An Intermittent Fluid System with Exponential On Times and Semi-Markov Input Rates

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Abstract

We consider a fluid system in which during off times the buffer content increases as a piecewise linear process according to some general semi-Markov process and during on times it decreases with a state dependent rate (or remains at zero). The lengths of off times are exponentially distributed. We show that such a system has a stationary distribution which satisfies a decomposition property where one component in the decomposition is associated with some dam process and the other with a clearing process. For the cases of constant and linear decrease rate the steady state Laplace Stieltjes transform (LST) and moments of the buffer content are computed explicitly.

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1 Introduction

We consider a fluid inventory system of which flow rates are functions of some semi-Markov process (SMP). This SMP is quite general other than the fact that there is a special state in which the sojourn time is exponentially distributed and is independent of the following state. When the SMP is in this special state, the inventory decreases according to some rate function, which may depend on the inventory content, as long as it is positive (otherwise it stays at zero). When the system is in any other state, it increases linearly with a rate depending on the state.

We apply PASTA (see [7]) and results from [4] to establish a decomposition result for the steady state distribution of the inventory content. In particular, the stationary distribution is a convolution of two distributions. The first is the steady state distribution of some dam process with compound Poisson input, and the second is a mixture of an atom at zero and the steady state distribution of some clearing process. The Laplace-Stieltjes transform (LST) and moments of the clearing process are completely characterized. For the dam process component we characterize the jump distribution. The complete steady state distribution of the inventory content can only be characterized when the steady state distribution of such a dam can be computed. Two such cases are when the release rate is constant and when the release rate is linear. For the first case the dam becomes the workload in an M/G/1 queue and in the second case the dam becomes a shot noise process. For both, the steady state distribution is well understood.

For earlier directly related papers see [4, 2] as well as further references in these papers. In [2] the busy period and the maximum buffer content for the present model are discussed, but the content process is only considered for the case where the SMP has three states. For the case with two increasing states and one decreasing state, the results in Theorem 3.3 of [2] (which were obtained via a different method) can be easily reproduced as a special case of the results established in the current paper.

The paper is organized as follows. In Section 2 we introduce the model and main notations. In Section 3 the main decomposition result is established. In Section 4 the steady state LST of the associated clearing process is determined. In Section 5 the jump size LST of the associated dam process is given. The special cases of constant and linear release rates are handled in Section 6. In Section 7 the busy period is considered and finally in Section 8 we discuss a generalization for the case of constant decrease rate but general decrease times.

2 The Model

Consider a Markov renewal process $\{(X_n, \tau_{n+1})|\ n \geq 0\}$, with finite state space $\{0, \ldots, K\}$, that is, satisfying

$$P[X_{n+1} = j, \tau_{n+1} \leq t|\ (X_0, \tau_1, \ldots, X_{n-1}, \tau_n) \in A, X_n = i] = p_{ij}F_{ij}(t) \tag{2.1}$$

where $P = \{p_{ij}|\ 0 \leq i, j \leq K\}$ is a stochastic matrix and

$$F_{ij}(t) = P[\tau_1 \leq t|\ X_0 = i, X_1 = j]$$

is some distribution function for any given i, j. We will use the following notations:

- $F_i(t) = \sum_{j=0}^{K} p_{ij}F_{ij}(t)$,
- $m_{ij}^{(k)} = \int_0^\infty x^k dF_{ij}(x)$,
- $m_{ij} = m_{ij}^{(1)}$,
- $m_i^{(k)} = \sum_{j=0}^{K} p_{ij}m_{ij}^{(k)}$,
- $m_i = m_i^{(1)}$,
- $E_i(\cdot) = E[\cdot|\ X_0 = i]$,
- $E_{ij}(\cdot) = E[\cdot|\ X_0 = i, X_1 = j]$,
- $\tau_{ij}(\alpha) = E_{ij}e^{-\alpha\tau_1}$,
- $\tau_i(\alpha) = E_i e^{-\alpha\tau_1} = \sum_{j=0}^{K} p_{ij}\tau_{ij}(\alpha)$,
- $\tau_i^e(\alpha) = \frac{1-\tau_i(\alpha)}{\alpha m_i}$ (LST of the associated stationary forward recurrence time).

We assume that P is irreducible and that 0 is a special state. In particular, for every j,

- $F_{0j}(t) = F_0(t) = 1 - e^{-\lambda t}$

That is, the sojourn time in state 0 is distributed $\exp(\lambda)$ independently from the following state.

With $T_0 = 0$ and $T_n = \sum_{k=1}^{n} \tau_k$ set $N(t) = \sup\{n|\ T_n \leq t\}$ and define the SMP (semi-Markov process) $X(t) = X_{N(t)}$. Now, let $a(0), \ldots, a(K)$ be positive numbers (input rates) and let $r : [0, \infty) \to [0, \infty)$ (output rate) where $r(0) = a(0)$, $r(w) > a(0)$ for $w > 0$ and r is almost surely continuous (w.r.t. Lebesgue measure). The buffer content process is of the following form

$$W(t) = W(0) + \int_0^t [a(X(s)) - r(W(s))1_{\{X(s)=0\}}]ds\ . \tag{2.2}$$

That is, the buffer content decreases only when the system is in state zero. Since $r(w) > a(0)$ for $w > 0$ and $r(0) = a(0)$ we can without loss of generality assume that $a(0) = 0$, so that $r(0) = 0$ and $r(w) > 0$ for $w > 0$. Also since the number of consecutive transitions that we make from 0 to 0 is geometric with probability of success $1 - p_{00}$, the total amount of time we spend in state 0 before there is a transition to a different state is also exponential. Thus there is no loss of generality in assuming that $p_{00} = 0$. From this point on we make these two assumptions ($a(0) = p_{00} = 0$) in order to simplify notations.

We assume that the process is stable, that is, has a steady state distribution. For the special case of $r(w) = r$ for $w > 0$ we will provide conditions for stability and for the special case of $r(w) = rw$ the process is always stable.

3 A Decomposition Result

We see that our process increases piecewise linearly when in states $1, \ldots, K$ and decreases according to some release rate when in state 0. It is easy to modify the results of [4] so that they apply to the case at hand. The reason why they do not apply directly is that in [4] it is assumed that during times when the process decreases, the decrease is not state dependent. To make the paper more self contained we observe that if we look at the process only during times of decrease, then we obtain a dam process with release rate $r(\cdot)$, with compound Poisson input and jumps which are distributed like the amount of work which is accumulated from the instant the process leaves state 0 until the first instant thereafter when it enters it again. We denote the Laplace transform of the steady state distribution of this process by $V(\alpha)$.

From PASTA (see [7]) it follows that the steady state distribution of the discrete time process embedded right before jumps also has the LST $V(\alpha)$. If we look at our original process, this is precisely the distribution of the discrete time process embedded at instants where the SMP leaves state 0 (instants where the sample path has local minima).

If we look at our process only during times of increases, an identical approach as in [4] gives that the stationary distribution of this process is a convolution of the distribution of the state of the process at the beginning of the cycle (i.e., with LST $V(\alpha)$) and the steady state distribution determined by

$$R(\alpha) = \frac{1}{ED} E \int_0^D e^{-\alpha \int_0^t a(X(s))ds} dt \tag{3.1}$$

where $D = \inf\{t|\ X(t) = 0\}$ and E is the expected value when we assume that the distribution of the initial state is $\{p_{0j}|\ 1 \leq j \leq K\}$. This stationary distribution is the stationary distribution of a clearing process, where during each cycle the process increases according to SMP and the clearing times are the times when SMP first reaches state 0.

With these observations it is now evident that the steady state distribution of the buffer content process is given by the following.

Theorem 3.1 *With $d = ED$, the LST of the steady state distribution of the buffer content process is given by*

$$w(\alpha) = V(\alpha) \frac{1 + \lambda d R(\alpha)}{1 + \lambda d} \ . \tag{3.2}$$

Proof: According to the above observations, the LST of the steady state distribution of the process embedded during times of decrease is $V(\alpha)$ and that of the process embedded during times of increase is $V(\alpha)R(\alpha)$. The fraction of time the process spends in the former is $\lambda^{-1}/(\lambda^{-1}+d) = 1/(1+\lambda d)$ and, thus, in the latter $\lambda d/(1+\lambda d)$. Therefore,

$$w(\alpha) = \frac{1}{1+\lambda d}V(\alpha) + \frac{\lambda d}{1+\lambda d}V(\alpha)R(\alpha) = V(\alpha)\frac{1+\lambda d R(\alpha)}{1+\lambda d} \tag{3.3}$$

and we are done. ■

We note that $\frac{1+\lambda d R(\alpha)}{1+\lambda d}$ is the LST of the mixture of an atom at zero and the steady state distribution of the associated clearing process. In light of Theorem 3.1 it remains to identify $R(\alpha)$, $d = ED$ and $V(\alpha)$. We begin with the first two, since this can be done without any further assumptions.

Finally it is clear that the mean of the steady state distribution is given by

$$(-V'(0)) + \frac{\lambda d}{1+\lambda d}(-R'(0)) \tag{3.4}$$

where $-V'(0)$ is the mean of the dam part and $-R'(0)$ is the mean of the clearing part.

4 The Steady State LST and Moments of the Associated Clearing Process

In this section we determine the steady state LST $R(\alpha)$ of the clearing process as introduced in (3.1). For $1 \le i \le K$, denote

$$\beta_i(\alpha) = E_i \int_0^D e^{-\alpha \int_0^t a(X(s))ds} dt \tag{4.1}$$

where we recall that $D = \inf\{t|\ X(t) = 0\}$. Then,

$$\begin{aligned} \beta_i(\alpha) &= E_i \int_0^{\tau_1} e^{-\alpha a(i)t} dt + E_i \int_{\tau_1}^D e^{-\alpha \int_0^t a(X(s))ds} dt \\ &= \frac{1 - E_i e^{-\alpha a(i)\tau_1}}{\alpha a(i)} + \sum_{j=1}^K p_{ij} E_{ij} e^{-\alpha a(i)\tau_1} \beta_j(\alpha) \qquad (4.2) \\ &= m_i \tau_i^e(\alpha a(i)) + \sum_{j=1}^K p_{ij}\tau_{ij}(\alpha a(i))\beta_j(\alpha)\ . \end{aligned}$$

Denoting $\tilde{P} = \{p_{ij}|\ 1 \le i,j \le K\}$, since P is irreducible, it follows that $\tilde{P}^n \to 0$ as $n \to \infty$ and thus, letting $B(\alpha) = \{p_{ij}\tau_{ij}(\alpha a(i))|\ 1 \le i,j \le K\}$, it follows that $B^n(\alpha) \to 0$ as $n \to \infty$. Therefore, $I - B(\alpha)$ is invertible and so (4.2) has a unique solution (see the paragraph following (5.2)).

In order to compute $d = ED$ we first compute $d_i = E_i D$. For this we simply set $\alpha = 0$ in (4.2) and obtain

$$d_i = m_i + \sum_{j=1}^{K} p_{ij} d_j \ . \tag{4.3}$$

This, of course, may be obtained via a direct argument. Since $I - \tilde{P}$ is invertible, (4.3) has the unique solution $(I - \tilde{P})^{-1}\tilde{m}$ where $\tilde{m} = \{m_i | \ 1 \leq i \leq K\}$ (column vector).

Clearly,

$$E \int_0^D e^{-\alpha \int_0^t a(X(s))ds} dt = \sum_{j=1}^{K} p_{0j} \beta_j(\alpha) \tag{4.4}$$

and similarly

$$ED = \sum_{j=1}^{K} p_{0j} d_j \tag{4.5}$$

so that

$$R(\alpha) = \frac{\sum_{j=1}^{K} p_{0j} \beta_j(\alpha)}{\sum_{j=1}^{K} p_{0j} d_j} \ . \tag{4.6}$$

Differentiating (4.2) n times gives

$$\beta_i^{(n)}(0) = m_i (\tau_i^e)^{(n)}(0) a^n(i) + \sum_{j=1}^{K} p_{ij} \sum_{k=0}^{n} \binom{n}{k} \tau_{ij}^{(k)}(0) \beta_j^{(n-k)}(0) a^k(i) \ , \tag{4.7}$$

thus, if we denote $b_i^{(n)} = (-1)^n \beta_i^{(n)}(0)$, note that $(\tau_i^e)^{(n)}(0) = (-1)^n m_i^{(n+1)}/((n+1)m_i)$ and $(-1)^k(-1)^{n-k} = (-1)^n$, we obtain

$$b_i^{(n)} = \frac{m_i^{(n+1)} a^n(i)}{n+1} + \sum_{j=1}^{K} p_{ij} \sum_{k=0}^{n} \binom{n}{k} m_{ij}^{(k)} a^k(i) b_j^{(n-k)} \ . \tag{4.8}$$

Since this can be rewritten as

$$b_i^{(n)} = \frac{m_i^{(n+1)} a^n(i)}{n+1} + \sum_{j=1}^{K} p_{ij} \sum_{k=1}^{n} \binom{n}{k} m_{ij}^{(k)} a^k(i) b_j^{(n-k)} + \sum_{j=1}^{K} p_{ij} b_j^{(n)} \tag{4.9}$$

then, given $\{b_j^{(k)} | \ 1 \leq j \leq K, 0 \leq k \leq n-1\}$, we can uniquely determine $b_j^{(n)}$ for $1 \leq j \leq K$. The nth moment of our clearing process is thus given by

$$\frac{\sum_{j=1}^{K} p_{0j} b_j^{(n)}}{\sum_{j=1}^{K} p_{0j} d_j} \ . \tag{4.10}$$

We complete this section by noting that the equation for determining the mean becomes

$$b_i^{(1)} = \frac{a(i) m_i^{(2)}}{2} + \sum_{j=1}^{K} a(i) p_{ij} m_{ij} d_j + \sum_{j=1}^{K} p_{ij} b_j^{(1)} \ . \tag{4.11}$$

5 Jump Size LST for the Associated Dam Process

As was observed in Section 3 one component in the decomposition is the stationary distribution of a Dam process with release rate $r(\cdot)$ and compound Poisson jumps. It is easy to see that the jump rate for this process is λ. Therefore, in order to characterize the process it suffices to determine the LST of the jump sizes. The jump sizes are distributed like

$$\int_0^D a(X(t))dt \ , \tag{5.1}$$

so that, denoting $\gamma_i(\alpha) = E_i e^{-\alpha \int_0^D a(X(t))dt}$ we have that

$$\begin{aligned} \gamma_i(\alpha) &= \sum_{j=0}^{K} E_i e^{-\alpha \int_0^D a(X(t))dt} 1_{\{X_1=j\}} \\ &= p_{i0}\tau_{i0}(\alpha a(i)) + \sum_{j=1}^{K} p_{ij}\tau_{ij}(\alpha a(i))\gamma_j(\alpha) \ . \end{aligned} \tag{5.2}$$

Due to precisely the same reasons as in (4.2), (5.2) has a unique solution. In particular, both equations have a solution of the form $(I - B(\alpha))^{-1}v(\alpha)$ for an appropriate vector $v(\alpha)$. As in (4.9), the moments can be determined via

$$c_i^{(n)} = p_{i0}m_{i0}^{(n)}a^n(i) + \sum_{j=1}^{K} p_{ij} \sum_{k=1}^{n} \binom{n}{k} m_{ij}^{(k)} a^k(i) c_j^{(n-k)} + \sum_{j=1}^{K} p_{ij}c_j^{(n)} \tag{5.3}$$

where $c_i^{(n)} = (-1)^n \gamma_i^{(n)}(0)$, so that the LST and moments of the jump size distribution are given by (4.6) and (4.10), respectively, except that β_j are replaced by γ_j and $b_j^{(n)}$ are replaced by $c_j^{(n)}$. Let us denote the LST of the jump size distribution by $C(\alpha)$.

6 The Cases of M/G/1 and Shot Noise

As discussed in the introduction, there are two choices of release rates where the steady state distribution of the Dam process (and thus of the process discussed in this paper) is readily available. The first is when $r(w) = r$ for $w > 0$. For this case the dam process becomes the workload process in some M/G/1 queue with service rate r. This is a special case of a reflected Lévy process with no negative jumps and the steady state LST for this process is well known (e.g., see [5]) and is given by the following Pollaczek-Khinchin formula:

$$V(\alpha) = \frac{\alpha(r - \lambda c)}{\alpha r - \lambda(1 - C(\alpha))} = \frac{1-\rho}{1-\rho C^e(\alpha)} \tag{6.1}$$

where $c = -C'(0)$ is the mean jump size, $\rho = \lambda c/r$ and $C^e(\alpha) = \frac{1-C(\alpha)}{\alpha c}$. The stability condition for this process is well known to be $\rho < 1$. The mean

$$-V'(0) = \frac{\rho}{1-\rho}\frac{c^{(2)}}{2c} = \frac{\lambda c^{(2)}}{2(r-\lambda)} \tag{6.2}$$

where $c^{(2)} = C''(0)$ is the second moment of the jump size distribution.

For the case $r(w) = rw$ the dam process becomes a shot noise process (e.g., see page 212 of [6]) . For such a process it is well known that

$$V(\alpha) = e^{-\int_0^\alpha \frac{\lambda(1-C(x))}{rx}dx} \tag{6.3}$$

and the mean is given by $\lambda c/r$.

7 Busy Period for the M/G/1 Case

It turns out that for the case of constant release rate, the LST of the busy period may be identified. This busy period is defined as the time that elapses from the first instant when the content process is at zero and the SMP makes a transition from state 0 to some other state, until the first time the content is zero again. Let us denote this LST by $\theta(\alpha)$. This LST has already been determined in even greater generality in [2]. In particular, $\theta(\alpha)$ is the unique solution of

$$\theta(\alpha) = Ee^{-\alpha D - (\alpha + \lambda(1-\theta(\alpha)))A(D)/r} \tag{7.1}$$

on $[0, 1]$, where $A(D) = \int_0^D a(X(t))dt$. Upon differentiation, this immidiately implies that the expected value is given by

$$-\theta'(0) = \frac{d + EA(D)/r}{1-\rho} \tag{7.2}$$

where

$$EA(D) = \frac{\sum_{k=1}^K p_{0j}c_j^{(1)}}{\sum_{k=1}^K p_{0j}d_j} \ . \tag{7.3}$$

In order to compute higher moments it is evident that it is necessary to first compute mixed moments of the form $E_i D^m A(D)^n$. For this purpose we note that in a similar way as for (4.2) and (5.2),

$$\delta_i(\alpha,\beta) = E_i e^{-\alpha D - \beta A(D)} = p_{i0}\tau_{i0}(\alpha + a(i)\beta) + \sum_{j=1}^K p_{ij}\tau_{ij}(\alpha + a(i)\beta)\delta_j(\alpha,\beta) \ . \tag{7.4}$$

Differentiation immediately implies that

$$\begin{aligned} E_i D^m A(D)^n &= p_{i0} m_{i0}^{(m+n)} a^n(i) \\ &+ \sum_{j=1}^K \sum_{k=0}^m \sum_{\ell=0}^n \binom{m}{k}\binom{n}{\ell} p_{ij} m_{ij}^{(k+\ell)} a^\ell(i) E_j D^{m-k} A(D)^{n-\ell} \ . \end{aligned} \tag{7.5}$$

As before these equations may be solved uniquely.

8 Linear Release Rate with General Decrease Durations

In this section we no longer assume that the period where the process is decreasing is exponential. Rather, we still assume that there is a single state for which the slope is -1 (without loss of generality), but now $F_{0j} = F_0$ is general but still independent of j. We still assume without loss of generality that $p_{00} = 0$. For this case, it follows from [4] that the steady state distribution of the inventory content is given by

$$\frac{m_0}{m_0 + d} F_u(t) + \frac{d}{m_0 + d} F_d(t) \ . \tag{8.1}$$

Here, F_u is the distribution of $W_u = (W + A(D) - \tau_0^e)^+$ and F_d is the distribution of $W_d = W + A(D)^*$ where $A(D)$ has the jump distribution, $A(D)^*$ has the steady state distribution of the associated clearing process, τ_0^e has the forward recurrence time distribution with respect to F_0 (that is, with density $(1 - F_0(\cdot))/m_0$) and W has the steady state distribution of the random walk embedded at instants where the sample path has local minima. W is actually the steady state distribution of the waiting time in a GI/GI/1 queue with interarrival times with distribution F_0 and service times distributed like $A(D)$. In fact, it is well known that W_u (steady state workload in a GI/GI/1 queue) has the alternative representation $W_u = \xi(W + A(D)^e)$ where ξ is an independent indicator with $P[\xi = 1] = 1 - P[\xi = 0] = \rho = EA(D)/m_0$ and $A(D)^e$ has the stationary forward recurrence time distribution with respect to $A(D)$ (e.g., see page 296 of [3] or Theorem 3.5 on page 189 of [1]). Thus the LST of the full process is given by

$$\frac{m_0}{m_0 + d}(1 - \rho) + Ee^{-\alpha W} \left(\frac{m_0}{m_0 + d} \rho \frac{1 - Ee^{-\alpha A(D)}}{\alpha EA(D)} + \frac{d}{m_0 + d} Ee^{-\alpha A(D)^*} \right) \tag{8.2}$$

or alternatively, by

$$\frac{m_0 - c}{m_0 + d} + Ee^{-\alpha W} \frac{(1 - C(\alpha))/\alpha + dR(\alpha)}{m_0 + d} \tag{8.3}$$

and we note that everything in this equation with the exception of the LST of W was computed in earlier sections. Thus, whenever the steady state distribution of W can be determined, then so can the steady state distribution for our intermittent process. Clearly when F_0 is the exponential distribution (as was the case before) the steady state distribution of W is well known (and discussed in Section 6).

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