Survival asymptotics for branching Brownian motion in a Poissonian trap field

János Engländer
Frank den Hollander

EURANDOM, P.O. Box 513,
5600 MB Eindhoven, The Netherlands

November 1, 2001

Abstract

In this paper we study a branching Brownian motion on $\mathbb{R}^d$ with branching rate β in a Poissonian trap field whose Borel intensity measure ν is such that $\mathrm{d}\nu/\mathrm{d}x$ decays radially with the distance to the origin as $\mathrm{d}\nu/\mathrm{d}x \sim \ell/|x|^{d-1}$, $|x| \to \infty$. The process starts with a single particle at the origin. The annealed probability that *none* of the particles hits a trap up to time t is shown to decay like $\exp[-I(\ell,\beta,d)t + o(t)]$ as $t \to \infty$, where the rate constant $I(\ell,\beta,d)$ is computed in terms of a variational problem. It turns out that this rate constant exhibits a crossover at a critical value $\ell_{cr} = \ell_{cr}(\beta,d)$. For $\ell < \ell_{cr}$ the optimal survival strategy is to empty a ball of radius $\sqrt{2\beta}\,t$ around the origin, to stay inside this ball and to branch at rate β. For $\ell > \ell_{cr}$, on the other hand, the optimal survival strategy depends on the dimension:

$d = 1$: suppress the branching until time t, empty a ball of radius $o(t)$ around the origin and stay inside this ball;

$d \geq 2$: suppress the branching until time $\eta^* t$ while moving (within a small empty tube) to a site at distance $c^* t$ away from the origin, empty a ball of radius $\sqrt{2\beta}\,(1-\eta^*)t$ around this site, and during the remaining time $(1-\eta^*)t$ stay inside this ball and branch at rate β. Here, $0 < \eta^* < 1$ and $c^* > 0$ are the minimisers of the variational problem for $I(\ell,\beta,d)$.

Remarkably, it turns out that η^* and c^* tend to a strictly positive limit as $\ell \downarrow \ell_{cr}$, i.e., the crossover at ℓ_{cr} is discontinuous. Moreover, $c^* > \sqrt{2\beta}\,(1-\eta^*)$ for all $\ell > \ell_{cr}$, i.e., the empty ball does not contain the origin.

In contrast, the annealed probability that *at least one* of the particles does not hit a trap up to time t is shown to decay to a strictly positive limit.

AMS 1991 *subject classifications.* 60J65, 60J80, 60F10, 82B44.
Key words and phrases. Branching Brownian motion, Poissonian traps, large deviations, survival probability, variational problem, optimal survival strategy.

Contents

1 Introduction and main theorems

In Sznitman [9] a whole range of questions is investigated that concern Brownian motion among Poissonian obstacles, in particular, survival probabilities and optimal survival strategies. The present paper generalises this setting by including branching and addresses a basic question in the same spirit.

Let $Z = (Z(t))_{t\geq 0}$ be the d-dimensional binary branching Brownian motion (BBM) with a spatially and temporally constant branching rate $\beta > 0$. The informal description of this process is as follows. A single particle starts at the origin, performs a Brownian motion on $\mathbb{R}^d$, after a mean–$1/\beta$ exponential time dies and produces two offspring, the two offspring perform independent Brownian motions from their birth location, die and produce two offspring after independent mean–$1/\beta$ exponential times, etc. Think of $Z(t)$ as the subset of $\mathbb{R}^d$ indicating the locations of the particles alive at time t. Write P_{δ_x} to denote the law of Z when the initial particle starts at x.

Let ω be the Poisson point process on $\mathbb{R}^d$ with a spatially dependent Borel intensity measure ν such that

$$\frac{\mathrm{d}\nu}{\mathrm{d}x} \sim \frac{\ell}{|x|^{d-1}}, \quad |x| \to \infty, \qquad \ell > 0, \tag{1.1}$$

i.e., the integral of $\mathrm{d}\nu/\mathrm{d}x$ over large spheres centered at the origin is asymptotically constant. Write $\mathbb{P}$ to denote the law of ω, and $\mathbb{E}$ to denote the corresponding expectation. For $a > 0$, let

$$K = K_a(\omega) = \bigcup_{x\in\mathrm{supp}(\omega)} B_a(x) \tag{1.2}$$

be the a-neighborhood of ω, which is to be thought of as a configuration of traps attached to ω (here $B_a(x)$ is the closed ball of radius a centered at x).

1.1 Hard killing rule: Theorems 1–3

For $A \subseteq \mathbb{R}^d$ Borel and $t \geq 0$, let $|Z(t) \cap A|$ be the number of particles located in A at time t and

$$|Z(t)| = \text{ the total number of particles at time } t. \tag{1.3}$$

For $t \geq 0$, let

$$R(t) = \bigcup_{s \in [0,t]} \mathrm{supp}(Z(s)) \tag{1.4}$$

denote the collection of all the particle trajectories up to time t (= the range of Z up to time t). Let T be the first time that Z hits a trap, i.e.,

$$\begin{aligned} T &= \inf\{t \geq 0 : |Z(t) \cap K| \geq 1\} \\ &= \inf\{t \geq 0 : R(t) \cap K \neq \emptyset\}. \end{aligned} \tag{1.5}$$

Thus, the event $\{T > t\}$ stands for the survival up to time t of Z among the Poissonian traps, i.e., no particle hits a trap up to time t. Our goal is to describe the asymptotic decay of the annealed survival probability $(\mathbb{E} \times P_{\delta_0})(T > t)$ as $t \to \infty$ and to identify the optimal survival strategy.

Our motivation comes from Engländer [1], where it was shown that if $d \geq 2$ and $\mathrm{d}\nu/\mathrm{d}x \equiv \ell$, then the annealed survival probability decays like $\exp[-\beta t + o(t)]$. Intuitively, this means that the system completely suppresses the branching up to time t in order to avoid the traps. The corresponding asymptotics for $d = 1$ was left as an open problem. In the present paper we will solve this problem. In addition, we will consider the higher-dimensional model with a decaying trap density field as in (1.1) and show that its behavior is partly similar to and partly different from the one-dimensional model.

To formulate our main results, we need some more notation. For $r, b \geq 0$, define

$$f_d(r,b) = \int_{B_r(0)} \frac{\mathrm{d}x}{|x + be|^{d-1}}, \tag{1.6}$$

where $e = (1, 0, \ldots, 0)$. For $\eta \in [0,1]$ and $c \in [0, \infty)$, let

$$\begin{aligned} k_{\beta,d}(\eta, c) &= \lim_{t \to \infty} \frac{1}{\ell t}\, \nu\left(B_{\sqrt{2\beta}\,(1-\eta)t}(ct)\right) \\ &= f_d\left(\sqrt{2\beta}\,(1-\eta), c\right) \end{aligned} \tag{1.7}$$

(recall (1.1)). Let

$$\ell^*_{cr} = \ell^*_{cr}(\beta, d) = \frac{1}{s_d}\sqrt{\frac{\beta}{2}} \tag{1.8}$$

with s_d the surface of the d-dimensional unit ball ($s_1 = 2$). Define

$$\ell_{cr} = \ell_{cr}(\beta, d) = \begin{cases} \ell_{cr}^* & \text{if } d = 1, \\ \alpha_d \ell_{cr}^* & \text{if } d \geq 2, \end{cases} \tag{1.9}$$

with

$$\alpha_d = \frac{-1 + \sqrt{1 + 4M_d^2}}{2M_d^2} \in (0, 1), \tag{1.10}$$

where

$$M_d = \frac{1}{2s_d} \max_{R \in (0,\infty)} [f_d(0, R) - f_d(1, R)]. \tag{1.11}$$

Theorem 1 (variational formula) *Fix d, β, a. For any ℓ,*

$$\lim_{t \to \infty} \frac{1}{t} \log(\mathbb{E} \times P_{\delta_0})(T > t) = -I(\ell, \beta, d), \tag{1.12}$$

where

$$I(\ell, \beta, d) = \min_{\eta \in [0,1],\, c \in [0,\infty)} \left\{ \beta\eta + \frac{c^2}{2\eta} + \ell k_{\beta,d}(\eta, c) \right\}. \tag{1.13}$$

(For $\eta = 0$ put $c = 0$ and $k_{\beta,d}(0, 0) = f_d(\sqrt{2\beta}, 0) = s_d\sqrt{2\beta}$.)

Theorem 2 (crossover) *Fix β, a.*
(i) For $d \geq 1$ and all $\ell \neq \ell_{cr}$, the variational problem in (1.13) has a unique pair of minimisers, denoted by $\eta^* = \eta^*(\ell, \beta, d)$ and $c^* = c^*(\ell, \beta, d)$.
(ii) For $d = 1$,

$$\begin{aligned} \ell \leq \ell_{cr} : \quad & I(\ell, \beta, d) = \beta \frac{\ell}{\ell_{cr}^*}, \\ \ell > \ell_{cr} : \quad & I(\ell, \beta, d) = \beta, \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} \ell < \ell_{cr} : \quad & \eta^* = 0,\ c^* = 0, \\ \ell > \ell_{cr} : \quad & \eta^* = 1,\ c^* = 0. \end{aligned} \tag{1.15}$$

(iii) For $d \geq 2$,

$$\begin{aligned} \ell \leq \ell_{cr} : \quad & I(\ell, \beta, d) = \beta \frac{\ell}{\ell_{cr}^*}, \\ \ell > \ell_{cr} : \quad & I(\ell, \beta, d) < \beta \left(1 \wedge \frac{\ell}{\ell_{cr}^*}\right), \end{aligned} \tag{1.16}$$

and

$$\begin{aligned} \ell < \ell_{cr}: &\quad \eta^* = 0,\ c^* = 0, \\ \ell > \ell_{cr}: &\quad 0 < \eta^* < 1,\ c^* > 0. \end{aligned} \tag{1.17}$$

(iv) For $d \geq 2$, $\ell \mapsto I(\ell, \beta, d)$ is continuous and strictly increasing, with $\lim_{\ell\to\infty} I(\ell, \beta, d) = \beta$ (see Fig. 1).
(v) For $d \geq 2$, $\ell \mapsto \eta^*(\ell, \beta, d)$ and $\ell \mapsto c^*(\ell, \beta, d)$ are both discontinuous at ℓ_{cr} and continuous on (ℓ_{cr}, ∞), with

$$\lim_{\ell\to\infty} \frac{1-\eta^*(\ell,\beta,d)}{c^*(\ell,\beta,d)} = 1, \qquad \lim_{\ell\to\infty} c^*(\ell,\beta,d) = 0. \tag{1.18}$$

Moreover, $c^* > \sqrt{2\beta}\,(1-\eta^*)$ for all $\ell > \ell_{cr}$.

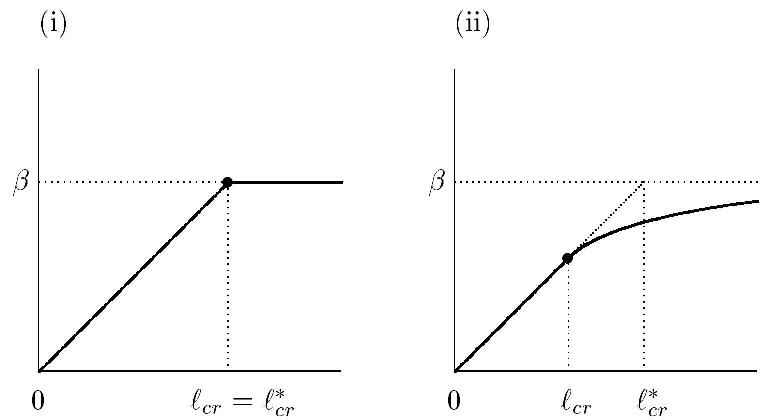

Fig. 1 $\ell \mapsto I(\ell, \beta, d)$ for: (i) $d = 1$; (ii) $d \geq 2$.

Theorem 3 (optimal survival strategy) *Fix β, a. For $r, b > 0$ and $t \geq 0$, define*

$$C(t; r, b) = \{\exists x_0 \in \mathbb{R}^d\colon\ |x_0| = b, B_{rt}(x_0 t) \cap K = \emptyset\}. \tag{1.19}$$

(i) For $d = 1$, $\ell < \ell_{cr}$ or $d \geq 2$, any ℓ, and $0 < \varepsilon < 1 - \eta^*$,

$$\begin{aligned} &\lim_{t\to\infty} (\mathbb{E} \times P_{\delta_0})\Big(C\left(t; \sqrt{2\beta}\,(1-\eta^*-\varepsilon)\,, c^*\right) \mid T > t\Big) = 1, \\ &\lim_{t\to\infty} (\mathbb{E} \times P_{\delta_0})\Big(|Z(t)| \geq \lfloor e^{\beta(1-\eta^*-\varepsilon)t} \rfloor \mid T > t\Big) = 1. \end{aligned} \tag{1.20}$$

(ii) For $d \geq 1$, $\ell < \ell_{cr}$ and $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{t\to\infty} (\mathbb{E} \times P_{\delta_0})\Big(B_{(1+\varepsilon)\sqrt{2\beta}\,t}(0) \cap K \neq \emptyset \mid T > t\Big) = 1, \\
&\lim_{t\to\infty} (\mathbb{E} \times P_{\delta_0})\Big(R(t) \subseteq B_{(1+\varepsilon)\sqrt{2\beta}\,t}(0) \mid T > t\Big) = 1, \\
&\lim_{t\to\infty} (\mathbb{E} \times P_{\delta_0})\Big(R(t) \nsubseteq B_{(1-\varepsilon)\sqrt{2\beta}\,t}(0) \mid T > t\Big) = 1.
\end{aligned} \tag{1.21}$$

(iii) For $d \geq 1$, $\ell > \ell_{cr}$ and $0 < \varepsilon < \eta^*$,

$$\lim_{t\to\infty} (\mathbb{E} \times P_{\delta_0})\Big(|Z((\eta^* - \varepsilon)t)| \leq \lfloor t^{d+\varepsilon} \rfloor \mid T > t\Big) = 1. \tag{1.22}$$

(iv) For $d = 1$, $\ell > \ell_{cr}$ and $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{t\to\infty} (\mathbb{E} \times P_{\delta_0})\Big(B_{\varepsilon t}(0) \cap K \neq \emptyset \mid T > t\Big) = 1, \\
&\lim_{t\to\infty} (\mathbb{E} \times P_{\delta_0})\Big(R(t) \subseteq B_{\varepsilon t}(0) \mid T > t\Big) = 1.
\end{aligned} \tag{1.23}$$

Theorem 1 can be explained as follows. Fix β, d and η, c.

– The probability to completely suppress the branching (i.e., only the initial particle is alive) up to time ηt is

$$\exp\left[-\beta\eta t + o(t)\right]. \tag{1.24}$$

– The probability for the initial particle to move to a site at distance ct from the origin during time ηt is

$$\exp\left[-\frac{c^2}{2\eta}t + o(t)\right]. \tag{1.25}$$

– Under (1.1), the probability to empty a $\sqrt{2\beta}\,(1-\eta)t$-ball around a site at distance ct from the origin is (see (1.7))

$$\exp\left[-\ell k_{\beta,d}(\eta, c)t + o(t)\right]. \tag{1.26}$$

The probability to empty a "small tube" connecting the origin with this site is $\exp[o(t)]$. For the initial particle to remain inside this tube up to time ηt is also $\exp[o(t)]$. (See Section 3.1.)

– The probability for the offspring of the initial particle present at time ηt to remain inside the $\sqrt{2\beta}\,(1-\eta)t$-ball during the remaining time $(1-\eta)t$ is $\exp[o(t)]$ as well. (See Section 2.)

The combined cost of these three large deviation events gives rise to the sum under the minimum in (1.13). The minimal cost is therefore determined by the minimisers of (1.13). (The proof of Theorem 1 shows that any other strategy has a probability that decays faster than exponential.)

Theorem 2 shows that (1.13) exhibits a crossover at the critical value $\ell_{cr} = \ell_{cr}(\beta, d)$ defined in (1.9), separating a low intensity from a high intensity regime. In the low intensity regime the minimisers are trivial, in the high intensity regime they are trivial only for $d = 1$. It is noteworthy for $d \geq 2$ that the minimisers are discontinuous at ℓ_{cr} and that in the high intensity regime the empty ball inside which the BBM branches freely does not contain the origin. Thus, at the crossover the centre of the empty ball jumps away from the origin and the radius jumps down.

Theorem 3 shows that the optimal strategy indeed is as described above (modulo fine details that are not seen on the exponential scale). We have:

- In the low intensity regime $\ell < \ell_{cr}$, the system empties a ball of radius $\sqrt{2\beta}\, t$, and until time t stays inside this ball and branches at rate β. The cost of this strategy is $\exp[-\ell s_d \sqrt{2\beta} + o(t)]$.

- In the high intensity regime $\ell > \ell_{cr}$:

$d = 1$: the system empties an $o(t)$-ball, and until time t suppresses the branching (i.e., produces a polynomial number of particles) and stays inside this ball. The cost of this strategy is $\exp[-\beta t + o(t)]$.

$d \geq 2$: the system suppresses the branching until time $\eta^* t$ while moving to a site at distance $c^* t$ from the origin (inside a small empty tube), empties a ball of radius $\sqrt{2\beta}\,(1 - \eta^*)t$ around this site, and during the remaining time $(1 - \eta^*)t$. stays inside this ball and branches at rate β. The cost of this strategy is $\exp[-\{\beta\eta^* + \frac{c^{*2}}{2\eta^*} + \ell k_{\beta,d}(\eta^*, c^*)\}t + o(t)]$.

1.2 Soft killing rule: Theorem 4

Suppose that, instead of considering the trapping time in (1.5), we kill the process when all the particles are absorbed by a trap. That is, if $Z^K = (Z^K(t))_{t \geq 0}$ denotes the BBM with killing at the boundary of the trap set K, then we define

$$\tilde{T} = \inf \left\{ t \geq 0 : |Z^K(t)| = 0 \right\} \tag{1.27}$$

and we pick $\{\tilde{T} > t\}$ as the survival up to time t. It turns out that then the survival probability does not even decay to zero.

Theorem 4 *Fix d, β, a. Then, for any Borel intensity measure ν,*

$$\lim_{t\to\infty} (\mathbb{E} \times P_{\delta_0})(\tilde{T} > t) > 0. \tag{1.28}$$

Theorem 4 follows from the assertion that the system may survive by emptying a ball with a finite radius $R > R_0$, where R_0 is chosen such that the branching rate β balances against the killing rate $\lambda(B_{R_0}(0))$, the principal Dirichlet eigenvalue of $-\Delta/2$ on the ball $B_{R_0}(0)$: $\lambda(B_{R_0}(0)) = \beta$. Indeed, as shown in Engländer and Kyprianou [2] Theorem 4(iv), for any $R > R_0$ there is a strictly positive probability that at all times at least one particle has not yet left $B_R(0)$. Consequently, the survival probability is bounded from below by $\exp[-\nu(B_{R_0}(0))]$.

1.3 Open problems

We conclude with some open problems. For the hard killing model:

- For $d = 1$ and $\ell > \ell_{cr}$, what is the radius of the $o(t)$-ball that is emptied and how many particles are there in this ball at time t?

- For $d \geq 2$ and $\ell > \ell_{cr}$, what is the optimal shape of the "small tube" in which the BBM moves away from the origin while suppressing the branching? How many particles are alive at time $\eta^* t$?

- What is the optimal survival strategy at $\ell = \ell_{cr}$?

- Instead of letting the trap density decay to zero at infinity, another way to make survival easier is by giving the Brownian motion an inward drift while keeping the trap density field constant. Suppose that $\mathrm{d}\nu/\mathrm{d}x \equiv \ell$ and that the inward drift *radially increases* like $\sim \kappa|x|^{d-1}$, $|x| \to \infty$, $\kappa > 0$. Is there again a crossover in ℓ at some critical value $\ell_{cr} = \ell_{cr}(\kappa, \beta, d)$? What is the optimal survival strategy?

For the soft killing model:

– What is the limit in (1.28), say, when $\mathrm{d}\nu/\mathrm{d}x$ is spherically symmetric?

– If the Brownian motion is given an outward drift, then for what values of the drift does the survival probability decay to zero?

Section 2 contains preparations. Theorems 1–3 are proved in Sections 3–5, respectively.

Finally, we refer the reader to Révész [8] for many interesting results on branching Brownian motion and branching random walk.

2 Preparations

In this section we formulate three preparatory propositions. Propositions 1–2 are needed in the proof of Proposition 3, which is itself needed in the proof of Theorem 1 in Section 3.

Recall from (1.3) that $|Z(t)|$ denotes the total number of particles at time t. The following result says that overproduction is superexponentially unlikely.

Proposition 1 *Let $\delta > 0$. Then*

$$\lim_{t\to\infty} \frac{1}{t} \log P_{\delta_0}\left(|Z(t)| > e^{(\beta+\delta)t}\right) = -\infty. \tag{2.1}$$

Proof of Proposition 1. Since $|Z(t)|$ under P_{δ_0} is a pure birth process (Yule's process) with $|Z(0)| = 1$, we have

$$P_{\delta_0}(|Z(t)| = k) = e^{-\beta t}(1 - e^{-\beta t})^{k-1}, \qquad k \in \mathbb{N},\ t \geq 0 \tag{2.2}$$

(see e.g. Karlin and Taylor [5], equation 8.11.5 and the discussion afterwards). Hence

$$P_{\delta_0}(|Z(t)| > l) = (1 - e^{-\beta t})^{l}, \qquad l \in \mathbb{N},\ t \geq 0, \tag{2.3}$$

giving (2.1). $\square$

For $B \subset \mathbb{R}^d$ open or closed, let η_B and $\hat\eta_B$ denote the first exit times from B for one Brownian motion W resp. for our BBM Z:

$$\begin{aligned} \eta_B &= \inf\{t \geq 0\colon W(t) \in B^c\}, \\ \hat\eta_B &= \inf\{t \geq 0\colon |Z(t) \cap B^c| \geq 1\}. \end{aligned} \tag{2.4}$$

The following result makes a comparison between these two quantities.

Proposition 2 *Let P_x denote the law of Brownian motion starting from x. For any $B \subset \mathbb{R}^d$ open or closed and any $x \in B$,*

$$P_{\delta_x}\Big(\hat\eta_B > t \mid |Z(t)| \le k\Big) \ge [P_x(\eta_B > t)]^k, \qquad k \in \mathbb{N}, t \ge 0. \tag{2.5}$$

Proof of Proposition 2. By an obvious monotonicity argument, it is enough to show that

$$P_{\delta_x}\Big(\hat\eta_B > t \mid |Z(t)| = k\Big) \ge [P_x(\eta_B > t)]^k, \qquad k \in \mathbb{N}, t \ge 0. \tag{2.6}$$

We will prove this inequality by induction on k. The statement is obviously true for $k = 1$. Assume that the statement is true for $1, 2, \dots, k-1$. Let σ_1 be the first branching time:

$$\sigma_1 = \inf\{t \ge 0 \colon |Z(t)| \ge 2\}. \tag{2.7}$$

By the Markov property, it suffices to prove the assertion conditioned on the event $\{\sigma_1 = s\}$ with $0 \le s \le t$ fixed. To that end, let $p_{x,s} = P_x(\eta_B > s)$ and

$$\tilde p(s, x, \mathrm{d}y) = P_x\Big(W(s) \in \mathrm{d}y \mid \eta_B > s\Big). \tag{2.8}$$

After time s the BBM evolves like two independent BBM's Z^1, Z^2 starting from $Z(s)$. For $i = 1, \dots, k-1$ and $y \in \mathbb{R}^d$, let

$$\begin{aligned} q_{i,k}(s, t-s) \quad &= \quad P_{\delta_y}\Big(|Z^1(t-s)| = i, |Z^2(t-s)| = k-i \\ &\qquad\qquad \mid |Z(t)| = k, \sigma_1 = s\Big) \end{aligned} \tag{2.9}$$

(which does not depend on y). Write $\hat\eta_B^1, \hat\eta_B^2$ to denote the analogues of $\hat{\eta}_B$ for Z^1, Z^2. Then

$$
\begin{aligned}
&P_{\delta_x}\Big(\hat{\eta}_B > t \mid |Z(t)| = k, \sigma_1 = s\Big) \\
&\quad = \int_B P_x(\eta_B > s, W(s) \in dy) \\
&\qquad \times P_{\delta_x}\Big(\hat{\eta}_B^1 > t-s, \hat{\eta}_B^2 > t-s \mid |Z(t)| = k, \sigma_1 = s\Big) \\
&\quad = p_{x,s} \int_B \tilde{p}(s,x,\mathrm{d}y) \sum_{i=1}^{k-1} q_{i,k}(s,t-s) \\
&\qquad \times P_{\delta_y}\Big(\hat{\eta}_B^1 > t-s \mid |Z^1(t-s)| = i\Big) \\
&\qquad \times P_{\delta_y}\Big(\hat{\eta}_B^2 > t-s \mid |Z^2(t-s)| = k-i\Big) \\
&\quad \geq p_{x,s} \int_B \tilde{p}(s,x,\mathrm{d}y) \sum_{i=1}^{k-1} q_{i,k}(s,t-s) \\
&\qquad \times [P_y(\eta_B > t-s)]^i [P_y(\eta_B > t-s)]^{k-i} \\
&\quad = p_{x,s} \int_B \tilde{p}(s,x,\mathrm{d}y)[P_y(\eta_B > t-s)]^k \\
&\quad \geq p_{x,s} \left[\int_B P_y(\eta_B > t-s)\tilde{p}(s,x,\mathrm{d}y)\right]^k, \qquad (2.10)
\end{aligned}
$$

where we use the induction hypothesis and Jensen's inequality. Replacing $p_{x,s}$ by $(p_{x,s})^k$, we obtain

$$
\begin{aligned}
&P_{\delta_x}\Big(\hat{\eta}_B > t \mid |Z(t)| = k, \sigma_1 = s\Big) \\
&\qquad \geq \left[p_{x,s} \int_B \tilde{p}(s,x,\mathrm{d}y) P_y(\eta_B > t-s)\right]^k. \qquad (2.11)
\end{aligned}
$$

By the Markov property, the right-hand side precisely equals $[P_x(\eta_B > t)]^k$, giving (2.5). $\square$

Recall from (1.4) that $R(t) = \cup_{s\in[0,t]} \operatorname{supp}(Z(s))$ denotes the range of Z up to time t. Let

$$
\begin{aligned}
M^+(t) &= \sup R(t) && \text{for } d = 1, \\
M^-(t) &= \inf R(t) && \text{for } d = 1,
\end{aligned}
$$

$$
M(t) = \inf\{r > 0 : R(t) \subseteq B_r(0)\} \quad \text{for } d \geq 1, \qquad (2.12)
$$

be the right-most and left-most point of $R(t)$ resp. the radius of the minimal ball containing $R(t)$. The following result identifies the typical behavior of these quantities as $t \to \infty$.

Proposition 3 *(i) For $d = 1$, $M^+(t)/t$ and $-M^-(t)/t$ converge to $\sqrt{2\beta}$ in P_{δ_0}-probability as $t \to \infty$.*
(ii) For $d \geq 1$, $M(t)/t$ converges to $\sqrt{2\beta}$ in P_{δ_0}-probability as $t \to \infty$.

Proof of Proposition 3. For (i), the reader is referred to McKean [6], [7] (see also Freidlin [3], Section 5.5 and equation 6.3.12). Turning to (ii), first note that the projection of Z onto the first coordinate axis is a one-dimensional BBM with branching rate β. Hence, the lower estimate for (ii) follows from (i) and the inequality

$$\begin{aligned} &P_{\delta_0}\Big(M(t)/t > \sqrt{2\beta} - \varepsilon\Big) \\ &\qquad \geq P^*_{\delta_0}\Big(M^+(t)/t > \sqrt{2\beta} - \varepsilon\Big) \qquad \forall \varepsilon > 0, t > 0, \end{aligned} \tag{2.13}$$

where $P^*_{\delta_0}$ denotes the law of the one-dimensional projection of Z. To prove the upper estimate for (ii), pick any $\varepsilon > 0$, abbreviate $B = B_{(\sqrt{2\beta}+\varepsilon)t}(0)$, and pick any $\delta > 0$ such that

$$\frac{1}{2}(\sqrt{2\beta} + \varepsilon)^2 > \beta + \delta. \tag{2.14}$$

Estimate (recall (2.4))

$$\begin{aligned} &P_{\delta_0}\Big(M(t)/t > \sqrt{2\beta} + \varepsilon\Big) \\ &\quad \leq P_{\delta_0}\Big(|Z(t)| > \lfloor e^{(\beta+\delta)t} \rfloor\Big) \\ &\qquad + P_{\delta_0}\Big(\hat{\eta}_B \leq t \mid |Z(t)| \leq \lfloor e^{(\beta+\delta)t} \rfloor\Big). \end{aligned} \tag{2.15}$$

By Proposition 1, the first term in the right-hand side of (2.15) tends to zero superexponentially fast. To handle the second term, we use Proposition 2 to estimate

$$P_{\delta_0}\Big(\hat{\eta}_B > t \mid |Z(t)| \leq \lfloor e^{(\beta+\delta)t} \rfloor\Big) \geq [P_0(\eta_B > t)]^{\lfloor e^{(\beta+\delta)t} \rfloor}. \tag{2.16}$$

Inserting the formula for Brownian hitting times (see e.g. Karlin and Taylor [5], equation 7.3.3), we find that for t large enough:

$$\begin{aligned} &P_{\delta_0}\Big(\hat{\eta}_B > t \mid |Z(t)| \leq \lfloor e^{(\beta+\delta)t} \rfloor\Big) \\ &\quad \geq \left[1 - \exp\left(-\frac{[(\sqrt{2\beta}+\varepsilon)t]^2}{2t}[1+o(1)]\right)\right]^{\lfloor e^{(\beta+\delta)t} \rfloor}. \end{aligned} \tag{2.17}$$

By (2.14), the right-hand side of (2.17) tends to 1 exponentially fast as $t \to \infty$, so that (2.15) yields

$$\lim_{t\to\infty} P_{\delta_0}\Big(M(t)/t > \sqrt{2\beta} + \varepsilon\Big) = 0 \qquad \forall \varepsilon > 0, \tag{2.18}$$

which completes the proof. □

In Proposition 3, (i) is stronger than (ii) for $d = 1$, since it says that the BBM reaches both ends of the interval $[-\sqrt{2\beta}t, \sqrt{2\beta}t]$.

3 Proof of Theorem 1

3.1 Proof of the lower bound

Fix β, d and η, c. We recall the type of strategy the lower bound is based on, as already explained heuristically in Section 1.1:

- *Completely suppress the branching up to time ηt* (i.e., only one Brownian particle is present up to time ηt). This event has probability

$$\exp\left[-\beta\eta t + o(t)\right]. \tag{3.1}$$

- *Move the single particle to a specific site at distance ct from the origin during time ηt.* This event has probability

$$\exp\left[-\frac{c^2}{2\eta}t + o(t)\right]. \tag{3.2}$$

- We may assume without loss of generality that the specific site has first coordinate ct and all other coordinates zero. Pick $t \mapsto r(t)$ such that $\lim_{t\to\infty} r(t) = \infty$ and $\lim_{t\to\infty} \frac{1}{t} r(t) = 0$. Pick $k > c$ and define the two-sided cylinder ("small tube") as

$$\begin{aligned} T_t \;=\; & \Big\{x = (x_1, \dots, x_d) \in \mathbb{R}^d : \\ & \qquad |x_1| \le kt,\ \sqrt{x_2^2 + \cdots + x_d^2} \le r(t)\Big\}. \end{aligned} \tag{3.3}$$

Since $r(t) = o(t)$, the probability to empty T_t is $\exp[o(t)]$ (recall (1.1)). Moreover, if W^1 denotes the first coordinate of the d-dimensional Brownian motion and

$$\begin{aligned} A_t &= \{ct \le W^1_{\eta t} \le ct + r(t)\}, \\ B_t &= \{|W^1_s| \le kt\ \forall\, 0 \le s \le \eta t\}, \end{aligned} \tag{3.4}$$

then

$$P(A_t \cap B_t) = \exp\left[-\frac{c^2}{2\eta}t + o(t)\right] \tag{3.5}$$

because

$$\begin{aligned}
&\exp\left[-\frac{c^2}{2\eta}t + o(t)\right] \\
&= P(A_t) \geq P(A_t \cap B_t) \geq P(A_t) - P(B_t^c) \\
&= \exp\left[-\frac{c^2}{2\eta}t + o(t)\right] - \exp\left[-\frac{k^2}{2\eta}t + o(t)\right] \\
&= \exp\left[-\frac{c^2}{2\eta}t + o(t)\right]. \qquad (3.6)
\end{aligned}$$

Decompose the Brownian motion W into an independent sum $W = W^1 + W^{d-1}$, and let

$$C_t = \{|W_s^{d-1}| \leq r(t) \ \forall\, 0 \leq s \leq \eta t\}. \tag{3.7}$$

Since $r(t) \to \infty$, we have $P(C_t) = \exp[o(t)]$. Combining this with (3.5) and using the independence of W^1 and W^{d-1}, we find

$$P\left(A_t \cap \ B_t \cap C_t\right) = \exp\left[-\frac{c^2}{2\eta}t + o(t)\right]. \tag{3.8}$$

In words, *emptying T_t, confining the Brownian particle to T_t up to time ηt and moving it to a specific site at distance $ct + o(t)$ from the origin at time ηt*, altogether costs $\exp[-\frac{c^2}{2\eta}t + o(t)]$.

- *Empty a $\sqrt{2\beta}\,(1-\eta)t$-ball around the specific site at distance $ct + o(t)$ from the origin.* (The tube T_t need not be disjoint from this ball, but this does not affect the argument.) Under (1.1), the probability of this event is

$$\exp\left[-\ell k_{\beta,d}(\eta, c)t + o(t)\right]. \tag{3.9}$$

- Finally, *require the branching system to stay inside the $\sqrt{2\beta}\,(1-\eta)t$-ball during the remaining time $(1-\eta)t$.* ¿From Proposition 3 it follows that the probability of this event is $\exp[o(t)]$.

Minimising the combined cost of all these events over the parameters η and c provides us with the lower estimate for the annealed survival probability.

3.2 Proof of the upper bound

Fix β, d and $\varepsilon > 0$ small. Recall that N_t is the number of particles at time t. For $t > 1$, define

$$\eta_t = \sup \left\{ \eta \in [0,1] \colon\ N_{\eta t} \le \lfloor t^{d+\varepsilon} \rfloor \right\}. \tag{3.10}$$

Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&(\mathbb{E}_\nu \times P_0)(T > t) \\
&\quad = \sum_{i=0}^{n-1} (\mathbb{E}_\nu \times P_0) \left(\{T > t\} \cap \left\{ \frac{i}{n} \le \eta_t < \frac{i+1}{n} \right\} \right) \\
&\quad \le \sum_{i=0}^{n-1} \exp \left[-\beta \frac{i}{n} t + o(t) \right] (\mathbb{E}_\nu \times P_t^{(i,n)})(T > t),
\end{aligned} \tag{3.11}$$

where we use (2.2) and introduce the conditional probabilities

$$P_t^{(i,n)}(\cdot) = P(\cdot \mid N_{\frac{i+1}{n}t} > \lfloor t^{d+\varepsilon} \rfloor), \qquad i = 0, 1, \dots, n-1. \tag{3.12}$$

Let $A_t^{(i,n)}$, $i = 0, 1, \dots, n-1$, denote the event that, among the $N_{\frac{i+1}{n}t}$ particles alive at time $\frac{i+1}{n}t$, there are $\le \lfloor t^{d+\varepsilon} \rfloor$ particles such that the ball with radius

$$\rho_t^{(i,n)} = (1-\varepsilon)\sqrt{2\beta} \left(1 - \frac{i+1}{n} \right) t \tag{3.13}$$

around the particle is non-empty (i.e., contains a point from ω). Trivially,

$$\begin{aligned}
(\mathbb{E}_\nu \times P_t^{(i,n)})(T > t) &\le (\mathbb{E}_\nu \times P_t^{(i,n)})(\{T > t\} \cap A_t^{(i,n)}) \\
&\quad + (\mathbb{E}_\nu \times P_t^{(i,n)})(T > t \mid [A_t^{(i,n)}]^c).
\end{aligned} \tag{3.14}$$

Consider the BBM's emanating from the particles alive at time $\frac{i+1}{n}t$. There is an obvious radial symmetry regarding each of these BBM's with respect to their starting points. Using this fact, along with their independence and Proposition 3, we conclude that the second term in the right-hand side of (3.14) is bounded above by

$$\left[1 - \frac{C_1}{[\rho_t^{(i,n)}]^{d-1}} \right]^{\lfloor t^{d+\varepsilon} \rfloor} \le \exp\left[-C_2(\varepsilon) t^{1+\varepsilon} \right] \tag{3.15}$$

uniformly in all parameters. Indeed, on the event $[A_t^{(i,n)}]^c$ there are $> \lfloor t^{d+\varepsilon} \rfloor$ balls containing a trap, in the remaining time $(1+\frac{i+1}{n})t$ the BBM emanating from the centre of each ball exits this ball with a probability tending to 1 as $t \to \infty$ (by Proposition 3 and (3.13)), and by radial symmetry the trap inside the ball has a probability $C_1/[\rho_t^{(i,n)}]^{d-1}$ to be hit by the BBM when exiting. The estimate in (3.15) is superexponentially small (SES).

To estimate the first term in the right-hand side of (3.14), randomly pick $\lfloor t^{d+\varepsilon} \rfloor + 1$ particles from the $N_{\frac{i+1}{n}t}$ particles alive at time $\frac{i+1}{n}t$, doing so independently of their spatial position and according to some probability distribution $\mathbb{Q}$. Here is a way how to realize $\mathbb{Q}$. Mark a random ancestral line by tossing a coin at each branching time and choosing the "nicer" or the "uglier" offspring according to the outcome. In this way we choose a "random" particle from the offspring. Repeat this procedure independently so many times until it produces $\lfloor t^{d+\varepsilon} \rfloor + 1$ different particles. Since the particles are chosen independently from the motion process, each of them is at a "random point" whose spatial distribution is identical to that of $W(\frac{i+1}{n}t)$, where W denotes the standard Brownian motion.

Consider now the centers of the empty balls at time $\frac{i+1}{n}t$ and let $x_0^{(i,n)}$ be the one closest to the origin. We have

$$
\begin{aligned}
&(\mathbb{E}_\nu \times P_t^{(i,n)}) \left(\left\{ |x_0^{(i,n)}| \geq ct \right\} \cap A_t^{(i,n)} \right) \\
&\leq (\mathbb{E}_\nu \times P_t^{(i,n)} \times \mathbb{Q}) \Big(\exists \text{ a random point at distance } \geq ct, \\
&\qquad \text{the ball with radius } \rho_t^{(i,n)} \text{around this random point is empty} \Big) \\
&\leq (\lfloor t^{d+\varepsilon} \rfloor + 1) \exp\left[-\frac{c^2}{2(i+1)/n} t + o(t) \right] \\
&\qquad \times \exp\left[-\ell f_d \left((1-\varepsilon)\sqrt{2\beta} \left(1 - \frac{i+1}{n} \right), c \right) t + o(t) \right]. \quad (3.16)
\end{aligned}
$$

Indeed, on the event $A_t^{(i,n)}$ the number of empty balls is $\leq \lfloor t^{d+\varepsilon} \rfloor$, so that at least one of the $\lfloor t^{d+\varepsilon} \rfloor + 1$ particles picked at random must have a non-empty ball around it. From (3.14) and (3.16) we now get

$$
\begin{aligned}
&(\mathbb{E}_\nu \times P_t^{(i,n)})(T > t) \\
&\leq \sum_{j=0}^{n-1} (\mathbb{E}_\nu \times P_t^{(i,n)}) \left(\left\{ \frac{j}{n}\sqrt{2\beta}\,t \leq |x_0^{(i)}| < \frac{j+1}{n}\sqrt{2\beta}\,t \right\} \cap A_t^{(i,n)} \right) \\
&\qquad + \exp[-\beta t + o(t)] + \mathrm{SES} \\
&\leq \sum_{j=0}^{n-1} (\lfloor t^{d+\varepsilon} \rfloor + 1) \exp\left[\frac{-\beta j^2/n^2}{(i+1)/n} t + o(t) \right] \\
&\qquad \times \exp\left[-\ell f_d \left((1-\varepsilon)\sqrt{2\beta}\left(1 - \frac{i+1}{n}\right), \frac{j}{n} \right) t + o(t) \right] \\
&\qquad + \exp[-\beta t + o(t)] + \mathrm{SES}.
\end{aligned} \tag{3.17}
$$

(The SES comes from the second term in the right-hand side of (3.14.) Substitute this estimate into (3.11), optimize over $i, j \in \{0, 1, \ldots, n-1\}$, let $n \to \infty$ followed by $\varepsilon \downarrow 0$, to obtain

$$\limsup_{t \to \infty} \frac{1}{t} \log(\mathbb{E}_\nu \times P_0)(T > t) \leq -I(\ell, \beta, d). \tag{3.18}$$

(In (3.17), put $\eta = i/n$ and $c = j/n$ before letting $n \to \infty$ and use the continuity of the functional from which the minimum is taken.) $\square$

4 Proof of Theorem 2

4.1 Proof of Theorem 2(i)

Let

$$F_d(\eta, c) = \beta\eta + \frac{c^2}{2\eta} + \ell f_d\left(\sqrt{2\beta}\,(1-\eta), c\right). \tag{4.1}$$

Then (1.13) reads (insert (1.7))

$$I(\ell, \beta, \eta) = \min_{\eta \in [0,1],\, c \in [0,\infty)} F_d(\eta, c). \tag{4.2}$$

Since F_d is continuous in both variables and diverges as $c \to \infty$, the minimisers η^*, c^* of (4.2) exist. We need to show that they are unique when $\ell \neq \ell_{cr}$.

$\underline{d=1}$: Since $f_1(r,b)=2r$, we have $f_1(\sqrt{2\beta}\,(1-\eta),c)=2\sqrt{2\beta}\,(1-\eta)$, which does not depend on c. Hence the minimum over c in (4.2) is taken at $c^*=0$, so that (4.2) reduces to

$$I(\ell,\beta,1)=\min_{\eta\in[0,1]}\left\{\beta\eta+\ell 2\sqrt{2\beta}\,(1-\eta)\right\}. \tag{4.3}$$

The function under the minimum in (4.3) is linear in η, and changes its slope from positive to negative as ℓ moves upwards through the critical value ℓ_{cr} given by $\beta=\ell_{cr}2\sqrt{2\beta}$. This identifies ℓ_{cr} as in (1.9). The minimiser of (4.3) changes from $\eta^*=0$ to $\eta^*=1$, proving (1.15), while $I(\ell,\beta,1)$ changes from $\ell 2\sqrt{2\beta}$ to β, proving (1.14).

$\underline{d\geq 2}$: We have

$$F_d(\eta,c)-F_d(0,0)=\beta\eta+\frac{c^2}{2\eta}+\ell A_{d,\beta}(\eta,c) \tag{4.4}$$

where

$$A_{\beta,d}(\eta,c)=f_d(\sqrt{2\beta}\,(1-\eta),c)-f_d(\sqrt{2\beta},0)\leq 0 \tag{4.5}$$

with equality if and only if $(\eta,c)=(0,0)$. (The latter statement is easily deduced from (1.6).) Suppose that $(\eta,c)=(0,0)$ is a minimiser when $\ell=\ell_0$. Then the right-hand side of (4.4) is nonnegative for all (η,c) when $\ell=\ell_0$. Consequently, for all $\ell<\ell_0$ the right-hand side of (4.4) is zero when $(\eta,c)=(0,0)$ and strictly positive otherwise. Therefore we conclude that there must exist an $\ell_{cr}\in[0,\infty]$ such that

(i) $(\eta,c)=(0,0)$ is the *unique* minimiser when $\ell<\ell_{cr}$.

(ii) $(\eta,c)=(0,0)$ is not a minimiser when $\ell>\ell_{cr}$.

In Section 4.2 we will identify ℓ_{cr} as given by (1.9–1.11), and this will show that actually $\ell_{cr}\in(0,\infty)$. It remains to prove that when $\ell>\ell_{cr}$ the minimisers are unique. This is done in Steps I–III below.

<u>I. Minimisers in the interior</u>: First note that the combination $\eta^*=0,c^*>0$ is not possible because $F(0,c)=\infty$ for all $c>0$ (see the remark below (1.13)). Also the combination $\eta^*>0,c^*=0$ is not possible because $F(\eta,0)$ takes its minimum either at $\eta=0$ or $\eta=1$, and so $\eta^*>0$ would imply $\eta^*=1$. However, $\eta^*=1$ can be excluded through following lemma.

Lemma 1 *For every $\ell_0>0$ there exists a $\delta_0=\delta_0(\ell_0)>0$ such that $\eta^*\leq 1-\delta_0$ for all $\ell\leq\ell_0$.*

Proof of Lemma 1. Since $F_d(1,c) = \beta + \frac{c^2}{2}$, a minimiser $\eta^* = 1$ would necessarily come with a minimiser $c^* = 0$, giving the value β for the minimum. However, it is possible to do better. Namely, note from (1.6) that $f_d(r,b) \sim v_d r^d / b^{d-1}$ as $r \downarrow 0$ and $r/b \downarrow 0$, with v_d the volume of the d-dimensional unit ball. Pick $\eta = 1-\delta$ and $c = \delta^{3/4}$. Then, for $\delta \downarrow 0$,

$$\begin{aligned} F_d(1-\delta, \delta^{3/4}) &= \beta(1-\delta) + \frac{\delta^{3/2}}{2(1-\delta)} + \ell f_d(\sqrt{2\beta}\,\delta, \delta^{3/4}) \\ &= \beta(1-\delta) + \left[\frac{1}{2}\delta^{3/2} + \ell v_d (2\beta)^{d/2} \delta^{(d+3)/4}\right](1+o(1)) \\ &= \beta(1-\delta) + o(\delta). \end{aligned} \tag{4.6}$$

For δ small enough, the right-hand side is strictly decreasing in δ, showing that the minimum cannot occur at $\delta = 0$. In fact, the above expansion shows that $\delta \geq \delta_0(\ell_0)$ for any $\ell \leq \ell_0$. □

Thus we may conclude that for $\ell > \ell_{cr}$ it is enough to consider $0 < \eta < 1$ and $c > 0$, and consequently we may look for the stationary points of F_d.

II. Stationary points: For $R \geq 0$, let

$$f_d(R) = \int_{B_R(0)} \frac{\mathrm{d}x}{|x+e|^{d-1}}. \tag{4.7}$$

Then we may write (4.1) as

$$F_d(\eta, c) = \beta\eta + \frac{c^2}{2\eta} + \ell c f_d\left(\frac{\sqrt{2\beta}\,(1-\eta)}{c}\right). \tag{4.8}$$

The stationary points are the solutions of the equations

$$\begin{aligned} 0 &= \beta - \frac{c^2}{2\eta^2} - \ell\sqrt{2\beta} f_d'\left(\frac{\sqrt{2\beta}\,(1-\eta)}{c}\right), \\ 0 &= \frac{c}{\eta} + \ell\Bigg[f_d\left(\frac{\sqrt{2\beta}\,(1-\eta)}{c}\right) \\ &\qquad - \frac{\sqrt{2\beta}\,(1-\eta)}{c} f_d'\left(\frac{\sqrt{2\beta}\,(1-\eta)}{c}\right)\Bigg]. \end{aligned} \tag{4.9}$$

Eliminating f_d', we obtain

$$\ell f_d\left(\frac{\sqrt{2\beta}\,(1-\eta^*)}{c^*}\right) = -\frac{c^*}{\eta^*} + \frac{1-\eta^*}{c^*}\left[\beta - \frac{c^{*2}}{2\eta^{*2}}\right] \tag{4.10}$$

and hence

$$F_d(\eta^*, c^*) = \beta - \frac{c^{*2}}{2\eta^{*2}}. \tag{4.11}$$

Putting

$$u = \frac{\sqrt{2\beta}\,(1-\eta)}{c}, \qquad v = \frac{c}{\sqrt{2\beta}\,\eta}, \tag{4.12}$$

we may rewrite (4.9) as

$$\begin{aligned} 0 &= \beta - \beta v^2 - \ell\sqrt{2\beta} f_d'(u), \\ 0 &= \sqrt{2\beta}\, v + \ell[f_d(u) - u f_d'(u)], \end{aligned} \tag{4.13}$$

and (4.11) as

$$F_d(u^*, v^*) = \beta(1 - v^{*2}). \tag{4.14}$$

III. Uniqueness: Suppose that (u_1, v_1) and (u_2, v_2) give the same minimum. Then, by (4.14), we have $v_1 = v_2$. Suppose that $u_1 \neq u_2$. Then from the first line of (4.13) it follows that $f_d'(u_1) = f_d'(u_2)$. Using this in the second line of (4.13), we get

$$f_d(u_1) - u_1 f_d'(u_1) = f_d(u_2) - u_2 f_d'(u_1), \tag{4.15}$$

or

$$\frac{f_d(u_1) - f_d(u_2)}{u_1 - u_2} = f_d'(u_1) = f_d'(u_2). \tag{4.16}$$

This in turn implies that there must exist a third value u_3, strictly between u_1 and u_2, such that

$$f_d'(u_3) = f_d'(u_1) = f_d'(u_2). \tag{4.17}$$

Uniqueness now follows from the following property of f_d, implying that f_d' does not attain the same value at three different points. (The singularity of f_d' at 1 does not affect the above argument.)

Lemma 2 *$R \mapsto f_d'(R)$ is strictly increasing on $(0,1)$, infinity at 1, and strictly decreasing on $(1,\infty)$.*

Proof of Lemma 2. Using polar coordinates, we can write (4.7) as

$$f_d(R) = C(d) \int_0^R \mathrm{d}r\ r^{d-1} \int_0^\pi \mathrm{d}\omega\ \left(1 + r^2 - 2r\cos\omega\right)^{-\frac{d-1}{2}}, \quad (4.18)$$

where only one angle variable ω appears because of radial symmetry. Hence

$$f_d'(R) = C(d) R^{d-1} \int_0^\pi \mathrm{d}\omega\ \left(1 + R^2 - 2R\cos\omega\right)^{-\frac{d-1}{2}}. \quad (4.19)$$

Set $S = 1/R$ to write

$$f_d'(1/S) = C(d) \int_0^\pi \mathrm{d}\omega\ \left(1 + S^2 - 2S\cos\omega\right)^{-\frac{d-1}{2}}. \quad (4.20)$$

For $S > 1$, the integrand is strictly decreasing in S for all $\omega \in [0, \pi]$. Therefore $S \mapsto f_d'(1/S)$ is strictly decreasing on $(1, \infty)$. At $S = 1$, it diverges. For $S < 1$, (4.20) equals

$$f_d'(1/S) = C(d)\pi F(\nu, \nu; 1, S^2) = C(d)\pi \sum_{k=0}^\infty S^{2k} \left(\frac{\prod_{l=0}^{k-1} (\nu + l)}{k!} \right)^2, \quad (4.21)$$

where F is the hypergeometric function and $\nu = \frac{d-1}{2}$ (see Gradshteyn and Ryzhik [4], 3.665.2 and 9.100). The summand is strictly increasing in S for all k. Therefore $S \mapsto f_d'(1/S)$ is strictly increasing on $(0, 1)$. □

4.2 Proof of Theorem 2(ii–iii)

Part (ii) is an immediate consequence of the calculation for $d = 1$ in Section 4.1. Part (iii) partly follows from the calculation for $d \geq 2$ in Section 4.1. The remaining items are proved here.

From (4.2) and (4.14) we see that

$$I(\ell, \beta, d) = \beta(1 - v^{*2}) \quad (4.22)$$

with v^* the maximal value of v on the curve in the (u, v)-plane given by the equation

$$\frac{\ell}{\sqrt{2\beta}} f_d(u) = -v + \frac{1}{2} u(1 - v^2), \quad (4.23)$$

which is obtained from (4.13) by eliminating $f_d'(u)$. Using (1.8), we can write the left-hand side of (4.23) as $\frac{\ell}{\ell^*_{cr}}\frac{1}{2s_d}f_d(u)$. Since $f_d(u) \sim s_d u$ as $u \to \infty$ by (4.7), we obtain from Lemma 2 that the left-hand side of (4.23) is strictly convex on $(0,1)$, has an infinite slope at 1, is strictly concave on $(1,\infty)$, and has the line $u \mapsto \frac{1}{2}\frac{\ell}{\ell^*_{cr}}u$ as its asymptote (see Fig. 2). Therefore we can identify ℓ_{cr} through the following formula:

$$\ell_{cr} = \sup\left\{\ell > 0\colon\ \frac{\ell}{\ell^*_{cr}}\frac{1}{s_d}f_d(u) > -\sqrt{1-\frac{\ell}{\ell^*_{cr}}} + \frac{1}{2}\frac{\ell}{\ell^*_{cr}}u \quad \forall u \in (0,\infty)\right\}. \tag{4.24}$$

Indeed, consider the line on the right-hand side of (4.23) when v is chosen such that $1 - v^2 = \frac{\ell}{\ell^*_{cr}}$, in which case it runs parallel to the asymptote of the curve on the left-hand side. If $\ell < \ell_{cr}$, then the line and the curve do not touch. As v is decreased, the line cuts the curve at infinity, indicating that v^* corresponds to $u^* = \infty$. On the other hand, if $\ell > \ell_{cr}$, then the line will touch the curve for a larger value of v, indicating that v^* corresponds to $u^* \in (0,\infty)$. In fact, v^* corresponds to $u^* \in (0,1)$ because the curve has infinite slope at $u = 1$ and is concave for $u \in (1,\infty)$.

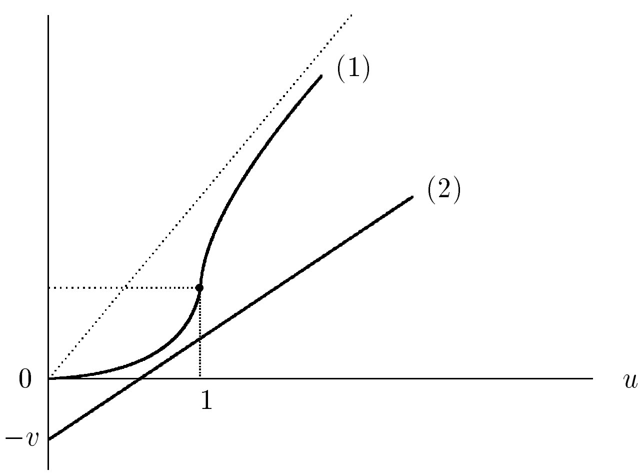

Fig. 2 Qualitative plot of:
(1) $u \mapsto \frac{\ell}{\ell^*_{cr}}\frac{1}{2s_d}f_d(u)$; (2) $u \mapsto -v + \frac{1}{2}u(1-v^2)$.
The dotted line has slope $\frac{1}{2}\frac{\ell}{\ell^*_{cr}}$.

Put

$$\hat{f}_d(u) = s_d u - f_d(u), \qquad M_d = \frac{1}{2s_d} \max_{u \in (0,\infty)} \hat{f}_d(u). \tag{4.25}$$

Then (4.24) reads

$$\ell_{cr} = \sup \left\{ \ell > 0\colon \sqrt{1 - \frac{\ell}{\ell^*_{cr}}} > \frac{\ell}{\ell^*_{cr}} M_d \right\}. \tag{4.26}$$

Hence

$$\sqrt{1 - \frac{\ell_{cr}}{\ell^*_{cr}}} = \frac{\ell_{cr}}{\ell^*_{cr}} M_d. \tag{4.27}$$

This completes the proof of (1.9–1.11).

We have now proved all the statements in Theorem 2(ii–iii) (see also Section 4.1).

4.3 Proof of Theorem 2 (iv–v)

The following properties hold:

(a) $\ell \mapsto u^*(\ell,\beta,d)$ and $\ell \mapsto v^*(\ell,\beta,d)$ are continuous on (ℓ_{cr},∞);

(b) $\ell \mapsto v^*(\ell,\beta,d)$ is strictly decreasing on (ℓ_{cr},∞);

(c)

$$\lim_{\ell \downarrow \ell_{cr}} u^*(\ell,\beta,d) = u_d, \qquad \lim_{\ell \downarrow \ell_{cr}} v^*(\ell,\beta,d) = \sqrt{1-\alpha_d}, \tag{4.28}$$

where u_d is the unique maximiser of the variational problem in (4.25) and α_d is given by (1.10);

(d) $\lim_{\ell\to\infty} u^*(\ell,\beta,d) = \lim_{\ell\to\infty} v^*(\ell,\beta,d) = 0$;

(e) $u^*(\ell,\beta,d) \in (0,1)$ for all $\ell \in (\ell_{cr},\infty)$.

These properties are easily deduced from Fig. 2. Note that curve (1) is ℓ times a function that does not depend on ℓ.

(iv) Items (a) and (b) in combination with (4.2) and (4.14) imply that $\ell \mapsto I(\ell,\beta,d)$ is continuous and strictly increasing on (ℓ_{cr},∞). To see that it is continuous at ℓ_{cr}, use item (c) to get $\lim_{\ell\downarrow\ell_{cr}} I(\ell,\beta,d) =$

$\beta\alpha_d = \beta\frac{\ell_{cr}}{\ell_{cr}^*}$ (recall (1.9)), which coincides with the limit from below. Item (d) in combination with (4.2) yields $\lim_{\ell\to\infty} I(\ell,\beta,d) = \beta$.

(v) Since (recall (4.12))

$$\eta^* = \frac{1}{1+u^*v^*}, \qquad c^* = \sqrt{2\beta}\,\frac{v^*}{1+u^*v^*}, \tag{4.29}$$

item (a) implies that $\ell \mapsto \eta^*(\ell,\beta,d)$ and $\ell \mapsto c^*(\ell,\beta,d)$ are continuous on (ℓ_{cr},∞). Clearly, (4.28) and (4.29) imply that η^* and c^* tend to a strictly positive limit as $\ell \downarrow \ell_{cr}$, which shows a discontinuity from their value zero for $\ell < \ell_{cr}$. Item (d) shows that $(1-\eta^*)/c^*$ and c^* tend to zero as $\ell \to \infty$. Finally, from item (e) we obtain that $c^* > \sqrt{2\beta}\,(1-\eta^*)$ (recall (4.12)).

5 Proof of Theorem 3

The proofs of the various statements in Theorem 3 all rely on the following simple consequence of Theorem 1. Let $\{E_t\}_{t\geq 0}$ be a family of measurable events satisfying

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}\times P_{\delta_0})(\{T>t\}\cap E_t^c) < -I(\ell,\beta,d). \tag{5.1}$$

Then

$$\lim_{t\to\infty} (\mathbb{E}\times P_{\delta_0})(E_t \mid T>t) = 1. \tag{5.2}$$

Since all the statements in Theorem 3 have the form of (5.2), they may be proved by showing the corresponding inequality of type (5.1). The proofs below are based on Section 3.2. We use the notations of that section freely.

5.1 Proof of Theorem 3(iii)

Let $\ell > \ell_{cr}$ and $0 < \varepsilon < \eta^*$. Abbreviate

$$K_t = \left\{ |Z((\eta^*-\varepsilon)t)| \leq \lfloor t^{d+\varepsilon} \rfloor \right\}. \tag{5.3}$$

Since $K_t^c = \{\eta_t < \eta^* - \varepsilon\}$ (recall (3.10)), we have similarly as in (3.11) that

$$
\begin{aligned}
&(\mathbb{E} \times P_{\delta_0})\,(\{T > t\} \cap K_t^c) \\
&\le \sum_{i=0}^{\lfloor n(\eta^*-\varepsilon)\rfloor -1} (\mathbb{E}_\nu \times P_0)\left(\{T > t\} \cap \left\{\frac{i}{n} \le \eta_t < \frac{i+1}{n}\right\}\right) \\
&\le \sum_{i=0}^{\lfloor n(\eta^*-\varepsilon)\rfloor -1} \exp\left[-\beta\frac{i}{n}t + o\,(t)\right] (\mathbb{E}_\nu \times P_t^{(i,n)})\,(T > t)\,. \qquad (5.4)
\end{aligned}
$$

To continue the estimate, substitute (3.17) into (5.4) and optimise over $j \in \{0, 1, \dots, n-1\}$, but with the constraint

$$
i \in \{0, 1, \dots, \lfloor n(\eta^* - \varepsilon)\rfloor - 1\}. \qquad (5.5)
$$

By Theorem 2(i), the variational problem defining $I(\ell, \beta, d)$ has a unique pair of minimisers. However, under the optimisation, the parameter $\eta = i/n$ is bounded away from η^* because of (5.5). Consequently,

$$
\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E} \times P_{\delta_0})\,(\{T > t\} \cap K_t^c) < -I(\ell, \beta, d). \qquad (5.6)
$$

5.2 Proof of Theorem 3(i)

The proof of the first limit in (1.20) is very similar to that of part (iii). First, recall that ct is the distance between the origin and the centre of the closest empty ball in the proof in Section 3.2. Let $\varepsilon' > 0$ and $\delta > 0$ be so small that

$$
\sqrt{2\beta}\,\varepsilon > \sqrt{2\beta}\,\varepsilon' + \delta. \qquad (5.7)
$$

Then, obviously, the event

$$
C_t = \{\exists x_0 \in \mathbb{R}^d\colon\, |\,|x_0| - c^*\,| < \delta,\, B_{\sqrt{2\beta}\,(1-\eta^*-\varepsilon')t}(x_0 t) \cap K = \emptyset\} \qquad (5.8)
$$

is contained in the event $C(t; \sqrt{2\beta}\,(1 - \eta^* - \varepsilon), c^*)$, so it suffices to prove the claim for C_t. Consider the optimisation procedure in the proof in Section 3.2, but now for the probability

$$
(\mathbb{E} \times P_{\delta_0})\,(\{T > t\} \cap C_t^c)\,. \qquad (5.9)
$$

Similarly to the proof of part (iii), the vector parameter $(\eta, c) = (i/n, j/n)$ is again bounded away from its optimal value. The difference is that, instead (5.5), now $(i/n, j/n)$ is bounded away from the set

$$(\eta^* - \varepsilon', \eta^* + \varepsilon') \times (c^* - \delta, c^* + \delta). \tag{5.10}$$

Again, it follows from the uniqueness of the minimisers that

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E} \times P_{\delta_0}) (\{T > t\} \cap C_t^c) < -I(\ell, \beta, d). \tag{5.11}$$

To prove the second limit in (1.20), abbreviate

$$\bar{K}_t = \left\{ |Z(t)| \geq \lfloor e^{\beta(1-\eta^*-\varepsilon)t} \rfloor \right\}. \tag{5.12}$$

First note that, by (2.2), for any $\varepsilon > 1/m$ and $k \geq 1 - \eta^* - 1/m$,

$$\begin{aligned} &\sup_{x\in\mathbb{R}^d} P_{\delta_x} \left(|Z\,(kt)\,| < \lfloor e^{\beta(1-\eta^*-\varepsilon)t} \rfloor \right) \\ &\qquad \leq e^{-\beta(\varepsilon-1/m)t}[1 + o(1)]. \end{aligned} \tag{5.13}$$

The probability $(\mathbb{E} \times P_{\delta_0})\,(T > t)$ was already estimated through (3.11) and (3.17). To estimate $(\mathbb{E} \times P_{\delta_0})\,(\{T > t\} \cap \bar{K}_t^c)$, use the analogue of (3.11), but modify the estimate in (3.17) as follows. First, observe that for

$$\frac{i+1}{n} \leq \eta^* + \frac{1}{m}$$

we can use the Markov property at time $\frac{i+1}{n}$ together with (5.13), to obtain an estimate that is actually stronger than the one in (3.17):

$$\begin{aligned} &(\mathbb{E}_\nu \times P_t^{(i,n)})(\{T > t\} \cap \bar{K}_t^c) \\ &\leq \exp\left[-\beta\left(\varepsilon - \frac{1}{m}\right) t + o(t)\right] \\ &\quad \times \sum_{j=0}^{n-1} (\lfloor t^{d+\varepsilon} \rfloor + 1) \exp\left[\frac{-\beta j^2/n^2}{(i+1)/n} t + o(t)\right] \\ &\quad \times \exp\left[-\ell f_d\left((1-\varepsilon)\sqrt{2\beta}\left(1 - \frac{i+1}{n}\right), \frac{j}{n}\right) t + o(t)\right] \\ &\quad + \exp[-\beta t + o(t)] + \mathrm{SES}. \end{aligned} \tag{5.14}$$

Compare now (3.17) with (5.14). The presence of the extra factor $\exp\left[-\beta\left(\varepsilon-\frac{1}{m}\right)t+o(t)\right]$ in (5.14) means that when the parameter $\eta=i/n$ is close to its optimal (for (3.17)) value η^*, the optimum obtained from (5.14) is strictly smaller than the one obtained from (3.17). Since, on the other hand, η^* is the unique minimiser for (3.17), this is already enough to conclude that

$$\limsup_{t\to\infty}\frac{1}{t}\log\left(\mathbb{E}\times P_{\delta_0}\right)\left(\{T>t\}\cap\bar{K}_t^c\right)<-I(\ell,\beta,d). \tag{5.15}$$

5.3 Proof of Theorem 3(ii)

To prove the second limit in (1.21), abbreviate

$$D_t=\{R(t)\subseteq B_{(1+\varepsilon)\sqrt{2\beta}\,t}(0)\} \tag{5.16}$$

and note that, since

$$\frac{1}{2}\Big((1+\varepsilon)\sqrt{2\beta}\Big)^2>\beta, \tag{5.17}$$

the same argument as in the proof of (2.18) gives us that

$$\limsup_{t\to\infty}\frac{1}{t}\log P_{\delta_0}(D_t^c)\leq-(1+\varepsilon)^2\beta+\beta=-(2+\varepsilon)\varepsilon\beta. \tag{5.18}$$

Pick $\varepsilon'>0$ such that $\varepsilon'\beta\frac{\ell}{\ell_{cr}^*}=(2+\varepsilon)\varepsilon\beta$. Then (5.18) says that

$$\limsup_{t\to\infty}\frac{1}{t}\log P_{\delta_0}(D_t^c)\leq-\varepsilon'\beta\frac{\ell}{\ell_{cr}^*}. \tag{5.19}$$

Using the first limit in (1.20) with $\varepsilon=\varepsilon'/2$, we find that (recall $\eta^*=c^*=0$ and (1.8))

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\left(\mathbb{E}\times P_{\delta_0}\right)\left(\{T>t\}\cap D_t^c\right)\\
&\leq\limsup_{t\to\infty}\frac{1}{t}\log\left(\mathbb{E}\times P_{\delta_0}\right)\Big(C\Big(t;\sqrt{2\beta}\left(1-\varepsilon'/2\right),0\Big)\cap D_t^c\Big)\\
&=\limsup_{t\to\infty}\frac{1}{t}\Big[\log\mathbb{P}\Big(C\Big(t;\sqrt{2\beta}\left(1-\varepsilon'/2\right),0\Big)\Big)+\log P_{\delta_0}\left(D_t^c\right)\Big]\\
&\leq-\left(1-\varepsilon'/2+\varepsilon'\right)\beta\frac{\ell}{\ell_{cr}^*}\\
&=-\left(1+\varepsilon'/2\right)I(\ell,\beta,d)\\
&<-I(\ell,\beta,d),
\end{aligned} \tag{5.20}$$

where the second inequality uses (1.1) and (5.19), and the second equality uses the first line of (1.16).

To prove the third limit in (1.21), let $0 < \varepsilon' < \varepsilon$ and abbreviate

$$\begin{aligned} A_t^1 &= \{B_{(1-\varepsilon)\sqrt{2\beta}\,t}(0) \cap K = \emptyset\}, \\ A_t^2 &= \{B_{(1-\varepsilon')\sqrt{2\beta}\,t}(0) \cap K = \emptyset\}, \\ D_t^1 &= \{R(t) \not\subseteq B_{(1-\varepsilon)\sqrt{2\beta}\,t}(0)\}. \end{aligned} \tag{5.21}$$

Estimate

$$\begin{aligned} (\mathbb{E} \times P_{\delta_0})(\{T > t\} \cap [D_t^1]^c) \leq (\mathbb{E} \times P_{\delta_0})(\{T > t\} \cap [D_t^1]^c \cap A_t^2) \\ +(\mathbb{E} \times P_{\delta_0})\left(\{T > t\} \cap [A_t^2]^c\right). \end{aligned} \tag{5.22}$$

From (5.11) we have that

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E} \times P_{\delta_0})\left(\{T > t\} \cap [A_t^2]^c\right) < -I(\ell, \beta, d). \tag{5.23}$$

Clearly,

$$\begin{aligned} (\mathbb{E} \times P_{\delta_0})([D_t^1]^c \cap A_t^1) &= P_{\delta_0}([D_t^1]^c)\mathbb{P}(A_t^1), \\ (\mathbb{E} \times P_{\delta_0})([D_t^1]^c \cap A_t^2) &= P_{\delta_0}([D_t^1]^c)\mathbb{P}(A_t^2), \end{aligned} \tag{5.24}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(A_t^2) < \lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(A_t^1). \tag{5.25}$$

Hence

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E} \times P_{\delta_0})([D_t^1]^c \cap A_t^2) \\ &\qquad < \limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E} \times P_{\delta_0})([D_t^1]^c \cap A_t^1) \\ &\qquad \leq -I(\ell, \beta, d), \end{aligned} \tag{5.26}$$

where the last inequality follows from Theorem 1 and the fact that $\{[D_t^1]^c \cap A_t^1\} \subseteq \{T > t\}$. By (5.22–5.23), and (5.26), we obtain that

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E} \times P_{\delta_0})(\{T > t\} \cap [D_t^1]^c) < -I(\ell, \beta, d). \tag{5.27}$$

The proof of the first limit in (1.21) is a slight adaptation of the previous argument. Let $0 < \varepsilon' < \varepsilon$. Let D_t be as in (5.16) but replace ε by ε', and abbreviate

$$\begin{aligned} A_t^1 &= \{B_{(1+\varepsilon)\sqrt{2\beta}\,t}(0) \cap K \neq \emptyset\}, \\ A_t^2 &= \{B_{(1+\varepsilon')\sqrt{2\beta}\,t}(0) \cap K \neq \emptyset\}. \end{aligned} \tag{5.28}$$

Estimate

$$\begin{aligned} (\mathbb{E} \times P_{\delta_0})(\{T > t\} \cap [A_t^1]^c) &\leq (\mathbb{E} \times P_{\delta_0})(\{T > t\} \cap D_t \cap [A_t^1]^c) \\ &\quad + (\mathbb{E} \times P_{\delta_0})\left(\{T > t\} \cap [D_t]^c\right). \end{aligned} \tag{5.29}$$

Now the statement follows from (5.29) and (5.20) along with the estimate

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E} \times P_{\delta_0})(D_t \cap [A_t^1]^c) \\ &\quad < \limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E} \times P_{\delta_0})(D_t \cap [A_t^2]^c) \\ &\quad \leq -I(\ell, \beta, d). \end{aligned} \tag{5.30}$$

5.4 Proof of Theorem 3(iv)

We will consider the two statements in the reversed order. For the second statement in (1.23), first note that, by Theorem 2(ii), we have $\eta^* = 1$. Now recall the definition of K_t from (5.3). The estimate in (5.6) with $\eta^* = 1$ says that the event $\{T > t\} \cap K_t^c$ has a probability that is smaller than the probability of $\{T > t\}$ on an exponential scale (let us call such an event *negligible*), i.e., considering survival, we may also assume that there are polynomially many particles only at time $t(1-\varepsilon)$ $(0 < \varepsilon < 1)$.

The strategy of the rest of the proof is to show two facts:
(a) *all the particles* have not left an $\varepsilon t/2$-ball around the origin up to time $t(1-\varepsilon)$ (let F_t denote this event);
(b) each BBM emanating from one of the "parent" particles at time $t(1-\varepsilon)$ will be contained in an $\varepsilon t/2$-ball around the position of the parent particle (let G_t denote this event).

For (a), note that trivially, $K_t \cap F_t^c$ has an exponentially small probability (because the polynomial factor does not affect the exponential estimate), but we must in fact show that $\{T > t\} \cap K_t \cap F_t^c$ is negligible. We now sketch how to modify (5.20) to prove this and leave the obvious details to the reader. To estimate $\{T > t\} \cap K_t \cap F_t^c$, replace $(1+\varepsilon)\sqrt{2\beta}$ by $\varepsilon/2$ and, instead of the first limit in (1.20) (regarding the existence of the empty ball), use Theorem 3(iii) along with the fact that the branching is independent of the motion.

For (b), we must show that $\{T > t\} \cap K_t \cap G_t^c$ is negligible. The proof is similar to the one in the previous paragraph: (5.20) should be appropriately modified. The difference is that now we must use the Markov property at time $t(1-\varepsilon)$ and deal with several particles at that time. However, this is no problem because on the event K_t we have polynomially many particles only. (The use of Theorem 3(iii) is just like in the previous paragraph.)

The first statement in (1.23) follows after replacing $(1+\varepsilon)\sqrt{2\beta}$ and $(1+\varepsilon')\sqrt{2\beta}$ by ε resp. ε' in (5.28–5.30), and using the second statement in (1.23) instead of (5.20).

References

[1] J. Engländer, On the volume of the supercritical super-Brownian sausage conditioned on survival, Stoch. Proc. Appl. 88 (2000) 225–243.

[2] J. Engländer and A.E. Kyprianou, Markov branching diffusions: martingales, Girsanov type theorems and applications to the long-term behavior, EURANDOM Report 2001–031.

[3] M. Freidlin, *Functional Integration and Partial Differential Equations*, Annals of Mathematics Studies 109, Princeton University Press, Princeton, 1985.

[4] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, 1980.

[5] S. Karlin and M. Taylor, *A First Course in Stochastic Processes*, Academic Press, New York, 1975.

[6] H.P. McKean, Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov, Comm. Pure Appl. Math. 28 (1975) 323–331.

[7] H.P. McKean, A correction to "Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov", Comm. Pure Appl. Math. 29 (1976) 553–554.

[8] P. Révész, *Random Walks of Infinitely Many Particles*, World Scientific, Singapore, 1994.

[9] A.-S. Sznitman, *Brownian Motion, Obstacles and Random Media*, Springer, Berlin, 1998.