + \frac{\varphi(t)^2}{t} \sup_{y \in \mathbb{R}^d} \log \mathbb{E} \left[\exp \left(-\frac{\beta t}{\varphi(t)^2} \int_{\mathbb{R}^d} (S_y^{\varphi(t)} \psi)^2 * W_1^- d\omega^\zeta \right) \right] \tag{1.13} \\
& \leq -\frac{1}{2} \|\nabla \psi\|_2^2 + \frac{m \varphi(t)^2}{t} \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left(\exp \left\{ -\frac{\beta t}{\varphi(t)^2} (S_y^{\varphi(t)} \psi)^2 * W_1^- \right\} - 1 \right) d\nu^\zeta \\
& = -\frac{1}{2} \|\nabla \psi\|_2^2 + \frac{m \varphi(t)^{d+2}}{t} \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left(\exp \left\{ \frac{-\beta t}{\varphi(t)^{d+2}} (S_{y/\varphi(t)}^1 \psi)^2 * W_{\varphi(t)}^- \right\} - 1 \right) d\nu^{\zeta/\varphi(t)} \\
& \xrightarrow{t \rightarrow \infty} \begin{cases} -\frac{1}{2} \|\nabla \psi\|_2^2 + m \Lambda_\psi(-\beta) & \text{for } \varphi(t) = t^{1/(d+2)}, \\ -\frac{1}{2} \|\nabla \psi\|_2^2 - m\beta & \text{for } \varphi(t) \gg t^{1/(d+2)}. \end{cases}
\end{aligned}$$

The assumptions $\log l(t) \ll t/\varphi(t)^2$ and $\varphi(t) \ll t^{1/2}$ imply $\log |Y_{R,t}^{\varphi,l}| + \log |\Psi| \ll t/\varphi(t)^2$. Combining (1.12) and (1.13) we obtain

$$\begin{aligned} & \liminf_{t \rightarrow \infty} -\frac{\varphi(t)^2}{t} \log \mathbb{E} \left[\exp \left(-t \lambda_{\beta \varphi(t)^{-2} V}(\mathcal{I}_l(t)) \right) \right] \\ & \geq \begin{cases} \min_{\psi \in \Psi} \frac{1}{2} \|\nabla \psi\|_2^2 - m \Lambda_\psi(-\beta) - 4\eta & \text{for } \varphi(t) = t^{1/(d+2)}, \\ \min_{\psi \in \Psi} \frac{1}{2} \|\nabla \psi\|_2^2 + m\beta - 4\eta & \text{for } t^{1/2} \gg \varphi(t) \gg t^{1/(d+2)}, \end{cases} \quad (1.14) \\ & \geq \begin{cases} mJ(\beta) - 4\eta & \text{for } \varphi(t) = t^{1/(d+2)}, \\ m\beta - 4\eta & \text{for } t^{1/2} \gg \varphi(t) \gg t^{1/(d+2)}. \end{cases} \end{aligned}$$

The claim (1.8) of Lemma 1.2 now follows by taking the limits $\eta \rightarrow 0$ and $\zeta \rightarrow 0$, i.e. $m \uparrow 1$. Lemma 1.2 is proved. \square

Proof of (0.13) and (0.16). For $\varphi(t) \ll t^{1/(d+2)}$ we have for all $\beta, \beta' > 0$ and all large t :

$$\frac{\beta}{\varphi(t)^2} V \geq \frac{\beta'}{t^{2/(d+2)}} V. \quad (1.15)$$

By monotonicity, this implies $\lambda_{\beta \varphi(t)^{-2} V}(\mathcal{I}_l(t)) \geq \lambda_{\beta' t^{-2/(d+2)} V}(\mathcal{I}_l(t))$. Hence, using (0.12) and Theorem 0.5 (which is proven in Section 3, below):

$$\liminf_{t \rightarrow \infty} -t^{-d/(d+2)} \log \mathbb{E} \left[\exp \left(-t \lambda_{\beta \varphi(t)^{-2} V}(\mathcal{I}_l(t)) \right) \right] \geq J(\beta') \xrightarrow{\beta' \rightarrow \infty} \tilde{c}(d, 1). \quad (1.16)$$

An analogous monotonicity estimate also holds true for the partition sums; this proves the lower bounds in (0.13) and (0.16).

To prove the upper bounds, we set $r(t) \stackrel{\text{def}}{=} t^{1/(d+2)} \tilde{r}_d$ and choose a length scale $l(t) \gg r(t)$. Then we have (where a denotes the minimal radius such that the support of W is contained in the ball $\bar{B}_a(0)$):

$$\begin{aligned} & \limsup_{t \rightarrow \infty} -t^{-d/(d+2)} \log \mathbb{E} \left[\exp \left(-t \lambda_{\beta \varphi(t)^{-2} V}(\mathcal{I}_l(t)) \right) \right] \\ & \leq \limsup_{t \rightarrow \infty} -t^{-d/(d+2)} \log \mathbb{E} \left[\exp \left(-t \lambda_{\beta \varphi(t)^{-2} V}(\mathcal{I}_l(t)) \right), \omega(B_{r(t)+a}(0)) = 0 \right] \\ & \leq \limsup_{t \rightarrow \infty} -t^{-d/(d+2)} \log \mathbb{E} \left[\exp \left(-t \lambda_{V \equiv 0}(B_{r(t)}(0)) \right), \omega(B_{r(t)+a}(0)) = 0 \right] \quad (1.17) \\ & = \limsup_{t \rightarrow \infty} t^{-d/(d+2)} \cdot \left(t \lambda_d r(t)^{-2} - \log \mathbb{P} \left[\omega(B_{r(t)+a}(0)) = 0 \right] \right) \\ & = \lambda_d \tilde{r}_d^{-2} + \tilde{r}_d^d |B_1(0)| = \tilde{c}(d, 1). \end{aligned}$$

The proof of the upper bound (0.16) is the same as in [12], Theorem 4.5.3, (4.5.33)–(4.5.36). This finishes the proofs. \square

2 ASYMPTOTIC BEHAVIOR OF THE PARTITION SUM

In this section, we prove (0.14) and (0.15) of Theorem 0.2. The main tool to obtain upper bounds in (0.14) and (0.15) is a change of measure, which transforms Brownian motion into a (stationary) diffusion process: Using this diffusion process as “strategy” for the Brownian particle turns out to be optimal (at least in the leading order) for survival among scaled Poissonian obstacles.

Proof of the upper bounds in (0.14) and (0.15). We treat both cases at the same time. Alternatively, the proof of (0.14) could be treated separately, simply using Jensen’s inequality.

Let $\phi \in \Phi$, $\phi \geq 0$. We first introduce a modification ϕ^ϵ of ϕ which is positive everywhere with an exponential decay at infinity: Let $\delta_1 \in C^\infty(\mathbb{R}^d)$ denote a fixed positive function with exponential decay at infinity and with $\|\delta_1\|_1 = 1$, to be explicit, say $\delta_1(x) = c_4 e^{-|x|}$ with a positive constant c_4 for all x outside a compact subset of \mathbb{R}^d . For every multiindex n , we get the following bound on the n -th derivative: There is a constant $c_5(n) > 0$ such that $|D_n \delta_1| \leq c_5(n) \delta_1$. For $\epsilon > 0$, we define the following approximation to Dirac’s δ : $\delta_\epsilon(x) \stackrel{\text{def}}{=} \epsilon^{-d} \delta_1(x/\epsilon)$. Let $\hat{f}(k) = \int e^{-ikx} f(x) dx$ denote the Fourier transform. Using the dominated convergence theorem we see:

$$\|\phi * \delta_\epsilon\|_2^2 = (2\pi)^{-d} \left\| \hat{\phi}(k) \hat{\delta}_1(\epsilon k) \right\|_2^2 \xrightarrow{\epsilon \rightarrow 0} (2\pi)^{-d} \left\| \hat{\phi} \right\|_2^2 = \|\phi\|_2^2 = 1, \quad (2.1)$$

and similarly

$$\|\nabla \phi * \delta_\epsilon\|_2^2 \xrightarrow{\epsilon \rightarrow 0} \|\nabla \phi\|_2^2. \quad (2.2)$$

Consequently $\phi^\epsilon \stackrel{\text{def}}{=} \|\phi * \delta_\epsilon\|_2^{-1} (\phi * \delta_\epsilon)$ satisfies $\|\nabla \phi^\epsilon\|_2^2 \xrightarrow{\epsilon \rightarrow 0} \|\nabla \phi\|_2^2$, $\phi^\epsilon > 0$, and there exist $\epsilon_0 > 0$, $r_0 > 0$ and $c_6 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ and $x \in \mathbb{R}^d$ with $|x| > r_0$: $\phi^\epsilon(x) \leq c_6 e^{-|x|}$. We get for all $\beta > 0$, using the dominated convergence theorem once more:

$$\frac{1}{2} \|\nabla \phi^\epsilon\|_2^2 - \int_{\mathbb{R}^d} (e^{-\beta(\phi^\epsilon)^2} - 1) dx \xrightarrow{\epsilon \rightarrow 0} \frac{1}{2} \|\nabla \phi\|_2^2 - \int_{\mathbb{R}^d} (e^{-\beta\phi^2} - 1) dx. \quad (2.3)$$

For $\epsilon > 0$, $t > 0$, we set $\phi_{\epsilon,t} \stackrel{\text{def}}{=} S_0^{\varphi(t)} \phi^\epsilon$. With ϵ , t being fixed for the moment, we define $b \stackrel{\text{def}}{=} \nabla \log \phi_{\epsilon,t}$. By a change of measure, we introduce a diffusion process with drift $b(Z_s)$ over the finite time horizon $t < \infty$: the bounds on the derivatives of δ_1 imply

$$\sup_{x \in \mathbb{R}^d} |D_n b(x)| < \infty \quad (2.4)$$

for every multindex n ; especially the Novikov condition (see e.g. [7], Corollary 3.5.13)

$$E_x \left[\exp \left(\frac{1}{2} \int_0^t |b(Z_s)|^2 ds \right) \right] < \infty \quad (2.5)$$

is satisfied. By the Cameron-Martin-Girsanov theorem,

$$\tilde{Z}_s = Z_s - \int_0^s b(Z_u) du \quad (2.6)$$

is a d -dimensional Brownian motion with respect to the probability measure

$$Q_x = \exp \left\{ \int_0^t b(Z_s) dZ_s - \frac{1}{2} \int_0^t |b(Z_s)|^2 ds \right\} P_x. \quad (2.7)$$

We denote the expectation operator with respect to Q_x by E_x^Q , while the symbol E_x is reserved for expectations with respect to P_x . We claim that $\phi_{\epsilon,t}^2 dx$ is an invariant distribution with respect to the transformed diffusion process, i.e. for every non-negative measurable test function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ we have for all $s \in [0, t]$:

$$\int_{\mathbb{R}^d} \phi_{\epsilon,t}(x)^2 E_x^Q[f(Z_s)] dx = \int_{\mathbb{R}^d} \phi_{\epsilon,t}(x)^2 f(x) dx. \quad (2.8)$$

It suffices to prove (2.8) for $f \in C_c^\infty(\mathbb{R}^d)$: In this case, the bounds (2.4) on the derivative of the drift imply that

$$g(x, s) \stackrel{\text{def}}{=} E_x^Q[f(Z_s)] \quad (2.9)$$

is a classical solution of the Cauchy problem

$$\frac{\partial g}{\partial s} = \frac{1}{2} \Delta g + b \cdot \nabla g, \quad (2.10)$$

$$g(x, 0) = f(x), \quad (2.11)$$

with bounded derivatives in x and s of every order (see e.g. [5], §5.3, Theorems 3.1 and 3.2, and [6], §6.4, §6.5). We use the heat equation (2.10) and integrate partially to get:

$$\begin{aligned} \frac{d}{ds} \int_{\mathbb{R}^d} g(x, s) \phi_{\epsilon,t}(x)^2 dx &= \int_{\mathbb{R}^d} \frac{\partial g}{\partial s}(x, s) \phi_{\epsilon,t}(x)^2 dx \\ &= \int_{\mathbb{R}^d} \left(\frac{1}{2} \Delta g(x, s) + \frac{\nabla \phi_{\epsilon,t}(x)}{\phi_{\epsilon,t}(x)} \cdot \nabla g(x, s) \right) \phi_{\epsilon,t}(x)^2 dx \\ &= \int_{\mathbb{R}^d} \nabla g(x, s) \phi_{\epsilon,t}(x) (\nabla \phi_{\epsilon,t}(x) - \nabla \phi_{\epsilon,t}(x)) dx = 0. \end{aligned} \quad (2.12)$$

The boundary terms of the partial integration vanish, since $\phi_{\epsilon,t}$ and its derivatives decay exponentially at infinity, while g and its derivatives are bounded. Our claim (2.8) is a consequence of (2.12).

The measure P_x is absolutely continuous with respect to Q_x with the Radon-Nikodym derivative

$$\begin{aligned} \frac{dP_x}{dQ_x} &= \exp \left\{ - \int_0^t b(Z_s) dZ_s + \frac{1}{2} \int_0^t |b(Z_s)|^2 ds \right\} \\ &= \exp \left\{ - \int_0^t b(Z_s) d\tilde{Z}_s - \frac{1}{2} \int_0^t |b(Z_s)|^2 ds \right\}. \end{aligned} \quad (2.13)$$

We remark that the stochastic integral in (2.13) remains unchanged when the underlying probability measure P_x is replaced by the equivalent measure Q_x . By translational invariance of the Poisson process we get

$$\begin{aligned} &\mathbb{E} \otimes E_0 \left[\exp \left(- \frac{\beta}{\varphi(t)^2} \int_0^t V(Z_s) ds \right) \right] \\ &= \mathbb{E} \left[\int_{\mathbb{R}^d} E_x \left[\exp \left(- \frac{\beta}{\varphi(t)^2} \int_0^t V(Z_s) ds \right) \right] \phi_{\epsilon,t}(x)^2 dx \right]. \end{aligned} \quad (2.14)$$

Define $Q \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} Q_x[\cdot] \phi_{\epsilon,t}(x)^2 dx$ to be the probability measure which makes $(Z_s)_{0 \leq s \leq t}$ a (stationary) diffusion process with starting distribution $\phi_{\epsilon,t}^2$ and drift b . We use (2.13), Jensen's inequality, and the fact that $(\int_0^s b(Z_u) d\tilde{Z}_u)_{0 \leq s \leq t}$ is a Q -martingale in the following estimate:

$$\begin{aligned} &\int_{\mathbb{R}^d} E_x \left[\exp \left(- \frac{\beta}{\varphi(t)^2} \int_0^t V(Z_s) ds \right) \right] \phi_{\epsilon,t}(x)^2 dx \\ &= E^Q \left[\exp \left(- \int_0^t b(Z_s) d\tilde{Z}_s - \frac{1}{2} \int_0^t |b(Z_s)|^2 ds - \frac{\beta}{\varphi(t)^2} \int_0^t V(Z_s) ds \right) \right] \\ &\geq \exp \left\{ E^Q \left[- \int_0^t b(Z_s) d\tilde{Z}_s - \frac{1}{2} \int_0^t |b(Z_s)|^2 ds - \frac{\beta}{\varphi(t)^2} \int_0^t V(Z_s) ds \right] \right\} \\ &= \exp \left\{ - \int_0^t E^Q \left[\frac{1}{2} |b(Z_s)|^2 + \frac{\beta}{\varphi(t)^2} V(Z_s) \right] ds \right\} \\ &\stackrel{(2.8)}{=} \exp \left\{ -t \int_{\mathbb{R}^d} \left(\frac{1}{2} |b(x)|^2 + \frac{\beta}{\varphi(t)^2} V(x) \right) \phi_{\epsilon,t}(x)^2 dx \right\} \\ &= \exp \left\{ -\frac{t}{2} \|\nabla \phi_{\epsilon,t}\|_2^2 - \frac{\beta t}{\varphi(t)^2} \int_{\mathbb{R}^d} V \phi_{\epsilon,t}^2 dx \right\}. \end{aligned} \quad (2.15)$$

Combining (2.15) with (2.14), we obtain, using the dominated convergence theorem (recall

that ϕ^ϵ decays exponentially fast at infinity) and (2.1)–(2.3):

$$\begin{aligned}
& -\frac{\varphi(t)^2}{t} \log \mathbb{E} \otimes E_0 \left[\exp \left(-\int_0^t \frac{\beta}{\varphi(t)^2} V(Z_s) ds \right) \right] \\
& \leq \frac{\varphi(t)^2}{2} \|\nabla \phi_{\epsilon,t}\|_2^2 - \frac{\varphi(t)^2}{t} \int_{\mathbb{R}^d} \left(\exp \left\{ -\frac{\beta t}{\varphi(t)^2} \phi_{\epsilon,t}^2 * W_1^- \right\} - 1 \right) dx \\
& = \frac{1}{2} \|\nabla \phi^\epsilon\|_2^2 - \frac{\varphi(t)^{d+2}}{t} \int_{\mathbb{R}^d} \left(\exp \left\{ -\frac{\beta t}{\varphi(t)^{d+2}} (\phi^\epsilon)^2 * W_{\varphi(t)}^- \right\} - 1 \right) dx \quad (2.16) \\
& \xrightarrow{t \rightarrow \infty} \begin{cases} \frac{1}{2} \|\nabla \phi^\epsilon\|_2^2 - \int_{\mathbb{R}^d} (\exp \{-\beta(\phi^\epsilon)^2\} - 1) dx & \text{for } \varphi(t) = t^{1/(d+2)}, \\ \frac{1}{2} \|\nabla \phi^\epsilon\|_2^2 + \beta \int_{\mathbb{R}^d} (\phi^\epsilon)^2 dx & \text{for } \varphi(t) \gg t^{1/(d+2)} \end{cases} \\
& \xrightarrow{\epsilon \rightarrow 0} \begin{cases} \frac{1}{2} \|\nabla \phi\|_2^2 - \Lambda_\phi(-\beta) & \text{for } \varphi(t) = t^{1/(d+2)}, \\ \frac{1}{2} \|\nabla \phi\|_2^2 + \beta & \text{for } \varphi(t) \gg t^{1/(d+2)}. \end{cases}
\end{aligned}$$

When we optimize over $\phi \in \Phi$, we get the two upper bounds in (0.14) and (0.15). \square

Proof of the lower bounds in (0.14) and (0.15). We treat both cases at the same time. We choose any scaling function $l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\log l(t) \ll t/\varphi(t)^2$ and $l(t) \gg t/\varphi(t)$ as $t \rightarrow \infty$; one possible choice is $l(t) = t$.

Let $T_{l(t)} \stackrel{\text{def}}{=} \inf\{s : Z_s \notin \mathcal{T}_{l(t)}\}$ denote the exit time from the box $\mathcal{T}_{l(t)}$. Since the potential V is bounded on compact domains, the random Schrödinger operator $-\Delta/2 + \beta\varphi(t)^{-2}V$ is essentially self-adjoint on $C_c^\infty(\mathcal{T}_{l(t)})$; for fixed $\beta > 0$ and scaling functions φ and l we denote its closure by H_t . The self-adjoint operator H_t is bounded from below: $H_t \geq \lambda_{\beta\varphi(t)^{-2}V}(\mathcal{T}_{l(t)})\mathbf{1}$; hence $e^{-tH_t} : L^2(\mathcal{T}_{l(t)}) \rightarrow L^2(\mathcal{T}_{l(t)})$ is a bounded, self-adjoint operator with

$$\|e^{-tH_t}\|_{L^2 \rightarrow L^2} \leq e^{-t\lambda_{\beta\varphi(t)^{-2}V}(\mathcal{T}_{l(t)})}, \quad (2.17)$$

we also refer to [12], Proposition 1.3.3. Let $f \in C_c^\infty(\mathbb{R}^d)$, $f \geq 0$, $\|f\|_1 = 1$ be any fixed test function. We choose a fixed $r > 0$ such that f is supported in \mathcal{T}_r . We get for $l(t) > r$, using the Feynman-Kac representation of e^{-tH_t} :

$$\begin{aligned}
& \int_{\mathbb{R}^d} f(x) E_x \left[\exp \left\{ -\frac{\beta}{\varphi(t)^2} \int_0^t V(Z_s) ds \right\} \right] dx \\
& \leq \int_{\mathbb{R}^d} f(x) \left(P_x [T_{l(t)} \leq t] + E_x \left[\exp \left\{ -\frac{\beta}{\varphi(t)^2} \int_0^t V(Z_s) ds \right\}, T_{l(t)} > t \right] \right) dx \\
& = \int_{\mathbb{R}^d} f(x) P_x [T_{l(t)} \leq t] dx + \langle \mathbf{1}_{\mathcal{T}_{l(t)}}, e^{-tH_t} f \rangle \quad (2.18) \\
& \leq P_0 [T_{l(t)-r} \leq t] + \|\mathbf{1}_{\mathcal{T}_{l(t)}}\|_2 \|e^{-tH_t}\|_{L^2 \rightarrow L^2} \|f\|_2 \\
& \leq 4d \exp\{-(l(t) - r)^2/(2t)\} + (2l(t))^{d/2} \|f\|_2 \exp\{-t\lambda_{\beta\varphi(t)^{-2}V}(\mathcal{T}_{l(t)})\}.
\end{aligned}$$

Using Lemma 1.2 and $l(t)^2/t \gg t/\varphi(t)^2$, we see that the first summand in the last sum is negligible as $t \rightarrow \infty$ compared to the expected value of the second one. We get, using translation invariance of the Poisson process, Lemma 1.2, and $\log l(t) \ll t/\varphi(t)^2$:

$$\begin{aligned}
& \liminf_{t \rightarrow \infty} -\frac{\varphi(t)^2}{t} \log \mathbb{E} \otimes E_0 \left[\exp \left\{ -\frac{\beta}{\varphi(t)^2} \int_0^t V(Z_s) ds \right\} \right] \\
&= \liminf_{t \rightarrow \infty} -\frac{\varphi(t)^2}{t} \log \mathbb{E} \left[\int_{\mathbb{R}^d} f(x) E_x \left[\exp \left\{ -\frac{\beta}{\varphi(t)^2} \int_0^t V(Z_s) ds \right\} \right] \right] dx \\
&\geq \liminf_{t \rightarrow \infty} -\frac{\varphi(t)^2}{t} \log \mathbb{E} \left[\exp \left\{ -t \lambda_{\beta \varphi(t)^{-2} V}(\mathcal{I}_{l(t)}) \right\} \right] \\
&= \begin{cases} J(\beta) & \text{for } \varphi(t) = t^{1/(d+2)}, \\ \beta & \text{for } t^{1/(d+2)} \ll \varphi(t) \ll (t/\log t)^{1/2}. \end{cases} \tag{2.19}
\end{aligned}$$

This finishes the proof of the lower bounds in (0.14) and (0.15). □

3 ANALYSIS OF THE VARIATIONAL PRINCIPLE

In this section, we prove Theorems 0.3 – 0.5. We start with some easy facts:

Lemma 3.1 *J is a concave, monotonically increasing function with $J(\beta) \leq \beta$.*

Proof of Lemma 3.1. For all $\phi \in \Phi$, the functions $\mathbb{R}^+ \ni \beta \mapsto \frac{1}{2} \|\nabla \phi\|_2^2 - \Lambda_\phi(-\beta)$ are concave and monotonically increasing. Consequently the infimum over $\phi \in \Phi$ has these properties, too. Furthermore, the fact $-\Lambda_\phi(-\beta) \leq \beta$ implies $J(\beta) \leq \inf_{\phi \in \Phi} \frac{1}{2} \|\nabla \phi\|_2^2 + \beta = \beta$. □

For $\phi \in \Phi$, $r > 0$ we introduce the scaled version $\phi_r(x) \stackrel{\text{def}}{=} r^{-d/2} \phi(x/r)$; its basic properties are collected in (3.26)–(3.27) in [9]. We get for every $\beta > 0$ and every dimension $d \geq 1$:

$$\begin{aligned}
J(\beta) &= \inf \left\{ \frac{1}{2} \|\nabla \phi_r\|_2^2 - \Lambda_{\phi_r}(-\beta) : r > 0, \phi \in \Phi, \|\nabla \phi\|_2 = 1 \right\} \\
&= \inf \left\{ \frac{1}{2r^2} - r^d \Lambda_\phi(-r^{-d}\beta) : r > 0, \phi \in \Phi, \|\nabla \phi\|_2 = 1 \right\}. \tag{3.1}
\end{aligned}$$

Finally we denote by r_d the radius of a d -dimensional ball of volume d . We set $c(d, 1) \stackrel{\text{def}}{=} r_d^{-2} \lambda_d$; this is the principal Dirichlet eigenvalue of $-\Delta/2$ on such a ball $B_{r_d}(0)$. These $c(d, 1)$ and r_d play an analogous role in the quenched case as $\tilde{c}(d, 1)$ and \tilde{r}_d in the annealed problem (see [10]).

3.1 The phase transition picture

Lemma 3.2

a) For $d = 1$, there are positive constants $c_1 \leq C_1$ and c_7 such that for all $\beta > 0$:

$$\beta - C_1\beta^4 \leq J(\beta) \leq \beta - c_1\beta^4 + c_7\beta^7. \quad (3.2)$$

b) For $d \geq 2$, there is $b_3(d) > 0$ such that for all $\beta \in (0, b_3(d)]$ we have $J(\beta) \geq \beta$.

Remark: We write the upper bound in (3.2) in this form, since it naturally arises in the proof.

Proof of Lemma 3.2. Let $\phi \in \Phi$ with $\|\nabla\phi\|_2 = 1$, and $r > 0$. For $y = \beta\phi_r(x)^2 \geq 0$ we integrate the inequalities $y - y^2/2 \leq 1 - e^{-y} \leq y - y^2/2 + y^3/6$ over $x \in \mathbb{R}^d$ and use $\|\nabla\phi_r\|_2^2 = r^{-2}$, $\|\phi_r\|_2^2 = 1$, $\|\phi_r\|_4^4 = r^{-d}\|\phi\|_4^4$, $\|\phi_r\|_6^6 = r^{-2d}\|\phi\|_6^6$ to get for $\beta > 0$:

$$\frac{1}{2r^2} + \beta - \frac{\beta^2}{2r^d}\|\phi\|_4^4 \leq \frac{1}{2}\|\nabla\phi_r\|_2^2 - \Lambda_{\phi_r}(-\beta) \leq \frac{1}{2r^2} + \beta - \frac{\beta^2}{2r^d}\|\phi\|_4^4 + \frac{\beta^3}{6r^{2d}}\|\phi\|_6^6. \quad (3.3)$$

The upper bound in (3.2) in the case $d = 1$ follows from the upper bound in (3.3) and from (3.1) when we set $r = 2\|\phi\|_4^{-4}\beta^{-2}$, $c_1 = \|\phi\|_4^8/8$, and $c_7 = \|\phi\|_4^8\|\phi\|_6^6/24$ for any fixed ϕ . To derive the lower bounds in the dimensions $d \leq 3$, we use the bound (3.8) in [10], which tells us (for $d < 4$)

$$c_8(d) \stackrel{\text{def}}{=} \sup \{ \|\phi\|_4^4 : \phi \in \Phi, \|\nabla\phi\|_2 = 1 \} < \infty. \quad (3.4)$$

We insert this bound into the lower bound in (3.3) and minimize over $r > 0$ (in view of (3.1)):

- For $d = 1$ we get the lower bound in (3.2) with $C_1 = c_8(d)^2/8$ and the minimizing radius $r = 2c_8(1)^{-1}\beta^{-2}$.
- For $d = 2$ and $\beta \leq b_3(2) \stackrel{\text{def}}{=} c_8(2)^{-1/2}$ we get $J(\beta) \geq \beta$; here the infimum over r of the lower bound in (3.3) is reached in the limit $r \rightarrow \infty$.
- For $d = 3$ we distinguish 2 cases:

Case 1. If $r \leq (2\beta)^{-1/2}$, then we get, using $\Lambda_\phi(-r^{-d}\beta) \leq 0$:

$$\frac{1}{2r^2} - r^d\Lambda_\phi(-r^{-d}\beta) \geq \frac{1}{2r^2} \geq \beta. \quad (3.5)$$

Case 2. We suppose $r > (2\beta)^{-1/2}$. For $\beta \leq b_3(3) \stackrel{\text{def}}{=} 2^{-1/5}c_8(3)^{-2/5}$ we get

$$\begin{aligned} \frac{1}{2r^2} - r^d\Lambda_\phi(-r^{-d}\beta) &\geq \frac{1}{2r^2} + \beta - \frac{\beta^2}{2r^3}c_8(3) \\ &\geq \beta + \frac{1}{2r^2}(1 - 2^{1/2}c_8(3)\beta^{5/2}) \geq \beta. \end{aligned} \quad (3.6)$$

For $d \geq 4$, the case $r \leq (2\beta)^{-1/2}$ is treated the same way as in the case $d = 3$. In the case $r > (2\beta)^{-1/2}$ we proceed as follows: Equation (3.19) in [10] tells us for $\sigma \leq 0$ (recall $d \geq 4$): $\Lambda_\phi(\sigma) \leq \sigma + c_9 |\sigma|^{d/(d-2)}$, where $c_9(d) \stackrel{\text{def}}{=} \left(\frac{2}{d}\right)^{\frac{2d-2}{d-2}} (d-2)^{\frac{2}{2-d}} \pi^{\frac{d+1}{2-d}} \Gamma\left(\frac{d}{2} + \frac{1}{2}\right)^{\frac{2}{d-2}}$, hence

$$r^d \Lambda_\phi(-r^{-d}\beta) \leq -\beta + c_9 r^{\frac{2d}{2-d}} \beta^{\frac{d}{d-2}}. \quad (3.7)$$

We define $b_3(d) \stackrel{\text{def}}{=} 2^{\frac{-d}{d+2}} c_9^{\frac{2-d}{d+2}}$. We use the bound (3.7), the hypothesis $\beta \leq b_3$, and the assumption $r > (2\beta)^{-1/2}$ to get

$$\begin{aligned} \frac{1}{2r^2} - r^d \Lambda_\phi(-r^{-d}\beta) &\geq \beta + r^{-2} \left(\frac{1}{2} - c_9 r^{\frac{4}{2-d}} \beta^{\frac{d}{d-2}} \right) \\ &\geq \beta + r^{-2} \left(\frac{1}{2} - c_9 (2\beta)^{\frac{2}{d-2}} \beta^{\frac{d}{d-2}} \right) = \beta + \frac{1}{2r^2} \left(1 - \left(\frac{\beta}{b_3} \right)^{\frac{d+2}{d-2}} \right) \\ &\geq \beta. \end{aligned} \quad (3.8)$$

The claim of Lemma 3.2 now follows for all dimensions using (3.1). □

3.2 Asymptotics in the large- β -region

Lemma 3.3 *There are positive constants $C_3(d)$ and $b_4(d)$ such that for all $\beta \geq b_4$ the following upper bound holds:*

$$J(\beta) \leq \tilde{c}(d, 1) - \frac{C_3}{\sqrt{\beta}}. \quad (3.9)$$

Proof of Lemma 3.3. By the upper bound in Lemma B.1 in Appendix B of [10] there are positive constants b_5 and c_{10} such that for all $\beta_1 \geq b_5$ there is a test function $\psi \in \Phi$ that fulfills

$$\frac{1}{2} \|\nabla \psi\|_2^2 + \beta_1 \left\| \psi \mathbf{1}_{\mathbb{R}^d \setminus B_{r_d}(0)} \right\|_2^2 \leq c(d, 1) - \frac{c_{10}}{\sqrt{\beta_1}}. \quad (3.10)$$

We define $C_3 \stackrel{\text{def}}{=} (r_d/\tilde{r}_d)^3 c_{10}$, $b_4 \stackrel{\text{def}}{=} (r_d/\tilde{r}_d)^2 b_5$. Given $\beta \geq b_4$, we set $\beta_1 = (\tilde{r}_d/r_d)^2 \beta \geq b_5$, choose $\psi \in \Phi$ as in (3.10), and scale: $\phi(x) = (r_d/\tilde{r}_d)^{d/2} \psi(xr_d/\tilde{r}_d)$. We obtain

$$\frac{1}{2} \|\nabla \phi\|_2^2 + \beta \left\| \phi \mathbf{1}_{\mathbb{R}^d \setminus B_{\tilde{r}_d}(0)} \right\|_2^2 \leq c(d, 1) \left(\frac{\tilde{r}_d}{r_d} \right)^{-2} - \frac{C_3}{\sqrt{\beta}}. \quad (3.11)$$

Using the inequality $-(e^{-\xi} - 1) \leq \xi \wedge 1$ we get, using (0.9), (0.10), and the definition of r_d and of $c(d, 1)$ in the last step:

$$\begin{aligned}
J(\beta) &\leq \frac{1}{2} \|\nabla \phi\|_2^2 - \int_{\mathbb{R}^d} (e^{-\beta \phi^2} - 1) dx \\
&\leq \frac{1}{2} \|\nabla \phi\|_2^2 + \beta \left\| \phi 1_{\mathbb{R}^d \setminus B_{\tilde{r}_d}(0)} \right\|_2^2 + \left\| 1_{B_{\tilde{r}_d}(0)} \right\|_2^2 \\
&\leq c(d, 1) \left(\frac{\tilde{r}_d}{r_d} \right)^{-2} - \frac{C_3}{\sqrt{\beta}} + \left(\frac{\tilde{r}_d}{r_d} \right)^d d = \tilde{c}(d, 1) - \frac{C_3}{\sqrt{\beta}}.
\end{aligned} \tag{3.12}$$

Lemma 3.3 is proved. □

Lemma 3.4 *There are positive constants $C_2(d)$ and $b_6(d)$ such that for all $\beta \geq b_6$ the following lower bound holds:*

$$J(\beta) \geq \tilde{c}(d, 1) - C_2 \sqrt{\frac{\log \beta}{\beta}}. \tag{3.13}$$

Proof of Lemma 3.4. This time we use the lower bound in Lemma B.1 in Appendix B of [10]. By the same scaling argument as the one leading to (3.11) we obtain: There are positive constants $c_{11}(d)$ and $b_7(d)$ such that for every radius $s > 0$, every $\beta_1 \geq s^{-2} b_7$, and every test function $\phi \in \Phi$ we have

$$\frac{1}{2} \|\nabla \phi\|_2^2 + \beta_1 \left\| \phi 1_{\mathbb{R}^d \setminus B_s(0)} \right\|_2^2 \geq c(d, 1) r_d^2 s^{-2} - c_{11} \beta_1^{-1/2} s^{-3}. \tag{3.14}$$

For fixed β_1 and ϕ , the left-hand side in (3.14) is a monotonically decreasing function of s . We choose a constant radius $\rho_d > 0$ so small that

$$c(d, 1) r_d^2 \rho_d^{-2} > \tilde{c}(d, 1), \tag{3.15}$$

and then a constant $b_8(d) \geq \rho_d^{-2} b_7$ so large that

$$c(d, 1) r_d^2 \rho_d^{-2} - c_{11} b_8^{-1/2} \rho_d^{-3} \geq \tilde{c}(d, 1). \tag{3.16}$$

Hence we get for all $\beta_1 \geq b_8$ and all $r \geq 0$, using (3.14) and (3.16):

$$\frac{1}{2} \|\nabla \phi\|_2^2 + \beta_1 \left\| \phi 1_{\mathbb{R}^d \setminus B_r(0)} \right\|_2^2 \geq \begin{cases} \tilde{c}(d, 1) & \text{for } r < \rho_d, \\ c(d, 1) r_d^2 r^{-2} - c_{11} \beta_1^{-1/2} r^{-3} & \text{for } r \geq \rho_d. \end{cases} \tag{3.17}$$

We choose a constant $b_9 \geq 4$ so large that $b_9 / \log b_9 \geq b_8$. Let $\beta \geq b_9$ and $\phi \in \Phi$. As in the quenched case (Lemma 3.6 in [10]) we use a rearrangement inequality: Let $\phi^\circ \in \Phi$ denote

the radially symmetric non-increasing rearrangement of $\phi \in \Phi$ (see [8], Section 3.3). Then $\Lambda_\phi = \Lambda_{\phi^\circ}$, and $\|\nabla\phi^\circ\|_2 \leq \|\nabla\phi\|_2$; see Lemma 7.17 in [8]. Let $r \geq 0$ denote the maximal radius such that $\beta\phi^\circ(x)^2 > \frac{1}{2}\log\beta$ holds for all $x \in B_r(0)$; consequently $\beta\phi^\circ(x)^2 \leq \frac{1}{2}\log\beta$ holds for all $x \in \mathbb{R}^d \setminus B_r(0)$, since ϕ° is radially symmetric non-increasing. We use the inequality

$$1 - e^y \geq \begin{cases} \frac{\beta^{-1/2}-1}{\frac{1}{2}\log\beta}y & \text{for } -\frac{1}{2}\log\beta \leq y \leq 0, \\ 1 - \beta^{-1/2} & \text{for } y \leq -\frac{1}{2}\log\beta; \end{cases} \quad (3.18)$$

and we abbreviate $\beta_1 = \frac{(1-\beta^{-1/2})\beta}{\frac{1}{2}\log\beta} \geq \frac{\beta}{\log\beta} \geq b_8$ (recall $\beta \geq b_9 \geq 4$) in the following estimate:

$$\begin{aligned} \frac{1}{2}\|\nabla\phi\|_2^2 - \Lambda_\phi(-\beta) &\geq \frac{1}{2}\|\nabla\phi^\circ\|_2^2 + \int_{\mathbb{R}^d} (1 - e^{-\beta(\phi^\circ)^2}) dx \\ &\geq \frac{1}{2}\|\nabla\phi^\circ\|_2^2 + \beta_1 \|\phi^\circ \mathbf{1}_{\mathbb{R}^d \setminus B_r(0)}\|_2^2 + (1 - \beta^{-1/2})|B_r(0)| \\ &\stackrel{(3.17)}{\geq} \begin{cases} \tilde{c}(d, 1) & \text{for } r < \rho_d, \\ c(d, 1)r_d^2 r^{-2} - c_{11}\beta_1^{-1/2}r^{-3} + (1 - \beta^{-1/2})|B_1(0)|r^d & \text{for } r \geq \rho_d. \end{cases} \end{aligned} \quad (3.19)$$

We estimate the last expression for sufficiently large β in the case $r \geq \rho_d$: We abbreviate $c_{12}(d) \stackrel{\text{def}}{=} c_{11}\rho_d^{-1}c(d, 1)^{-1}r_d^{-2}$, $c_{13}(d) \stackrel{\text{def}}{=} c_{12} \vee 1$, and $C_2(d) \stackrel{\text{def}}{=} \tilde{c}(d, 1)c_{13}$. Then we choose $b_6 \geq b_9$ so large that $c_{13}(\log b_6)^{1/2}b_6^{-1/2} \leq 1$. Assume $\beta \geq b_6$. We estimate (recall the definition (0.10) of $\tilde{c}(d, 1)$):

$$\begin{aligned} &c(d, 1)r_d^2 r^{-2} - c_{11}\beta_1^{-1/2}r^{-3} + (1 - \beta^{-1/2})|B_1(0)|r^d \\ &\geq (1 - c_{12}\beta_1^{-1/2})c(d, 1)r_d^2 r^{-2} + (1 - \beta^{-1/2})|B_1(0)|r^d \\ &\geq (1 - c_{13}(\log\beta)^{1/2}\beta^{-1/2})(c(d, 1)r_d^2 r^{-2} + |B_1(0)|r^d) \\ &\geq (1 - c_{13}(\log\beta)^{1/2}\beta^{-1/2})\tilde{c}(d, 1) = \tilde{c}(d, 1) - C_2(\log\beta)^{1/2}\beta^{-1/2}. \end{aligned} \quad (3.20)$$

We used in the last inequality $1 - c_{13}(\log\beta)^{1/2}\beta^{-1/2} \geq 0$, which follows from the choice of b_6 and from $(\log\beta)\beta^{-1} \leq (\log b_6)b_6^{-1}$; recall $\beta \geq b_6 \geq 4$. The estimates (3.19), (3.20) and the definition (0.8) of J together yield the claim (3.13) of Lemma 3.4. \square

Proof of Theorem 0.3. On the one hand, Lemma 3.2 and Lemma 3.1 imply $J(\beta) = \beta$ for $0 < \beta \leq b_3$, $d \geq 2$. On the other hand, Lemma 3.3 has the consequence $J(\beta) < \beta$ for large β . These two facts together with the the concavity of J (see Lemma 3.1) imply Theorem 0.3. \square

Proof of Theorem 0.4. It only remains to show $J(\beta) < \beta$ for all $\beta > 0$ in dimension $d = 1$. To see this, we observe $J(\beta) \xrightarrow{\beta \downarrow 0} 0$ and $J(\beta) < \beta$ for sufficiently small $\beta > 0$ as a consequence of the bounds (3.2), and use the concavity of J . □

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