ESTIMATES FOR THE DISTRIBUTION OF SUMS AND MAXIMA OF SUMS OF RANDOM VARIABLES WITHOUT THE CRAMER CONDITION

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§1. Introduction

1.1. Let $X_1, X_2, \ldots$ be identically distributed independent random variables with distribution function $F(t) = \mathbf{P}(X_1 < t)$, and let

$$S_n = \sum_{j=1}^{n} X_j, \quad \overline{S}_n(a) = \max_{k \leq n}(S_k - ak), \quad \overline{S}_n = \overline{S}_n(0),$$

$$B_j(v) = \{X_j \leq y + vg(j)\}, \quad B(v) = \bigcap_{j=1}^{n} B_j(v), \quad v \geq 0,$$

(1.1)

where the function g will be chosen in dependence on F.

The main purpose of this article consists in evaluating the probabilities

$$\mathbf{P}(S_n > x), \quad \mathbf{P}(\overline{S}_n(a) > x), \quad \text{and} \quad \mathbf{P}(\overline{S}_n(a) > x; B(v)) \tag{1.2}$$

as $x \to \infty$. The probabilities $\mathbf{P}(\overline{S}_n(a) > x; B(v))$ play an important role in finding the exact asymptotics of $\mathbf{P}(\overline{S}_n(a) > x)$ (see, for instance, [1–4]).

Concerning the distribution of X_j, we assume that the "tails"

$$F(-t) = \mathbf{P}(X_j < -t), \quad 1 - F(t) = \mathbf{P}(X_j \geq t), \quad t > 0,$$

are majorized or minorized by a regularly varying or semiexponential function.

Here a function $V(t)$, $t > 0$, is called *regularly varying* or *regular* if

$$V(t) = t^{-\beta} L(t), \quad \beta > 0, \tag{1.3}$$

where $L(t)$ is a slowly varying function as $t \to \infty$.

A function $V(t)$ is called *semiexponential* if

$$V(t) = e^{-t^{\alpha} L(t)}, \quad \alpha \in (0, 1), \tag{1.4}$$

with $L(t)$ having the same meaning.

Majorants (or minorants) for the positive tails $1 - F(t)$ are denoted by $V(t)$ and those for the negative tails $F(-t)$, by $W(t)$. Moreover, $W(t)$ is assumed to belong to the class (1.3):

$$W(t) = t^{-\alpha} L_W(t). \tag{1.5}$$

Use of the same symbol α here and in (1.4) does not lead to misunderstanding, since the corresponding considerations belong to different sections.

Henceforth we use the following conditions:

$[M^+]$ $1 - F(t) \le V(t), \ t > 0;$
$[M_+]$ $1 - F(t) \ge V(t), \ t > 0;$
$[M^-]$ $F(-t) \le W(t), \ t > 0;$
$[M_-]$ $F(-t) \ge W(t), \ t > 0,$

where $V(t)$ has the form (1.3) or (1.4), and $W(t)$ has the form (1.5).

Studying the exact asymptotics of $\mathbf{P}(S_n > x)$ and $\mathbf{P}(\overline{S}_n(a) > x)$, we will also use the tail regularity condition:

$[R]$ $1 - F(t) = V(t), \quad t > 0,$

that is the intersection of conditions $[M^+]$ and $[M_+]$.

Since we study large deviation probabilities on the *positive* half-axis $t > 0$, the main parameter by which we classify different cases is the parameter β of the majorants $V(t)$ in (1.3) or the presence of majorants like (1.4).

In § 2–§ 5 we obtain upper estimates for the probabilities under study in the following four cases:

$$
\begin{aligned}
&1. \quad \beta < 1 \text{ or } \alpha < 1; \\
&2. \quad \beta \in (1, 2), \quad \mathbf{E}|X_j| < \infty; \\
&3. \quad \beta > 2, \quad \mathbf{E}X_j^2 < \infty; \\
&4. \quad \text{the dominant } V(t) \text{ is semiexponential and } \mathbf{E}X_j^2 < \infty.
\end{aligned}
\tag{1.6}
$$

In § 6 we obtain lower estimates and derive some corollaries for the exact asymptotics of $\mathbf{P}(S_n > x)$ and $\mathbf{P}(\overline{S}_n(a) > x)$.

In § 7 we find out conditions for uniform relative convergence to a stable law and establish the law of the iterated logarithm for the sums S_n in the case of $\mathbf{E}X_j^2 = \infty$.

Inequalities for sums of random variables, close to those of § 3 and § 4, were obtained in [5] and [6] ($\mathbf{E}X_j^2 < \infty$ in [5]). However, in some sense the inequalities of [5] and [6] are in a "less final" form and require extra efforts for deriving simple explicit estimates. Some of these inequalities were extended in [7–10] to the maxima of successive sums in the case of $\mathbf{E}X_j^2 < \infty$. As D. A. Korshunov communicated to the author, the asymptotics of $\mathbf{P}(\overline{S}_n(a) > x)$ for $a > 0$ and under rather general assumptions was obtained in [11]. Thus, some results of the present article are known (this mainly concerns corollaries to the main assertions). We still exhibit them to make exposition more complete and systematic. More precise bibliographical comments are made in due course.

The results of the present article have been used and will be used for finding approximations for $\mathbf{P}(S_n > x)$ and $\mathbf{P}(\overline{S}_n(a) > x)$, asymptotic expansions inclusively, in much the same manner as in [1–4]. See § 6 for bibliographical comments on the exact asymptotics found in the article.

Now, we say a few words about the methods for obtaining some of the main inequalities. The upper estimates of § 2–§ 5 differ essentially, still having much in common. In particular, they rest upon inequalities of the same type for truncated random variables. The scheme of the proof of these inequalities is the same (see also [5] and [7]) and proceeds as follows:

Consider the random variables $X_j^{(y)}$ "truncated" at level $y > 0$ and having the distribution function

$$
\mathbf{P}\big(X_j^{(y)} < t\big) = \frac{F(t)}{F(y)}, \quad t \le y.
$$

Denote by $S_n^{(y)}$ and $\overline{S}_n^{(y)}$ the sums and maxima of these sums corresponding to $X_j^{(y)}$. Then it is obvious that the probability

$$
P \equiv \mathbf{P}(\overline{S}_n > x, B(0))
\tag{1.7}
$$

in (1.2) for $v = 0$ equals

$$
P = F^n(y)\mathbf{P}(\overline{S}_n^{(y)} > x).
$$

By [8, Chapter 4, Theorem 16], for every $\mu \ge 0$ we have

$$
\mathbf{P}(\overline{S}_n^{(y)} > x) \le e^{-\mu x}[\max(1, \mathbf{E}e^{\mu X_1^{(y)}})]^n.
$$

Since

$$\mathbf{E}e^{\mu X_1^{(y)}} = F^{-1}(y)\int\limits_{\infty}^{y} e^{\mu t}dF(t),$$

we come to the following inequality that underlies many of the forthcoming considerations:

$$P \le e^{-\mu x}[\max(F(y), R(\mu, y)]^n \le e^{-\mu x}\max(1, R^n(\mu, y)), \tag{1.8}$$

where

$$R(\mu, y) = \int\limits_{-\infty}^{y} e^{\mu t}dF(t).$$

The problem is thus reduced to estimating $R(\mu, y)$. Estimates differ in each of the cases (1.6).

Throughout the sequel, the letter c with or without indices stands for constants which are not the same if used in different places. The relation $a_n \sim b_n$ as $n \to \infty$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

§2. Upper Estimates in the Cases of $\beta < 1$ or $\alpha < 1$

In the case when $\beta < 1$ or $\alpha < 1$ (on assuming $[M^+]$ or $[M_-]$ respectively), estimates for the probabilities (1.2) may essentially differ depending on the interrelation between the negative and positive tails. Therefore, we distinguish the two possibilities:

1. $\beta < 1$ and the tail $F(-t)$, $t > 0$, is arbitrary.
2. $\alpha < 1$ and the tail $F(-t)$ is essentially "heavier" than the tail $1 - F(t)$, $t > 0$.

Estimates in the first case are factually estimates for the sums S_n when $X_n \ge 0$ ($F(0) = 0$).

2.1. The case in which the negative tail is arbitrary and $\beta < 1$. As mentioned, this and many subsequent sections rest on estimates for the probability P in (1.7).

In §2–§5 we choose the truncation level y in (1.8) such that $y \le x$ and the ratio

$$r = \frac{x}{y}$$

is bounded, so that the growth of x and y as $y \to \infty$ is the same up to a bounded factor.

Theorem 2.1. *Assume that condition $[M^+]$ is satisfied and $\beta < 1$. Then there exists a constant c such that the following inequality holds for the probability P in (1.7):*

$$P \le c[nV(y)]^r, \quad r = \frac{x}{y}. \tag{2.1}$$

The constant c in (2.1) can be replaced by $\left(\frac{e}{r}\right)^r + \varepsilon(nV(y))$, where $\varepsilon(\cdot)$ is a bounded function such that $\varepsilon(v) \downarrow 0$ as $v \downarrow 0$.

Note that (2.1) makes sense only for bounded $nV(y)$, say, $nV(y) < 1$, and we assume without loss of generality that this inequality is always satisfied. If $nV(y) \to 0$ ($nV(x) \to 0$) then we deal with the *domain of large deviations* of $\overline{S}_n$. This follows from the fact that, by Corollary 2.1 below, $\mathbf{P}(\overline{S}_n > x) \to 0$ whenever $nV(x) \to 0$. The domain of large deviations can also be characterized immediately in terms of inequalities for x. To this end, we introduce the inverse function $V^{(-1)}$ of V and put $N(n) = V^{(-1)}(1/n)$, so that $N = N(n)$ is a solution to the equation $nV(x) = 1$. Now, we put $x = sN(n)$ and observe that the increase of s amounts to the decrease of $nV(x)$. Indeed, put for brevity

$$\Pi = \Pi(x) = nV(x).$$

Then for a fixed s

$$\Pi(sN(n)) \sim s^{-\beta}\Pi(N(n)) = s^{-\beta}$$

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and the domain $\Pi < \varepsilon$ is equivalent to the domain $s \gtrsim \varepsilon^{-1/\beta}$. By the properties of slowly varying functions, as s increases the value of $\Pi(x)$ for every fixed $\delta > 0$ lies within the bounds

$$s^{-\beta-\delta} \leq \Pi(x) \leq s^{-\beta+\delta}.$$

Similar "reverse" inequalities are valid for s.

Thus, in our case the order of large deviations is determined by the parameter $s = \frac{x}{N(n)}$ or the value of $\Pi = nV(x)$.

Corollary 2.1. *Assume that condition* $[M^+]$ *is satisfied and* $\beta < 1$. *Then there exists a function* $\varphi(t) \downarrow 0$, $t \downarrow 0$, *such that*

$$\sup_{x:\Pi\leq\varepsilon} \frac{\mathbf{P}(\overline{S}_n > x)}{\Pi} \leq 1 + \varphi(\varepsilon), \quad \Pi = nV(x), \tag{2.2}$$

or equivalently

$$\sup_{x:s\geq t} \frac{\mathbf{P}(\overline{S}_n > x)}{\Pi} \leq 1 + \varphi\left(\frac{1}{t}\right).$$

Proof. Put

$$y = x\left(1 + \frac{1}{\sqrt{|\ln\varepsilon|}}\right)^{-1}.$$

Then for $\Pi \leq \varepsilon$ we have

$$r = 1 + \frac{1}{\sqrt{|\ln\varepsilon|}}, \quad \Pi^{r-1} \leq \exp\left\{\frac{\ln\varepsilon}{\sqrt{|\ln\varepsilon|}}\right\} = \exp\{-\sqrt{|\ln\varepsilon|}\} \to 0$$

as $\varepsilon \to 0$. Moreover,

$$\frac{L(y)}{L(x)} \to 1, \quad x > (nV(x))^{-1/2\beta} \text{ as } x \to \infty.$$

Hence, there is a function $\varphi_1(t) \downarrow 0$, $t \downarrow 0$, such that for $\Pi \leq \varepsilon$

$$\frac{L(y)}{L(x)} \leq 1 + \varphi_1(\varepsilon), \quad \frac{V(y)}{V(x)} \leq \left(1 + \frac{1}{\sqrt{|\ln\varepsilon|}}\right)^\beta (1 + \varphi_1(\varepsilon)) \equiv 1 + \varphi_2(\varepsilon).$$

By Theorem 2.1, we then have

$$\mathbf{P}(\overline{S}_n > x) \leq \mathbf{P}(\overline{B}(0)) + P \leq nV(y) + c[nV(y)]^r$$

$$\leq \Pi[1 + \varphi_2 + c\Pi^{r-1}(1 + \varphi_2)^r] \leq \Pi[1 + \varphi_2 + c_1 e^{-\sqrt{|\ln\varepsilon|}}],$$

completing the proof of the corollary.

Proof of Theorem 2.1. We have to evaluate

$$P = \mathbf{P}(\overline{S}_n > x, B), \quad B = B(0) = \bigcap_{j=1}^n B_j, \quad B_j = \{X_j \leq y\}$$

(cf. (1.7)). As mentioned, our arguments base on (1.8). To use this inequality, we have to estimate

$$R(\mu, y) = \int_{-\infty}^y e^{\mu t}dF(t) = I_1 + I_2 + I_3, \tag{2.3}$$

where for $\mu \geq 0$ and $M = \frac{2\beta}{\mu} < y$

$$I_1 = \int_{-\infty}^{0} e^{\mu t}\, dF(t) \leq F(0),$$

(2.4)

$$I_2 = \int_{0}^{M} e^{\mu t}\, dF(t) \leq 1 - F(0) - e^{2\beta}F(M) + \mu \int_{0}^{M} V(t)e^{\mu t}\, dt.$$

For $\beta < 1$, the last integral increases unboundedly as $\mu \to 0$ but does not exceed

$$\frac{e^{2\beta} M V(M)}{\beta + 1}\left(1 + o(1)\right) \leq \frac{c}{\mu} V\left(\frac{1}{\mu}\right);$$

so

$$I_2 \leq 1 - F(0) + cV\left(\frac{1}{\mu}\right).$$

(2.5)

(Observe that for $\beta > 1$ we have $I_2 \leq 1 - F(0) + c\mu$ (see below), so that (2.5) fails for $\beta > 1$.)

Now, we evaluate

$$I_3 = \int_{M}^{y} e^{\mu t}\, dF(t) \leq V(M)e^{2\beta} + \mu \int_{M}^{y} V(t)e^{\mu t}\, dt \equiv V(M)e^{2\beta} + \mu I_3^0.$$

(2.6)

To this end, we first consider the ratio of the values of the integrand of I_3^0 at the points $t \in [M, y]$ and $t + 1/\mu$:

$$\frac{V(t)e^{\mu t}}{V(t + 1/\mu)e^{\mu t + 1}} \sim e^{-1}\left(1 + \frac{1}{\mu t}\right)^{\beta} \leq e^{-1}\left(1 + \frac{1}{2\beta}\right)^{\beta} < e^{-1/2} < 1.$$

This means that the integrals $I_{3,k}$ defined like I_3^0 but calculated over the subintervals $\left(y - \frac{k}{\mu}, y - \frac{k+1}{\mu}\right)$, $k = 0, 1, \ldots$, of $[M, y]$ are dominated by a geometric progression with denominator e^{-1}. Therefore, the main contribution to I_3^0 is made by the first integrals $I_{3,0}, I_{3,1}, \ldots$.

Henceforth we choose μ so that $\lambda = \mu y \to \infty$ ($y \gg 1/\mu$). Substituting $(t - y)\mu = u$, we obtain

$$\mu I_3^0 = e^{\mu y} \int_{0}^{(y-M)\mu} V\left(y - \frac{u}{\mu}\right) e^{-u}\, du,$$

where $V(y - u/\mu) \sim V(y)$ for $u/\mu = o(y)$ (or $u = o(\lambda)$) and so by the Lebesgue dominated convergence theorem we have

$$\mu I_3^0 \sim e^{\mu y} V(y).$$

Furthermore, it is easy to indicate a function $\varphi(\lambda) \downarrow 0$, $\lambda \uparrow \infty$, such that

$$\mu I_3^0 \leq e^{\mu y} V(y)(1 + \varphi(\lambda)).$$

(2.7)

Summing up (2.4)–(2.7), we obtain

$$R(\mu, y) \leq 1 + cV\left(\frac{1}{\mu}\right) + e^{\mu y} V(y)(1 + \varphi(\lambda)).$$

(2.8)

Hence,
$$R^n(\mu, y) \leq \exp\left\{ ncV\left(\frac{1}{\mu}\right) + nV(y)e^\lambda (1 + \varphi(\lambda)) \right\}. \tag{2.9}$$

Now, choose μ (or λ) as "almost minimizing"
$$-\mu x + \Pi(y)e^\lambda \quad (\Pi(y) = nV(y)).$$

To this end, put
$$\lambda = \ln \frac{r}{\Pi(y)} \equiv \ln T, \tag{2.10}$$

where for brevity we introduce the notation
$$T = \frac{r}{\Pi(y)} = \frac{r}{nV(y)}.$$

Observe that, with this choice of λ (or $\mu = \ln T/y$) and for $nV(y) \to 0$, we have $T \to \infty$, $\lambda = \mu y \to \infty$; so the above-made assumption $y \gg 1/\mu$ is satisfied. From (1.8), (2.9), and (2.10) we deduce that
$$\ln P \leq -x\mu + cnV\left(\frac{1}{\mu}\right) + \Pi(y)e^\lambda (1 + \varphi(\lambda)), \tag{2.11}$$

where $\Pi(y)e^\lambda = r$, and for every $\delta > 0$ and sufficiently large y
$$nV\left(\frac{1}{\mu}\right) \leq c_1 nV\left(\frac{y}{|\ln nV(y)|}\right) \leq c_1 nV(y) |\ln nV(y)|^{\beta+\delta} \leq c_2 \frac{(\ln T)^{\beta+\delta}}{T}.$$

Therefore, (2.11) implies that
$$\ln P \leq -r \ln T + r + \varphi_1(T),$$

where $\varphi_1(T) \downarrow 0$ as $T \to \infty$, and without loss of generality we may assume that $\ln T \geq 1$. This finishes the proof of the theorem.

REMARK 2.1. If the function $L(t)$ is differentiable, $L'(t) = o\left(\frac{L(t)}{t}\right)$, and $1 - F(t) = V(t)$, then the estimate for I_3 in (2.6) can be refined:
$$I_3 \leq \gamma \frac{V(y)}{\mu y}$$

for every $\gamma > \beta$ and all y large enough. This allows us to strengthen Theorem 2.1 and obtain the estimate
$$P \leq c \left[\frac{nV(y)}{|\ln nV(y)|}\right]^r.$$

2.2. The case when the negative tail admits a minorant with exponent $\alpha < 1$ and is much "heavier" than the positive tail. If in this section we had again used inequalities like
$$\mathbf{P}(\overline{S}_n > x) \leq \mathbf{P}(\overline{B}) + P$$

for the sets $B = B(0)$ (cf. the proof of Corollary 2.1) then we would fail in obtaining the desired estimates for $\mathbf{P}(\overline{S}_n > x)$ with the right-hand side independent of n. For this reason, we turn to using these inequalities for the sets $B(v)$ with $v > 0$ and for $g(j) = j^{1/\gamma}$, $\gamma \in (\alpha, \beta)$ (see (1.1)).

To simplify calculations, in this section we additionally assume that in (1.3) and (1.5)
$$L(t) = L + o(1), \quad L_W(t) = L_W + o(1) \tag{2.12}$$

as $t \to \infty$.

Theorem 2.2. *Assume that conditions* $[M^+]$ *and* $[M_-]$ *are satisfied and that* $V(t)$ *and* $W(t)$ *are defined by* (1.3) *and* (1.5) *with* $\alpha < \min(1, \beta)$ *and satisfy* (2.12). *Then for a suitable* v, $y \to \infty$, *and for all* n

$$P(v) \equiv \mathbf{P}(\overline{S}_n > x, B(v)) \leq cy^{(\gamma-\beta)r}, \quad r = \frac{x}{y}, \tag{2.13}$$

where the fixed $\gamma \in (\alpha, \beta)$ *can be chosen arbitrarily close to* α.

Moreover,

$$\mathbf{P}(\overline{S}_n > x) \leq cx^{-\beta} \min(n, x^\gamma) \tag{2.14}$$

for every fixed $\gamma > \alpha$.

Corollary 2.2. *If conditions* $[M^+]$ *and* $[M_-]$ *are satisfied,* $\alpha < \min(1, \beta)$, *and the functions* $V(t)$ *and* $W(t)$ *have the form* (1.3) *and* (1.5) *respectively, then* $\overline{S}_\infty$ *is a proper random variable.*

The claim of the corollary is obvious from (2.14).

We first obtain estimates for the probability $P = P(0)$ in (1.7).

Lemma 2.1. *Suppose that the conditions of Theorem 2.2 are satisfied. Then for all* n

$$P \leq c(1 - V(y))^n y^{-r(\beta-\alpha)} (\ln y)^{-r\alpha} \leq cy^{-r(\beta-\alpha)} (\ln y)^{-r\alpha}, \quad r = \frac{x}{y}. \tag{2.15}$$

PROOF. In view of inequalities (1.8), the problem is again reduced to evaluating $R(\mu, y)$ in (2.3) for the same splitting of this integral into the subintegrals I_1, I_2, and I_3. Here for $\mu \to 0$ and $\alpha < 1$

$$I_1 = F(0) - \mu \int_{-\infty}^0 F(t)e^{\mu t}\, dt \leq F(0) - \int_0^\infty e^{-u} W\left(\frac{u}{\mu}\right) du;$$

$$\int_0^\infty e^{-u} W\left(\frac{u}{\mu}\right) du \sim W\left(\frac{1}{\mu}\right) \int_0^\infty e^{-u} u^{-\alpha}\, du = \Gamma(1-\alpha) W\left(\frac{1}{\mu}\right),$$

so that

$$I_1 \leq F(0) - \Gamma(1-\alpha) W\left(\frac{1}{\mu}\right)(1 + o(1)), \tag{2.16}$$

where $\Gamma(\cdot)$ is the Γ-function. Estimates for the integrals I_2 and I_3 for $\beta < 1$ remain the same as in (2.5) and (2.7). Therefore,

$$R(\mu, y) \leq 1 - \Gamma(1-\alpha) W\left(\frac{1}{\mu}\right)(1 + o(1)) + cV\left(\frac{1}{\mu}\right) + V(y)e^{\mu y}(1 + \varphi(\mu y)), \tag{2.17}$$

where $V\left(\frac{1}{\mu}\right) = o\left(W\left(\frac{1}{\mu}\right)\right)$.

If $\beta > 1$ then instead of the summand with $V\left(\frac{1}{\mu}\right)$ in (2.17) we have $c\mu$ (see a remark on (2.5)). Since $\mu = o\left(W\left(\frac{1}{\mu}\right)\right)$, all subsequent arguments relying on (2.17) are preserved. For $\beta = 1$, instead of the summand with $V\left(\frac{1}{\mu}\right)$ in (2.17) we have $c\mu \ln \frac{1}{\mu}$, again with the obvious validity of the relation $\mu \ln \frac{1}{\mu} = o\left(W\left(\frac{1}{\mu}\right)\right)$ and preservation of the subsequent considerations.

Now, we choose μ so that

$$\Gamma(1-\alpha) W\left(\frac{1}{\mu}\right) = V(y)e^{\mu y}. \tag{2.18}$$

To simplify search for a solution μ, we use conditions (2.12). Then for $y \gg 1/\mu$ equation (2.18) takes the form

$$y^\beta \mu^\alpha = ce^{\mu y}(1 + o(1)). \tag{2.19}$$

Put $\mu y = \lambda$. Then (2.19) can be written down as

$$\alpha \ln \lambda + (\beta - \alpha) \ln y = \lambda + c_1 + o(1).$$

Whence we see that we "almost satisfy" (2.19) by setting

$$\lambda = (\beta - \alpha) \ln y + \alpha \ln \ln y + c_2.$$

With this choice of λ (or μ) we have $R(\mu, y) \le 1 + o\left(W\left(\frac{1}{\mu}\right)\right)$. It is easy to see that c_2 can always be chosen so that

$$R(\mu, y) \le 1 - V\left(\frac{1}{\mu}\right) \le 1 - V(y).$$

Therefore, by (1.8)

$$P \le (1 - V(y))^n e^{-r[(\beta-\alpha)\ln y + \alpha \ln \ln y + c_2]} = c(1 - V(y))^n y^{-r(\beta-\alpha)} (\ln y)^{-r\alpha},$$

which completes the proof of Lemma 2.1.

PROOF OF THEOREM 2.2.   We first evaluate

$$P(v) \equiv \mathbf{P}(\overline{S}_n > x, B(v)). \tag{2.20}$$

We put

$$m_1 = g^{(-1)}(x) = x^\gamma, \quad m_k = x^\gamma \rho^{k-1}, \quad \rho > 1, \quad M_0 = 0,$$

$$M_k = \sum_{j=1}^{k} m_j \equiv x^\gamma \rho_k, \quad \rho_k = \frac{\rho^k - 1}{\rho - 1} \ge \rho^{k-1}, \quad k = 1, 2, \ldots; \tag{2.21}$$

$$x_k = x + g(M_{k-1}) = x\left(1 + \rho_{k-1}^{1/\gamma}\right), \quad y_k = y + vg(M_k) = y\left(1 + vr\rho_k^{1/\gamma}\right).$$

Then for $n > M_1$ we have

$$P(v) \le \mathbf{P}\left(\overline{S}_{m_1} > x_1; \bigcap_{j=1}^{m_1}\{X_j \le y_1\}\right) + \mathbf{P}\left(S_{M_1} > -M_1^{1/\gamma}; \bigcap_{j=1}^{M_1}\{X_j \le y_1\}\right)$$

$$+ \mathbf{P}\left(\overline{S}_n > x, \ \overline{S}_{m_1} \le x, \ S_{M_1} \le -M_1^{1/\gamma}; \ B(v)\right). \tag{2.22}$$

For the same reasons, the last probability for $n > M_2$ does not exceed

$$\mathbf{P}\left(\overline{S}_{m_2} > x_2; \bigcap_{j=1}^{m_2}\{X_j \le y_2\}\right) + \mathbf{P}\left(S_{M_2} > -M_2^{1/\gamma}; \bigcap_{j=1}^{M_2}\{X_j \le y_2\}\right)$$

$$+ \mathbf{P}\left(\overline{S}_n > x, \ \overline{S}_{M_2} \le x, \ S_{M_2} \le -M_2^{1/\gamma}; \ B(v)\right),$$

etc. To evaluate $P(v)$, we thus have to estimate

$$\sum_{k=1}^{\nu} \mathbf{P}\left(\overline{S}_{m_k} > x_k; \bigcap_{j=1}^{m_k}\{X_j \le y_k\}\right) \tag{2.23}$$

and

$$\sum_{k=1}^{\nu} \mathbf{P}\left(S_{M_k} > -M_k^{1/\gamma}; \bigcap_{j=1}^{M_k}\{X_j \le y_k\}\right) \tag{2.24}$$

for $\nu = \min\{k : M_k \geq n\}$. In view of Lemma 2.1, for y large enough the first sum does not exceed

$$\sum_k y_k^{-r_k(\beta - \alpha)}, \tag{2.25}$$

where

$$r_k = \frac{x_k}{y_k} = \frac{x\left(1 + \rho_{k-1}^{1/\gamma}\right)}{y\left(1 + v\rho_k^{1/\gamma}\right)} \geq r - \varepsilon, \quad \varepsilon > 0,$$

for all k and a suitable $v = v(r, \rho, \varepsilon)$. Therefore, the sums in (2.23) and (2.25) do not exceed

$$\sum_k y_k^{-(r-\varepsilon)(\beta - \alpha)}. \tag{2.26}$$

However, ρ_k increases faster than a geometric progression (see (2.21)), and the same can be said about the sequence $1 + rv\rho_k^{1/\gamma}$ (see the definition of y_k). Therefore, the sums in (2.23), (2.25), and (2.26) do not exceed $cy^{-(r-\varepsilon)(\beta - \alpha)}$.

Now, we estimate the sum in (2.24). For brevity, put $M_k^{1/\gamma} = z_k$. Denote by η the number of the events $\{X_j \leq 0\}$ in M_k trials. Then

$$\mathbf{P}\left(S_{M_k} > -z_k; \bigcap_{j=1}^{M_k}\{X_j \leq y_k\}\right) = \sum_{i=1}^{M_k} \mathbf{P}(\eta = i)\mathbf{P}\left(S_{M_k} > -z_k; \bigcap_{j=1}^{M_k}\{X_j \leq y_k\}/\eta = i\right)$$

$$= \sum_{i=[M_k p_1]}^{[M_k p_2]} + O\left(e^{-\delta M_k}\right), \tag{2.27}$$

where $p_1 = F(0) - \varphi$, $p_2 = F(0) + \varphi$, $\varphi > 0$, and $\delta = \delta(\varphi) > 0$. From now on, let $\eta = i \in [p_1 M_k, p_2 M_k]$ be fixed. Then

$$S_{M_k} = S_i^- + S_{M_k - i}^+,$$

where S_i^- is the sum of the independent random variables X_j^- with the distribution function $\frac{F(t)}{F(0)}$, $t \leq 0$, and $S_{M_k - i}^+$ is the similar sum of the random variables X_j^+ with the distribution function $\frac{F(t) - F(0)}{1 - F(0)}$, $t \geq 0$. Therefore, the ith summand in (2.27) does not exceed

$$\mathbf{P}(S_i^- > -2z_k) + \mathbf{P}\left(S_{M_k - i}^+ > z_k; \bigcap_{j=1}^{M_k - i}\{X_j^+ \leq y_k\}\right). \tag{2.28}$$

By Theorem 2.1, the second summand does not exceed $[M_k V(y_k)]^{r_k^*}$, where (see (2.21))

$$r_k^* = \frac{M_k^{1/\gamma}}{y_k} = \frac{x\rho_k^{1/\gamma}}{y\left(1 + vr\rho_k^{1/\gamma}\right)} \geq \frac{r}{1 + vr} > r - \varepsilon$$

for $v < \frac{\varepsilon}{r}$. Hence, the second summand in (2.28) does not exceed

$$[M_k V(y_k)]^{r-\varepsilon} \leq c\left[x^\gamma \rho_k\left(y\rho_k^{1/\gamma}\right)^{-\beta}\right]^{r-\varepsilon} = c_1\left[y^{\gamma - \beta}\rho^{1 - \beta/\gamma}\right]^{r-\varepsilon}$$

uniformly in i. However, $\rho_k \geq \rho^{k-1}$, $\rho > 1$, and $\gamma < \beta$. Therefore, the sum of these k summands (see (2.27) and (2.24)) does not exceed $c_2 y^{(\gamma - \beta)(r-\varepsilon)}$.

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Now, we evaluate the first summand in (2.28), setting for brevity $M_k = n$ and $i = np$, $p \in [p_1, p_2]$. For the event under the probability sign we have the inclusion

$$\left\{ S_{np}^- > -2n^{1/\gamma} \right\} \subset \bigcap_{j=1}^{np} \left\{ X_j > -2n^{1/\gamma} \right\};$$

hence,

$$\mathbf{P}(S_{np}^- > -2n^{1/\gamma}) \le (1 - W(2n^{1/\gamma}))^{np} \le (1 - cn^{-\frac{\alpha}{\gamma}})^{np} < e^{-cp_1 n^{1-\frac{\alpha}{\gamma}}} \qquad (2.29)$$

uniformly in $i \in [np_1, np_2]$. Again using the fact that the numbers $n = M_k$ grow as a geometric progression, $M_1 = x^\gamma$, and that $1 - \alpha/\gamma > 0$, we conclude that the sum in (2.24) does not exceed $e^{-cy^{\gamma-\alpha}}$.

Now, observe that, for ε small enough, $(\gamma - \beta)(r - \varepsilon)$ can be written down as $(\gamma' - \beta)r$, where $\gamma' < \alpha$, as well as γ, can be chosen arbitrarily close to α.

Combining the above estimates, we arrive at (2.13).

To derive the second assertion of Theorem 2.2, we have to evaluate

$$\mathbf{P}(\overline{B}(v)) \le \sum_{j=1}^{n} \mathbf{P}(X_j > y + vj^{1/\gamma}) = \sum_{j=1}^{n} V(y + vj^{1/\gamma}) \le \int_0^n V(y + vt^{1/\gamma}) \, dt. \qquad (2.30)$$

If $n \le y^\gamma$ then the integral does not exceed $cny^{-\beta}$. If $n \ge y^\gamma$ then we should write the integral in (2.30) as the sum $\int_0^{y^\gamma} + \int_{y^\gamma}^n$, where the first integral has been already estimated and is at most $cy^{\gamma-\beta}$. The second integral does not exceed

$$c \int_{y^\gamma}^\infty t^{-\beta/\gamma} \, dt = c_1 y^{\gamma-\beta}.$$

Thus,

$$\mathbf{P}(\overline{B}(v)) \le cy^{-\beta} \min(y^\gamma, n). \qquad (2.31)$$

On putting $r = \frac{\beta}{\beta - \gamma} + \varepsilon$ in (2.13), we obtain the same estimate for $\mathbf{P}(\overline{S}_n > x)$ as in (2.31):

$$\mathbf{P}(\overline{S}_n > x) \le cx^{-\beta} \min(x^\gamma, n).$$

The proof of the theorem is over.

REMARK 2.2. It is easy to see that, by slightly complicating calculations, we can make estimates (2.13) and (2.14) more precise. If we put $g(j) = j^{1/\alpha} \ln^{-b} j$ and $m_1 = x^\alpha \ln^b x$, then the parameter γ in (2.13) and (2.14) can be replaced with α, but the right-hand sides of these inequalities then acquire a logarithmic factor. Indeed, the only place in the proof of Theorem 2.2 which is sensible of the approximation of the parameter γ to α is the estimate for the first summand in (2.28). However, this estimate is exponential (see (2.29) and below). Therefore, we can achieve a power-like character of decay of the estimate by a suitable choice of b in the definition of the function g. The estimates for $\mathbf{P}(\overline{B}(v))$ change accordingly. Therefore, in fact we have the estimate

$$\mathbf{P}(\overline{S}_n > x) \le cx^{-\beta} \min(n, x^\alpha \ln^{b_1} x) \qquad (2.32)$$

for a suitable $b_1 > 0$.

We can eliminate conditions (2.12) but again for the price of complicating calculations; moreover, the right-hand sides in (2.13), (2.14), and (2.32) change slightly. Since inequalities (2.13) suffice for the further derivation of the exact asymptotics of $\mathbf{P}(\overline{S}_n > x)$, we refrain from implementing the mentioned complications in the proof of the theorem.

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§3. The Case of $\beta \in (1,2)$ and $\mathbf{E}|X_j| < \infty$

Here we assume without loss of generality that $\mathbf{E}X_j = 0$. We introduce into consideration the "weakest" among the conditions $[M^-]$ in the case of the existence of $\mathbf{E}X_j$. It corresponds to $\alpha = 1$:

$[M_1^-]$ $W(t) = \frac{c_1}{t(\ln t)^2}$.

3.1. Estimates for the distribution of $\overline{S}_n$.

Theorem 3.1. *Assume that conditions $[M^\pm]$ are satisfied with $\beta \in (1,2)$, $\alpha > 1$, and that*

$$W(t) \le cV(t). \tag{3.1}$$

Then inequality (2.1) of Theorem 2.1 and inequality (2.2) of Corollary 2.1 remain true.

If (3.1) fails then (2.1) remains valid for all n and y such that

$$nW\left(\frac{y}{|\ln nV(y)|}\right) < 1. \tag{3.2}$$

For validity of (3.2), either of the following two conditions is sufficient:

$$nW(y) < 1, \quad |\ln nV(y)| < [nW(y)]^{-\frac{1}{\alpha+\varepsilon}} \tag{3.3}$$

for some $\varepsilon > 0$ and $c_2, c_3 < \infty$, or

$$nW\left(\frac{y}{\ln y}\right) < 1. \tag{3.4}$$

If conditions $[M^+]$ and $[M_1^-]$ are satisfied then (2.1) and (2.2) are valid for $n < cx$.

We can replace 1 on the right-hand sides of (3.2)–(3.4) with an arbitrary fixed constant c_1. However, for the values of y such that, say, $nW(y) > 1$, $nV(y) > 1$, inequality (2.1) is as a rule trivial; after the replacement of 1 in (3.2) with c_1, the constant c in (2.1) admits the representation $\left(\frac{e}{r}\right)^r e^{c_1} + \varepsilon(nV(y))$ (cf. Theorem 2.1).

Corollary 3.1. *If conditions $[M^\pm]$ are satisfied then the analog of Corollary 2.1 looks like*

$$\sup_{x:\Pi \le \varepsilon, \Pi_W \le 1} \frac{\mathbf{P}(\overline{S}_n > x)}{\Pi} \le 1 + \varphi(\varepsilon), \tag{3.5}$$

where $\Pi = nV(x)$, $\Pi_W = nW\left(\frac{x}{|\ln nV(x)|}\right)$, and $\varphi(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.

The proof repeats that of Corollary 2.1 with obvious modifications.

Corollary 3.2. *Under the conditions $[M^\pm]$ and $n \le x^\gamma$, $1 < \gamma < \min(\alpha, \beta)$, inequalities (2.1) and (2.2) are valid always. They are also valid under the conditions $[M^+]$, $[M_1^-]$, and $n < cx$.*

This corollary is obvious because

$$y^\gamma V(y) \to 0, \quad y^\gamma W\left(\frac{y}{\ln y}\right) \to 0$$

as $y \to \infty$. Without loss of generality we may assume that $y > e$. Therefore, inequality (3.4) and, in consequence, (3.2) and (3.5) are satisfied.

REMARK 3.1. Conditions (3.2)–(3.4) seem essential for (2.1) and (2.2), since for $W(y) \gg V(y)$ and under regularity conditions like $[R]$ the deviations y for which $nW(y) > 1$ even in the case of $nV(y) \to 0$ fall into the region of "normal deviations," where for the distribution of the normalized sums S_n effective is the approximation by the limit stable law with parameters $(\alpha, -1)$.

PROOF OF THEOREM 3.1 repeats mostly that of Theorem 2.1. To use (1.8), we again estimate $R(\mu, y)$ in (2.3), where under the new conditions

$$I_1 = \int_{-\infty}^{0} e^{\mu t} dF(t) = F(0) + \mu \int_{-\infty}^{0} t \, dF(t) + \int_{-\infty}^{0} (e^{\mu t} - 1 - \mu t) dF(t).$$

Here

$$\int_{-\infty}^{0} (e^{\mu t} - 1 - \mu t) dF(t) = \mu \int_{-\infty}^{0} (1 - e^{\mu t}) F(t) dt.$$

Since the integrand of the last integral is negative for $\mu > 0$, the integral does not exceed

$$\mu \int_{0}^{\infty} W(t)(1 - e^{-\mu t}) dt = \mu^2 \int_{0}^{\infty} \widetilde{W}(t) e^{-\mu t} dt,$$

where for $\alpha > 1$

$$\widetilde{W}(t) = \int_{t}^{\infty} W(u) \, du \sim \frac{t W(t)}{\alpha - 1} \quad \text{as} \quad t \to \infty,$$

$$\mu^2 \int_{0}^{\infty} \widetilde{W}(t) e^{-\mu t} \, dt \sim \mu \widetilde{W}\left(\frac{1}{\mu}\right) \int_{0}^{\infty} e^{-u} u^{-\alpha+1} \, du \sim \frac{W\left(\frac{1}{\mu}\right)}{\alpha - 1} \Gamma(2 - \alpha) \quad \text{as } \mu \to 0.$$

Thus,

$$I_1 \le F(0) + \mu \int_{-\infty}^{0} t \, dF(t) + \frac{W\left(\frac{1}{\mu}\right)}{\alpha - 1} \Gamma(2 - \alpha)(1 + o(1)).$$

With the same choice of $M = \frac{2\beta}{\mu}$, we have the estimate

$$I_2 = \int_{0}^{M} e^{\mu t} \, dF(t) \le 1 - F(0) + \mu \int_{0}^{\infty} t \, dF(t) + \int_{0}^{M} (e^{\mu t} - 1 - \mu t) \, dF(t),$$

where

$$\int_{0}^{M} (e^{\mu t} - 1 - \mu t) \, dF(t) \le \mu \int_{0}^{M} (e^{\mu t} - 1) V(t) \, dt \le \mu(e^{2\beta} - 1)\widetilde{V}(M),$$

$$\widetilde{V}(t) = \int_{t}^{\infty} V(u) \, du \sim \frac{V(t)t}{\beta - 1}.$$

Therefore,

$$I_2 \le 1 - F(0) + \mu \int_{0}^{\infty} t \, dF(t) + \frac{(e^{2\beta} - 1)V\left(\frac{2\beta}{\mu}\right)}{\beta - 1},$$

$$I_1 + I_2 \le 1 + c_1 W\left(\frac{1}{\mu}\right) + c_2 V\left(\frac{1}{\mu}\right). \tag{3.6}$$

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Here the estimate for I_3 (see (2.6) and (2.7)) is the same as in Theorem 2.1. As a result, under the conditions of the current section we obtain

$$R(\mu, y) \le 1 + c_1 W\left(\frac{1}{\mu}\right) + c_2 V\left(\frac{1}{\mu}\right) + V(y)e^{\mu y}(1 + \varphi(\lambda)), \tag{3.7}$$

where $\lambda = \mu y$ (cf. (2.8)). Here the choice of an optimal μ is implemented by analogy to the preceding case (see (2.10)). Therefore, by analogy to (2.11) we obtain

$$\ln P \le -x\mu + c_1 n W\left(\frac{1}{\mu}\right) + c_2 n V\left(\frac{1}{\mu}\right) + e^\lambda \Pi(y)(1 + \varphi(\lambda))$$

$$\le -r \ln T + r + \varphi_1(T) + c_3 n W\left(\frac{y}{|\ln n V(y)|}\right).$$

If (3.1) is satisfied then we can eliminate the last summand on the right-hand side (ascribing it to $\varphi_1(T)$). If (3.2) holds then

$$\ln P \le -r \ln T + c.$$

This proves (2.1) and so (2.2).

Verify that (3.3) and (3.4) are sufficient for (3.2). Assume (3.3). Then

$$nW\left(\frac{y}{|\ln n V(y)|}\right) \le nW(y)|\ln n V(y)|^{\alpha+\varepsilon/2} \le [nW(y)]^{1-\frac{\alpha+\varepsilon/2}{\alpha+\varepsilon}} = [nW(y)]^{\frac{\varepsilon}{2(\alpha+\varepsilon)}} < 1.$$

If (3.4) holds then

$$nW\left(\frac{y}{|\ln n V(y)|}\right) \le nW\left(\frac{y}{|\ln V(y)|}\right) \le cnW\left(\frac{y}{\ln y}\right) < c.$$

Moreover, as mentioned, all above arguments remain valid after replacement of 1 on the right-hand side of (3.2) with c.

Now, suppose that condition $[M_1^-]$ ($\alpha = 1$) is satisfied. In this case, it is only the estimate for I_1 that is changed,

$$\widetilde{W}(t) = \int_t^\infty W(u)du = \frac{c_1}{\ln t},$$

and the summand $c_1 W(1/\mu)$ in (3.7) is replaced with $\mu \widetilde{W}(1/\mu) = \frac{c_1 \mu}{|\ln \mu|}$. Moreover, the sufficient conditions (3.2)–(3.4) transform into $n < cy$ ($n < cx$).

The proof of the theorem is over.

3.2. Estimates for the distribution of $\overline{S}_n(a)$, $a > 0$. Now, we evaluate

$$\mathbf{P}(\overline{S}_n(a) > x) \quad \text{and} \quad \mathbf{P}(\overline{S}_n(a) > x, B(v)),$$

where as before $\mathbf{E}X_j = 0$ and

$$\overline{S}_n(a) = \max_{k \le n}(S_k - ak), \ a > 0, \quad B(v) = \bigcap_{j=1}^n B_j(v).$$

Under the conditions of the current section, we put $g(j) = j$, so that

$$B_j(v) = \{X_j \le y + vj\}, \ v > 0.$$

Clearly, $\overline{S}_n(a)$ is nothing but the value of $\overline{S}_n$ for the summands with a negative mean.

Theorem 3.2. *Assume that conditions $[M^+]$ and $[M_1^-]$ are satisfied. Then for all n and for $v \leq \frac{a}{2r}$*

$$\mathbf{P}(\overline{S}_n(a) > x; \; B(v)) \leq c[mV(x)]^{r_1}, \tag{3.8}$$

where $m = \min(n, x)$ and $r_1 = \frac{r}{1+vr} \geq \frac{r}{1+a/2}$.

Corollary 3.3. *Assume that conditions $[M^+]$ and $[M_1^-]$ are satisfied. Then for all n*

$$\mathbf{P}(\overline{S}_n(a) > x) \leq cmV(x). \tag{3.9}$$

As D. A. Korshunov informed us, a more precise result was obtained in [11], wherein it was established that, for all so-called strongly subexponential tails $1 - F(t) = V(t)$, we have

$$\mathbf{P}(\overline{S}_n(a) > x) \sim \frac{1}{a} \int_0^{an} V(x + u)\, du \tag{3.10}$$

for $x \to \infty$ and all n (for $n = \infty$ this relation follows also from [12] and partially from [8]). Using the sufficient conditions of [11], one can show that distributions (1.3) belong to the class of strongly subexponential distributions. If we also account for the asymptotics of the integral in (3.10) then we found out that (3.9) ensues from (3.10).

A still more precise asymptotic representation for $\mathbf{P}(\overline{S}_n(a) > x)$ was obtained in [4], wherein Theorem 3.2 was also established. Nevertheless, we exhibit the proof of this theorem below to make exposition systematic.

PROOF OF COROLLARY 3.3. Inequality (3.9) follows from (3.8) and the relations

$$\mathbf{P}(\overline{S}_n(a) > x) \leq \mathbf{P}(\overline{B}(v)) + \mathbf{P}(\overline{S}_n(a) > x; B(v)),$$

$$\mathbf{P}(\overline{B}(v)) \leq \sum_{j=1}^{n} V(y + jv) \leq \int_0^{nv} V(y + u)\, du \leq cmV(x).$$

PROOF OF THEOREM 3.2. First of all, observe that, estimating $\mathbf{P}(\overline{S}_n(a) > x)$, without loss of generality we may assume that condition $[M^-]$ is satisfied. Indeed, introduce the random variables $^{(y)}X_j = \max(-y, X_j) + a_y$, $a_y = \mathbf{E}(X_j + y; X_j \leq -y) < a$, that are the centered "cuts" of X_j at the level $-y$, $y > 0$, and furnish the notations $\overline{S}_n(a)$ corresponding to the quantities $^{(y)}X_j$ with the left superscript (y). Then it is obvious that

$$\overline{S}_n(a) \leq {}^{(y)}\overline{S}_n(a + a_y),$$

where $a_y \to 0$ as $y \to \infty$, and all conditions like $[M^-]$ for $^{(y)}X_j$ are satisfied. We thus obtain a sought estimate for $\mathbf{P}(\overline{S}_n(a) > x)$ if we assume that $[M^-]$ is satisfied and "slightly decrease" the value of a.

Now, we turn to the proof of the theorem. For $n \leq x$ the claim follows from Theorem 3.1 and Corollary 3.2. Now, suppose that $x > n$. Without loss of generality, we assume that x is an integer. Then

$$P(x, y, n) \equiv \mathbf{P}(\overline{S}_n(a) > x; B(v)) \leq \mathbf{P}(\overline{S}_x(a) > x; B(v)) + \mathbf{P}\left(S_x > \frac{ax}{2}; B(v)\right)$$

$$+ \mathbf{P}\left(\overline{S}_x(a) \leq x, \; S_x \leq \frac{ax}{2}, \overline{S}_n(a) > x; \; B(v)\right) \equiv p_1 + p_2 + p_3. \tag{3.11}$$

Here

$$B(v) \subset \bigcap_{j=1}^{x} \{X_j \leq y + vx\} \equiv B^*,$$

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and Theorem 3.1 yields

$$p_1 \equiv \mathbf{P}(\overline{S}_x(a) > x, \ B(v)) \le \mathbf{P}(\overline{S}_x > x, B^*) \le (xV(x))^{r_1}, \quad r_1 = \frac{x}{y_1}, \quad y_1 = y + vx.$$

Similarly,

$$p_2 \equiv \mathbf{P}\left(S_x > \frac{ax}{2}, \ B(v)\right) \le \left(xV\left(\frac{ax}{2}\right)\right)^{r_1} \le c(xV(x))^{r_1}; \tag{3.12}$$

$$p_3 \le \mathbf{P}(\overline{S}_{n-x}(a) > x_1; \ X_j \le y_1 + jv, j = 1, \dots, n - x) = P(x_1, y_1, n - x), \tag{3.13}$$

where $x_1 = x(1 + a/2) \equiv xA$ and $y_1 = y + vx = y(1 + vr)$. If $n < x + x_1$ then the estimates are accomplished by applying Theorem 3.1 to $P(x_1, y_1, n - x)$. If $n > x + x_1$ then we should continue the recurrent estimation by using the inequalities (here we may again assume without loss of generality that x_1 is an integer)

$$P(x, y, n) \le (1 + c)(xV(x))^{r_1} + P(x_1, y_1, n - x)$$

that follow from (3.11) and (3.12).

As a result, for some $\nu > 1$ we obtain

$$P(x, y, n) \le (1 + c) \sum_{k=0}^{\nu} (x_k V(x_k))^{r_{k+1}},$$

where

$$x_k = xA^k,$$

$$y_k = y_{k-1} + vx_{k-1} = y + vx(A^{k-1} + A^{k-2} + \cdots + 1) = y\left(1 + vr\frac{A^k - 1}{A - 1}\right),$$

$$r_k = \frac{x_{k-1}}{y_k} = \frac{xA^{k-1}(A - 1)}{y(A - 1 + rvA^k - vr)} = \frac{r(A - 1)}{vrA + (A - 1 - vr)A^{1-k}}.$$

If $v \le \frac{a}{2r}$ then r_k does not decrease and $\min r_k$ coincides with

$$r_1 = \frac{r}{1 + vr} \ge \frac{r}{1 + a/2} = \frac{r}{A}.$$

Since

$$1 > x_k V(x_k) \sim A^{k(1-\beta)} xV(x)$$

as $x \to \infty$, it follows that

$$P(x, y, n) \le \frac{(1 + c)(xV(x))^{r_1}}{1 - A^{r_1(1-\beta)}}(1 + o(1)),$$

which completes the proof of the theorem.

§4. The Case of $\beta > 2$ and $\mathbf{E}X_j^2 < \infty$

Under the conditions of the current section, we may assume without loss of generality that

$$\mathbf{E}X_1 = 0, \quad \mathbf{E}X_1^2 = 1.$$

Conditions on the negative tails are not needed in this section.

4.1. Estimates for the distribution of $\overline{S}_n$. In the sequel we need the values $N = N(n)$ that characterize the region of deviations of S_n where the asymptotics of $\mathbf{P}(S_n > x)$ changes from the "normal" asymptotics $1 - \Phi\left(\frac{x}{\sqrt{n}}\right)$ to the asymptotics $nV(x)$ describing $\mathbf{P}(S_n > x)$ for x large enough.

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More precisely, we define N as a deviation for which the asymptotics $e^{-\frac{x^2}{2n}(1+o(1))}$ and $nV(x)$ "almost coincide"; i.e., we put

$$N = \sqrt{(\beta - 2)n \ln n}$$

which is a main part of a solution to the equation $-\frac{x^2}{2n} = \ln n - \beta \ln x = \ln nV(x)$.

Under the conditions of the previous sections, the role of deviations N at which the approximation by a stable law is replaced with the approximation by the quantity $nV(x)$ was played by deviations of order $n^{1/\gamma}$, $\gamma = \min(\alpha, \beta)$ (a solution to the equation $nV(x) = 1$ or $nW(x) = 1$). We also note that in this section we always assume that $x \to \infty$ $(y \to \infty)$; moreover, the deviations x (or y) always exceed $\sqrt{n}$, so that we always have

$$nV(x) \to 0 \quad (nV(y) \to 0)$$

as $x \to \infty$.

REMARK 4.1. To avoid a confusion for $n = 1$, it would be more convenient to put $N = \sqrt{(\beta - 2)n \ln(n+1)}$ (the value $n = 1$ is not excluded). Since $P = 0$ for $n = 1$ in the most interesting case of $r \geq 1$, we can assume that $n \geq 2$ wherever a confusion with $\ln n$ may arise.

Theorem 4.1. *Assume that condition $[M^+]$ is satisfied, $\beta > 2$, $\mathbf{E}X_1 = 0$, and $\mathbf{E}X_1^2 < \infty$. Then*
1. *For every fixed $h > 1$ and all sufficiently large $y = sN$, $s^2 \geq \frac{h}{2}$,*

$$P \leq T^{-r+\theta}, \tag{4.1}$$

where

$$T = \frac{r}{nV(y)}, \quad \theta = \frac{h}{4s^2}\left(1 + b\frac{\ln s}{\ln n}\right), \quad b = \frac{2\beta}{\beta - 2}, \quad N = \sqrt{(\beta - 2)n \ln n}, \quad r = \frac{x}{y}.$$

2. *For every fixed $h > 1$, $y = x$, $\frac{1}{\ln n} < s^2 < \frac{h}{2}$, and all sufficiently large n*

$$P \leq e^{-\frac{x^2}{2nh}}. \tag{4.2}$$

Corollary 4.1. (a) *If $s \to \infty$ then for every $\varepsilon > 0$*

$$P \leq T^{-r+\varepsilon}. \tag{4.3}$$

(b) *If $s^2 > \ln n$ then*

$$P \leq cT^{-r}. \tag{4.4}$$

Corollary 4.2. (a) *If $s \to \infty$ then for every $\delta > 0$ and all sufficiently large x*

$$\mathbf{P}(\overline{S}_n > x) \leq nV(x)(1 + \delta). \tag{4.5}$$

(b) *If $s^2 \geq h/2$ then*

$$\mathbf{P}(\overline{S}_n > x) \leq cnV(x). \tag{4.6}$$

(c) *For every fixed $h > 1$, $1/\ln n < s^2 < h/2$, and all sufficiently large n*

$$\mathbf{P}(\overline{S}_n > x) \leq e^{-\frac{x^2}{2nh}} \tag{4.7}$$

REMARK 4.2. It is easy to verify that, as in Corollary 2.1, there exists a function $\varphi(t) \downarrow 0$, $t \uparrow \infty$, such that the following relation holds alongside (4.5):

$$\sup_{x:s \geq t} \frac{\mathbf{P}(\overline{S}_n > x)}{nV(x)} \leq 1 + \varphi(t).$$

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PROOF OF COROLLARY 4.1. The first assertion is obvious from (4.1). Prove the second. Since $y = sN$, the following estimates for T are obvious:

$$T < c_1 s^{\beta+\varepsilon} n^{\frac{\beta+\varepsilon}{2}-1}$$

for every $\varepsilon > 0$. Whence we obtain

$$\ln T^\theta \le \frac{h}{4s^2}\left(1 + b\frac{\ln s}{\ln n}\right)\left[\ln c_1 + (\beta + \varepsilon)\ln s + \left(\frac{\beta+\varepsilon}{2} - 1\right)\ln n\right].$$

Clearly, for $s^2 > \ln n$ the right-hand side of this inequality is bounded. This completes the proof of Corollary 4.1.

PROOF OF COROLLARY 4.2. The proof rests again on the inequality

$$\mathbf{P}(\overline{S}_n > x) \le nV(y) + P.$$

Item (a) follows from (4.3) on putting $r = 1 + \varepsilon$.

Prove (b). If $s \to \infty$ then (b) follows from (a). If s is bounded then by necessity $n \to \infty$ as $x \to \infty$ and $\theta \le \frac{h}{2s^2}$. On putting $r = 1 + \frac{h}{2s^2}$, we obtain

$$\mathbf{P}(\overline{S}_n > x) \le 2nV\left(\frac{x}{1 + h/2s^2}\right) \le cnV(x).$$

Item (c) follows from the inequalities (see (4.2))

$$\mathbf{P}(\overline{S}_n > x) \le nV(x) + e^{-\frac{x^2}{2nh}}, \tag{4.8}$$

where for $s^2 < h/2$

$$e^{-\frac{x^2}{2nh}} > \exp\left\{-\frac{h}{2}\frac{(\beta-2)n\ln n}{2nh}\right\} = n^{-\frac{\beta-2}{4}}, \quad nV(x) \le cn^{1-\frac{\beta}{2}+\varepsilon}$$

for every $\varepsilon > 0$ and all n large enough. Therefore, the second summand on the right-hand side of (4.8) is dominating. Slightly changing h if need be, we arrive at (c), finishing the proof of Corollary 4.2.

PROOF OF THEOREM 4.1. We proceed along the same lines as in the proofs of the preceding theorems. The proof bases again on inequality (1.8) and estimates for $R(\mu, y)$. However, here we partition $R(\mu, y)$ into subintegrals otherwise (cf. (2.3)). Put $M(v) = \frac{v}{\mu}$, so that $M = M(2\beta)$ (cf. (2.4)). Then

$$R(\mu, y) = I_1 + I_2,$$

where

$$I_1 = \int_{-\infty}^{M(\varepsilon)} e^{\mu t}\, dF(t) = \int_{\infty}^{M(\varepsilon)}\left(1 + \mu t + \frac{\mu^2 t^2}{2}e^{\mu\theta(t)}\right) dF(t), \quad 0 \le \frac{\theta(t)}{t} \le 1. \tag{4.9}$$

Here

$$\int_{-\infty}^{M(\varepsilon)} dF(t) = 1 - V(M(\varepsilon)) \le 1,$$

$$\int_{-\infty}^{M(\varepsilon)} t\, dF(t) = -\int_{M(\varepsilon)}^{\infty} t\, dF(t) \le 0, \tag{4.10}$$

$$\int_{-\infty}^{M(\varepsilon)} t^2 e^{\mu\theta(t)}\, dF(t) \le e^\varepsilon \int_{-\infty}^{M(\varepsilon)} t^2\, dF(t) \le e^\varepsilon \equiv h. \tag{4.11}$$

Hence,

$$I_1 \le 1 + \frac{\mu^2 h}{2}. \tag{4.12}$$

Now, we estimate

$$I_2 = -\int_{M(\varepsilon)}^{y} e^{\mu t}\, dF(t) \le V(M(\varepsilon))e^\varepsilon + \mu \int_{M(\varepsilon)}^{y} V(t)e^{\mu t}\, dt. \tag{4.13}$$

We first consider

$$I_{2,1} = \mu \int_{M(\varepsilon)}^{M} V(t)e^{\mu t} dt$$

for $M(\varepsilon) < M < y$. For $t = \frac{v}{\mu}$ we have

$$V(t)e^{\mu t} = V\left(\frac{v}{\mu}\right) e^v \sim V\left(\frac{1}{\mu}\right) g(v),$$

where the function $g(v) = v^{-\beta} e^v$ is convex on $(0, \infty)$. Therefore,

$$I_{2,1} \le \frac{\mu}{2}(M - M(\varepsilon))V\left(\frac{1}{\mu}\right)(g(\varepsilon) + g(2\beta)) \le cV\left(\frac{1}{\mu}\right). \tag{4.14}$$

Estimation of

$$I_{2,2} = \mu \int_{M}^{y} V(t)e^{\mu t} dt$$

is carried out in the same way as that of I_3^0 in (2.6) and (2.7), and yields

$$I_{2,2} \le V(y)e^{\mu y}(1 + \varphi(\lambda)), \quad \lambda = \mu y, \tag{4.15}$$

$\varphi(\lambda) \downarrow 0$ as $\lambda \uparrow \infty$. Summing up (4.12)–(4.14), we obtain

$$R(\mu, y) \le 1 + \frac{\mu^2 h}{2} + cV\left(\frac{1}{\mu}\right) + V(y)e^{\mu y}(1 + \varphi), \tag{4.16}$$

$$R^n(\mu, y) \le \exp\left\{\frac{n\mu^2 h}{2} + cnV\left(\frac{1}{\mu}\right) + nV(y)e^{\mu y}(1 + \varphi)\right\}. \tag{4.17}$$

As μ we first take the value (see (2.10))

$$\mu = \frac{1}{y}\ln T,$$

where $T = \frac{r}{nV(y)}$. Then by analogy to (2.9) we obtain

$$R^n(\mu, y) \le \exp\left\{\frac{n\mu^2 h}{2} + cnV\left(\frac{1}{\mu}\right) + r(1 + \varphi)\right\}, \tag{4.18}$$

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where as before

$$nV\left(\frac{1}{\mu}\right) \sim nV\left(\frac{y}{\ln T}\right) \sim cnV\left(\frac{y}{|\ln nV(y)|}\right) \le cnV(y)|\ln nV(y)|^{\beta+\varepsilon} \to 0,$$

since $nV(y) \to 0$. Therefore (cf. (2.11)),

$$\ln P \le -r\ln T + r + \frac{nh}{2y^2}\ln^2 T + \varphi_1(T) = \ln T\left[-r + \frac{nh}{2y^2}\ln T\right] + \varphi_1(T), \qquad (4.19)$$

where $\varphi_1(T) \downarrow 0$ as $T \uparrow \infty$. For $y = sN$, $N = \sqrt{(2-\beta)n\ln n}$, we have

$$\ln T = -\ln nV(y) + O(1) = -\ln n + \beta\ln s + \frac{\beta}{2}\ln n + O(\ln L(sN)) + O(1)$$

$$= \frac{\beta-2}{2}\ln n\left[1 + b\frac{\ln s}{\ln n}\right](1 + o(1)), \qquad (4.20)$$

where $b = \frac{2\beta}{\beta-2}$. Hence,

$$\frac{nh}{2y^2}\ln T = \frac{h}{4s^2}\left[1 + b\frac{\ln s}{\ln n}\right](1 + o(1)), \quad \ln P \le -\ln T\left[r - \frac{h'}{4s^2}\left(1 + b\frac{\ln s}{\ln n}\right)\right]$$

for every $h' < h < 1$ and all y large enough. This proves the first assertion of Theorem 4.1.

Now, we consider "mild" values of s, for example, such as

$$\frac{1}{\ln n} \le s^2 < \frac{h}{2}.$$

This corresponds to the following range of the values of y:

$$\frac{h(\beta-2)}{2}n\ln n > y^2 > (\beta-2)n.$$

Here we take as μ the value

$$\mu = \frac{x}{nh} \to 0 \text{ as } n \to \infty.$$

Then

$$\ln P \le -\mu x + \frac{n\mu^2 h}{2} + cnV\left(\frac{1}{\mu}\right) + nV(y)e^{\mu y}(1 + \varphi). \qquad (4.21)$$

Here for $x = y$ it is obvious that

$$nV\left(\frac{1}{\mu}\right) \le cnV\left(\sqrt{\frac{n}{\ln n}}\right) \to 0 \qquad (4.22)$$

as $n \to \infty$. Next, from (4.20) we easily derive that

$$nV(y) \le n^{\frac{2-\beta}{2}}.$$

Moreover,

$$\mu y = \frac{y^2}{nh} = \frac{s^2(\beta-2)\ln n}{h}.$$

Therefore,

$$\frac{\gamma nV(y)e^{\mu y}}{\mu y} \le cn^{1-\beta/2+\frac{s^2(\beta-2)}{h}} \to 0 \qquad (4.23)$$

for $s^2 < h/2$.

Summing up (4.15)–(4.17), we obtain

$$\ln P \leq -\frac{x^2}{2nh} + o(1).$$

The summand $o(1)$ can be eliminated by changing $h > 1$. This proves (4.2) and finishes the proof of the theorem.

4.2. Estimates for the distribution of $\overline{S}_n(a)$, $a > 0$. This section differs slightly from Section 3.2. As in that section, here we put

$$B(v) = \bigcap_{j=1}^{n} B_j(v), \quad B_j(v) = \{X_j \leq y + vj\}, \quad v > 0.$$

Theorem 4.2. *Assume that condition $[M^+]$ is satisfied, $\beta > 2$, $\mathbf{E}X_j = 0$, and $\mathbf{E}X_j^2 < \infty$. Then for all n and x*

$$\mathbf{P}(\overline{S}_n(a) > x; \ B(v)) \leq c[mV(x)]^{r_1}, \tag{4.24}$$

where $m = \min(n, x)$, $r_1 = \frac{r}{1+vr}$, $r = \frac{x}{y}$, and $v \leq \frac{a}{2r}$.

Corollary 4.3. *Assume that condition $[M^+]$ is satisfied. Then for all n and x*

$$\mathbf{P}(\overline{S}_n(a) > x) \leq cmV(x). \tag{4.25}$$

See also a remark on Corollary 3.3.

The proof of Corollary 4.3 is the same as that of Corollary 3.3. We merely note that (4.24) and (4.25) should be considered only at large x when $mV(x) < 1$. The remarks of §3 on Corollary 3.3 remain completely valid.

The proof of Theorem 4.2 as well almost completely repeats that of Theorem 3.2. We merely have to note that here the proof of (4.24) uses Theorem 4.1 for $n \leq cx$ and so the condition $s^2 > \ln n$ in item (b) of Corollary 4.1 is satisfied; $P < cT^{-r}$. In other aspects the consideration remains the same.

§5. Semiexponential Tails, $\mathbf{E}X_1^2 < \infty$

In this section we consider semiexponential tails $V(x)$ of the form (1.4):

$$V(t) = e^{-l(t)}, \quad l(t) = t^\alpha L(t),$$

where $\alpha \in (0, 1)$ and $L(t)$ is a slowly varying function.

We need a smoothness condition on $l(t)$, although it seems unessential for the final results.

$[D]$. *The function $l(x + t)$ as $x \to \infty$ and $t = o(x)$ admits the representation*

$$l(x + t) = l(x) + tl'(x)(1 + o(1)), \tag{5.1}$$

where $l'(x) \sim \frac{\alpha l(x)}{x}$.

For simplicity, we may assume that the function $L(t)$ is differentiable and $L'(t) = o\left(\frac{L(t)}{t}\right)$. Then $[D]$ is always satisfied and $l'(x)$ can be identified with the derivative of l.

5.1. Estimates for the distribution of $\overline{S}_n$. We again introduce into consideration the function $N = N(n)$ (see §4) that characterizes the region of deviations x where the "normal" asymptotics $e^{-\frac{x^2}{2n}}$ and the asymptotics $nV(x)$ give just about the same result. More precisely, we define N as a solution to the equation

$$\frac{N^2}{2n} = -\ln nV(N). \tag{5.2}$$

From the standpoint of the asymptotics of $N(n)$, this is the same as a solution of the equation

$$\frac{N^2}{2n} = -\ln V(N) = l(N).$$

It is slightly more convenient to consider the equation

$$N^2 = nl(N) \tag{5.3}$$

whose solution differs from a solution of the original equation by a bounded factor. It is easy to see that in our case $N = N(n)$ has the form

$$N = n^{\frac{1}{2-\alpha}} L_1(n), \tag{5.4}$$

where $L_1(n)$ is a slowly varying function.

The domain of deviations $x \le N(n)$ can be called "*Cramer-type*" (or *normal*); the domain $N(n) < x \le N_2(n)$, where $N_2(n)$ is defined below, is *intermediate*; and the domain $x > N_2(n)$ is the domain wherein effective is the "*maximal jump principle*" (in this domain the asymptotics of $\mathbf{P}(\overline{S}_n > x)$ coincides with the asymptotics of $\mathbf{P}(\max_{k \le n} X_k > x) \sim nV(x)$; see [1] for more details).

Put

$$w(t) = -t^{-2} \ln V(t) = t^{-2} l(t) = t^{\alpha-2} L(t). \tag{5.5}$$

We may assume without loss of generality that $w(t) \downarrow$. Then equation (5.3) can be written as $w(N) = \frac{1}{n}$, and $N(n)$ is nothing but the value of the inverse function $w^{(-1)}$ of w at the point $\frac{1}{n}$:

$$N(n) = w^{(-1)}\left(\frac{1}{n}\right).$$

It is easy to see that if L satisfies the condition

$$L\big(tL^{\frac{1}{2-\alpha}}(t)\big) \sim L(t) \text{ as } t \to \infty, \tag{5.6}$$

then $w^{(-1)}(u)$ has the form

$$w^{(-1)}(u) \sim u^{\frac{1}{\alpha-2}} L^{\frac{1}{2-\alpha}}(u^{\frac{1}{\alpha-2}}),$$

so that $L_1(n) \sim L^{\frac{1}{2-\alpha}}\big(n^{\frac{1}{2-\alpha}}\big)$.

Observe that condition (5.6) is rather loose but it is valid not always. For example, it fails for the slowly varying function $L(t) = \exp\{\ln t / \ln \ln t\}$.

Since the boundary $N(n)$ of the Cramer-type domain of deviations depends on n, it can be equivalently characterized in terms of n: $n \ge \frac{1}{w(x)}$ for the Cramer-type domain, and $n < \frac{1}{w(x)}$ for the intermediate domain. Thus, as a characteristics of deviations we can take both the number $s = \frac{x}{N(n)}$ (cf. §4; $s \le 1$ for the Cramer-type domain) and the number $\sigma = \sigma(x) = nw(x)$ ($\sigma > 1$ for the Cramer-type domain); $\sigma \sim s^{\alpha-2}$ as $n \to \infty$. In some cases it is more convenient to use the characteristics σ (we often omit the argument x; if it differs from x then we will indicate it).

Theorem 5.1. *Assume that condition* $[M^+]$ *is satisfied, the function* l *satisfies condition* $[D]$, *and the function* w *is defined by* (5.5). *Then there exists a constant* c *(whose explicit form can be easily found from the proof) such that for every fixed* $h > 1$, *all* n, *and all sufficiently large* y

$$P \le c[nV(y)]^{r - \frac{h\sigma(y)}{2}}, \quad r = \frac{x}{y}. \tag{5.7}$$

If for arbitrary fixed $h > 1$ *and* $\varepsilon > 0$ *we have* $\sigma h \ge 1 + \varepsilon$ *then for* $y = x$ *and all sufficiently large* n

$$P \le e^{-\frac{x^2}{2nh}}. \tag{5.8}$$

If the deviations y are characterized by the relation $y = sN(n)$ then (5.7) holds with $\sigma(y)$ replaced by $s^{\alpha-2}(1 + o(1))$. If $y = x$, $s^{2-\alpha} < h$, then (5.8) holds.

We state some corollaries to Theorem 5.1.

Alongside the function $w(t)$ (see (5.5)), define the function

$$w_2(t) = w(t)l(t) = t^{-2}l^2(t) = t^{2\alpha-2}L^2(t), \tag{5.9}$$

considering it like $w(t)$ monotone decreasing, so that the inverse function $w_2^{(-1)}(\cdot)$ is well defined. Put

$$N_2(n) = w_2^{(-1)}\left(\frac{1}{n}\right) = n^{\frac{1}{2-2\alpha}}L_2(n), \tag{5.10}$$

where L_2 is a slowly varying function which, like L_1, can be found explicitly under the additional assumption (5.6).

Next, let r_0 be a minimal solution to the equation

$$1 = r - \frac{\sigma h}{2}r^{2-\alpha}$$

which exists always if $\sigma h < 2^{\alpha-1}$. Put $\sigma^* = r_0 - 1 \sim \frac{\sigma h}{2}$, equivalence as $\sigma \to 0$. Here and below $h > 1$ is as before an arbitrary fixed number.

Corollary 5.1. 1. If $\sigma h < 2^{\alpha-1}$ then

$$\mathbf{P}(\overline{S}_n > x) \leq cnV(x)^{(1+\sigma^*)^{-\alpha}}. \tag{5.11}$$

If $\sigma l(x) \leq c$ or, which is the same, $x \geq c_2 N_2(x)$ then

$$\mathbf{P}(\overline{S}_n > x) \leq c_1 nV(x).$$

If $\sigma l(x) \to 0$ $(x \gg N_2(n))$ then

$$\mathbf{P}(\overline{S}_n > x) < nV(x)\bigl(1 + o(1)\bigr).$$

2. If $\sigma h \geq 2^{\alpha-1}$ then

$$\mathbf{P}(\overline{S}_n > x) < cnV(x)^{\frac{1}{2\sigma h}}. \tag{5.12}$$

Let $h > 1$ and $\varepsilon > 0$ be arbitrary fixed numbers. If $\sigma h \geq 1 + \varepsilon$ then $l(x) > 2\ln n$ and for all sufficiently large n

$$\mathbf{P}(\overline{S}_n > x) \leq e^{-\frac{x^2}{2nh}} = V(x)^{\frac{1}{2\sigma h}}. \tag{5.13}$$

The condition $l(x) > 2\ln n$ in the last assertion seems redundant.

REMARK 5.1. As in Corollaries 2.1 and 4.2 (see also Remark 4.1), it is easy to verify that there exists a function $\varphi(t) \downarrow 0$, $t \uparrow \infty$, such that for $x = sN_2(n)$

$$\sup_{x:s>t} \frac{\mathbf{P}(\overline{S}_n > x)}{nV(x)} \leq 1 + \varphi(t).$$

PROOF OF THEOREM 5.1. The scheme of the proof is former. Again inequality (1.8) plays the main role. The estimate for the integral I_1 in (4.9) remains the same as in Theorem 4.1 for $M(\varepsilon) = \varepsilon/\mu$ (see (4.9)). Put $e^\varepsilon = h$. Then (see (4.12))

$$I_1 < 1 + \frac{\mu^2 h}{2}. \tag{5.14}$$

The estimate for

$$I_2 = \int\limits_{M(\varepsilon)}^{y} e^{\mu t} dF(t) \leq V(M(\varepsilon))h + \mu \int\limits_{M(\varepsilon)}^{y} V(t)e^{\mu t} dt \tag{5.15}$$

(see (4.13)) changes slightly.

Put

$$f(t) = -l(t) + \mu t.$$

If $[D]$ holds then for $t \ll \mu^{\frac{1}{\alpha-1}}$ (μ is small) this function decreases; for $t \gg \mu^{\frac{1}{\alpha-1}}$ it increases.

Assume for simplicity that l is a continuously differentiable function and $l'(t) \downarrow$ (by $[D]$ $l'(t) \sim \frac{\alpha l(t)}{t}$). Then the minimum of $f(t)$ is attained at the point $t_0 = \zeta(\mu)$, where $\zeta = (l')^{(-1)}$ is the inverse function of $l'(\cdot)$ on the interval (t_0, ∞), so that $l'(\zeta(\mu)) = \mu$, $\zeta(s) = s^{\frac{1}{\alpha-1}}L^*(s)$, and $L^*(s)$ is a slowly varying function. Put

$$\mu = v l'(y), \tag{5.16}$$

where $v > 1$ will be specified later. Clearly, for $v > 1$ we have $\zeta(\mu) < y$. Observe that for $v \approx 1/\alpha > 1$ the value

$$f(y) = -l(y) + v l'(y) y \approx l(y)(v\alpha - 1)$$

can be made small and $e^{f(y)}$, "comparable with 1."

Also, observe that

$$y = \zeta\left(\frac{\mu}{v}\right) \sim v^{\frac{1}{1-\alpha}}\zeta(\mu), \quad v > 1. \tag{5.17}$$

In the forthcoming considerations, we will bear in mind the fact that $v \geq 1 + \varepsilon$, $\varepsilon > 0$. Put

$$M = b\zeta(\mu),$$

where b is an arbitrary point in the interval $(1, \alpha^{\frac{1}{\alpha-1}})$, for example, its midpoint. Then, on the one hand, for $t \geq M$

$$f'(t) \geq f'(M) = -b^{\alpha-1}l'(\zeta(\mu)) + \mu \sim \mu(1 - b^{\alpha-1}) = c\mu, \ c > 0. \tag{5.18}$$

On the other hand, putting $\zeta(\mu) = \zeta$ for brevity, we have

$$f(M) \sim -l(b\zeta) + \mu b\zeta \sim -\frac{l'(b\zeta)b\zeta}{\alpha} + \mu b\zeta \sim \left(1 - \frac{b^{\alpha-1}}{\alpha}\right)\mu b\zeta, \tag{5.19}$$

where $b^{\alpha-1} > \alpha$. Since

$$\mu\zeta(\mu) > \mu^{\frac{\alpha}{\alpha-1}+\delta}, \quad l(M(\varepsilon)) = l\left(\frac{\varepsilon}{\mu}\right) > \mu^{-\alpha+\delta}$$

for every $\delta > 0$ and all μ small enough; in view of the above-mentioned properties of the function f, the integral

$$\int\limits_{M(\varepsilon)}^{M} e^{f(t)} dt,$$

which is a part of the integral on the right-hand side of (5.15), is estimated by

$$\int\limits_{M(\varepsilon)}^{M} e^{f(t)} dt \leq M e^{-\mu^{-\alpha+\delta}} = b\zeta(\mu)e^{-\mu^{-\alpha+\delta}} = o(\mu^2) \tag{5.20}$$

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as $\mu \to 0$. Obviously, the summand $V(M(\varepsilon))h$ in (5.15) admits a similar estimate.

To calculate the other part $\int_M^y e^{f(t)}dt$ of the integral in (5.15) in the case of $y > M$ (or, which is the same, $v^{\frac{1}{1-\alpha}} > b$), we use inequality (5.18) which means that, while calculating the integral, it suffices to consider only its part over a "neighborhood" of the point y. For $t = y - u$, $u = o(y)$, $U = o(y)$, and $U \gg \mu^{-1}$, by (5.16) we have

$$f(t) - f(y) = l(y) - l(y - u) - \mu u = l'(y)u(1 + o(1)) - \mu u \sim \mu u \left(\frac{1}{v} - 1\right).$$

Therefore,

$$\mu \int_{y-U}^{y} e^{f(t)}dt \sim \mu e^{f(y)} \int_0^U e^{\mu u(1/v-1)}du \le e^{f(y)}\frac{v}{v-1}. \tag{5.21}$$

The integral $\int_M^{y-U}$ is estimated similarly and gives $o(e^{f(y)})$.

We can now estimate $R(\mu, y)$. Combining (5.14), (5.15), (5.20), and (5.21), we obtain

$$R(\mu, y) \le 1 + \frac{\mu^2 h}{2}(1 + o(1)) + \frac{ve^{f(y)}}{v-1}(1 + o(1)),$$
$$R^n(\mu, y) \le \exp\left\{\frac{nh\mu^2}{2}(1 + o(1)) + \frac{vn}{v-1}e^{f(y)}(1 + o(1))\right\}. \tag{5.22}$$

Put

$$\mu = \frac{1}{y}\ln T, \quad T = \frac{r(1-\alpha)}{nV(y)} \equiv \frac{c}{nV(y)}$$

and note that for $\sigma(y) = nw(y) > \frac{2r}{h}$ inequality (5.7) becomes trivial (its right-hand side increases unboundedly). For the deviations y satisfying $\sigma(y) \le \frac{2r}{h}$, we have $n \le c_1 y^{2-\alpha+\varepsilon}$ for all $\varepsilon > 0$, and so

$$\mu = \frac{1}{y}\ln T \sim -\frac{1}{y}\ln nV(y) \ge \frac{l(y)}{y}(1 + o(1)) \sim \frac{l'(\alpha)}{\alpha}.$$

This means that in (5.16) $v \approx 1/\alpha > 1$ and all assumptions imposed on μ and v are satisfied. As before, we now find that

$$\ln P \le -\mu x + \frac{nh\mu^2}{2}(1 + o(1)) + \frac{nV(y)}{r(1-\alpha)}e^{\mu y}(1 + o(1)), \tag{5.23}$$

$$\frac{nV(y)}{r(1-\alpha)}e^{\mu y} = r, \quad -\mu x + \frac{nh\mu^2}{2} = (-r + \rho)\ln T,$$

where

$$\rho = \frac{nh\ln T}{2y^2} = -\frac{nh(\ln nV(y) + \ln c)}{2y^2}, \quad c = r(1-\alpha).$$

Here for $nw(y) = \sigma(y)$ we have (see (5.5))

$$-\frac{\ln V(y)}{y^2} = l(y)y^{-2} = w(y) = \frac{\sigma(y)}{n}. \tag{5.24}$$

Therefore, assuming for simplicity that $cn \ge 1$, we obtain

$$\rho \le \frac{\sigma(y)h}{2}, \tag{5.25}$$

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$$P \leq c_1[nV(y)]^{r - \frac{h\sigma(y)}{2}}$$

(if $cn < 1$ then we should add $o(1)$ to the right-hand side of (5.25), removing this summand afterwards by slightly increasing h). This proves the first part of the theorem.

Now, consider the Cramer-type domain of deviations where $\sigma = nw(x)$ may be large. Here we put $y = x$,

$$\mu = \frac{x}{nh},$$

so that

$$\mu = \frac{xw(x)}{\sigma h} = \frac{l(x)}{x\sigma h} \sim \frac{l'(x)}{\alpha\sigma h}, \quad v \sim \frac{1}{\alpha\sigma h}$$

(see (5.16)) and the assumption $v > 1$ presumed in (5.16) may fail at large σ. If $v > b^{1-\alpha}$ (or, which is the same, $x = y > M$, see (5.17)) then all above-obtained estimates for $R(\mu, y)$ are preserved and we again have (5.22). However, if $v \leq b^{1-\alpha}$ then $\int_M^y e^{f(t)} dt$ in the preceding calculations disappear together with the last summand on the right-hand side of (5.22). In this case we readily come to the second assertion of the theorem.

Thus, we are left with settling the case of $v = \frac{1}{\alpha\sigma h} > b^{1-\alpha} > 1$ and estimating in this case the last summand in (5.22) whose logarithm for $\mu = \frac{y}{nh} = \frac{x}{nh}$ equals

$$H = \mu y + \ln nV(y) + O(1) = \frac{x^2}{n}\left(\frac{1}{h} - w(y)n\right) + \ln n + O(1) = \frac{x^2}{n}\left(\frac{1}{h} - \sigma\right) + \ln n + O(1).$$

If $\sigma h \geq 1 + \varepsilon$ and $x^2 \gg n\ln n$ then $H \to -\infty$ as $n \to \infty$. However, if $c\sqrt{n} < x < n^{1/2+\varepsilon}$ for a sufficiently small $\varepsilon > 0$ then

$$\sigma = nw(x) \to \infty, \quad \frac{x^2\sigma}{n} = x^2 w(x) \gg \ln n \to \infty$$

and hence $H \to -\infty$ again. Thus, the last summand in (5.22), (5.23) is negligible and

$$\ln P \leq -\frac{x^2}{2nh}(1 + o(1)) + o(1),$$

where the summand $o(1)$ can be eliminated by slightly increasing h. The proof of the theorem is over.

PROOF OF COROLLARY 5.1. We have

$$\mathbf{P}(\overline{S}_n > x) \leq nV(y) + c[nV(y)]^{r - \frac{h\sigma(y)}{2}}. \tag{5.26}$$

Our purpose is to choose y (or $r = \frac{x}{y}$) as optimal as possible. Observe that for $x \to \infty$ and r comparable with 1 we have

$$\sigma(y) = \sigma\left(\frac{x}{r}\right) \sim r^{2-\alpha}\sigma, \quad l(y) \sim r^{-\alpha}l(x).$$

The second summand in (5.26) has the exponent

$$-l(x)\left[r^{1-\alpha} - \frac{h\sigma}{2}r^{2-2\alpha}\right](1 + o(1)), \quad \sigma = \sigma(x),$$

attaining its minimum in a neighborhood of the point $\widehat{r} = (\sigma h)^{\frac{1}{\alpha-1}}$ and equal to

$$-l(x)\frac{1}{2\sigma h}(1 + o(1)).$$

Therefore, if $\widehat{r}^{\,-\alpha} = (\sigma h)^{-\frac{\alpha}{\alpha-1}} \geq \frac{1}{2\sigma h}$ or, which is the same, $\sigma h \geq 2^{\alpha-1}$, then the exponent of the second summand in (5.26) is greater than or equal to the exponent of the first summand, and a sought r_0 can be taken to be $r_0 = \widehat{r}$. Furthermore, the power of n in the second summand of (5.26) equals $\frac{1}{2\sigma h} \leq 2^{-\alpha} < 1$. Hence, from (5.26) we obtain

$$\mathbf{P}(\overline{S}_n > x) \leq cnV(x)^{\frac{(1+o(1))}{2\sigma h}},$$

where the factor $1 + o(1)$ can be eliminated by slightly increasing h. This proves (5.12).

If $\sigma h < 2^{\alpha-1}$ then as r_0 we should take a number for which the summands on the right-hand side of (5.26) become approximately the same; i.e., we should take r_0 to be equal to a minimal solution of the equation

$$1 = r - \frac{\sigma h}{2} r^{2-\alpha};$$

namely,

$$r_0 = 1 + \frac{\sigma h}{2} + (2-\alpha)\left(\frac{\sigma h}{2}\right)^2 + \cdots \equiv 1 + \sigma^*,$$

$\sigma^* \sim \frac{\sigma h}{2}$ as $\sigma \to 0$. In this case

$$\mathbf{P}(\overline{S}_n > x) \leq cnV\left(\frac{x}{1+\sigma^*}\right) = cnV(x)^{(1+\sigma^*)^{-\alpha}(1+o(1))},$$

where the factor $1 + o(1)$ can be eliminated again by slightly changing h. This proves (5.11). The last two inequalities of Corollary 5.1 are obvious consequences of (5.11), since in the first of them

$$V(x)^{(1+\sigma^*)^{-\alpha}} = e^{-l(x)(1+\sigma^*)^{-\alpha}} \leq e^{-l(x)+O(l(x)\sigma^*)},$$

where $l(x)\sigma^* \sim l(x)\frac{\sigma h}{2} < \frac{ch}{2}$, while $l(x)\sigma^* \to 0$ in the second.

Assertion (5.13) follows from (5.26) with $x = y$, (5.8), and the fact that for $\sigma h \geq 1 + \varepsilon$

$$e^{-\frac{x^2}{2nh}} = e^{-\frac{l(x)}{2\sigma h}} \geq V(x)e^{l(x)\frac{1+2\varepsilon}{2+2\varepsilon}} \gg nV(x)$$

when $l(x) > 2\ln n$.

5.2. Estimates for the distribution of $\overline{S}_n(a)$. Like in Sections 3.2 and 4.2, the purpose of this section is to evaluate

$$P(a, v) \equiv \mathbf{P}(\overline{S}_n(a) > x, \ B(v)),$$

where

$$B(v) = \bigcap_{j=1}^{n} B_j(v), \quad B_j(v) = \{X_j \leq y + vj\}.$$

Put

$$z = z(x) = \frac{1}{l'(x)} \sim \frac{x}{\alpha l(x)} = o(x). \tag{5.27}$$

Then, by [D], $z(x)$ is the increment of x for which $l(x + zt) - l(x) \approx t$ or, which is the same, $V(x + zt) \sim e^t V(x)$. If the argument of the function $z(\cdot)$ differs from x then we indicate it.

Theorem 5.2. *Assume that $\delta \in (0, 1)$ and $\varepsilon \in (0, 1)$ are fixed and $v \leq \frac{a(1-\delta)}{r}$. Then for $y \geq \varepsilon x$*

$$P(a, v) \leq c \min(z^{r+1}(y), n^r)V^r(y), \ r = \frac{x}{y}. \tag{5.28}$$

We note that estimate (5.28) is not uniprovable and that $z^{r+1}(y)$ can be replaced with $z(y)^r$. This circumstance relates to the fact that in the proof of the theorem we use coarse inequalities (5.33). The proof of a precise estimate requires extra effort. On the other hand, in the sequel for searching a precise asymptotics for $\mathbf{P}(\overline{S}_n(a) > x)$ (see [1]), inequality (5.28) turns out to be sufficient.

In connection with the indicated shortcoming of inequality (5.28), we cannot extract from it the following assertion whose proof is arranged otherwise.

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Theorem 5.3.

$$\mathbf{P}(\overline{S}_n(a) > x) \le cmV(x), \ m = \min(z, n), \ z = z(x). \tag{5.29}$$

To prove Theorem 5.2, we need an auxiliary assertion. Put

$$S(k, r) = \sum_{j=1}^{n} j^k V^r(y + vj). \tag{5.30}$$

Lemma 5.1.

$$S(k, r) \le ck! \min\left(A^{k+1}, \frac{n^{k+1}}{(k+1)!}\right) V^r(y), \tag{5.31}$$

where $A = \frac{z(y)}{rv}$, and c can be chosen arbitrarily close to 1 as $n \to \infty$.

PROOF. Obviously,

$$S(k, r) \le cI(k, r),$$

where

$$I(k, r) = \int_0^n t^k V^r(y + vt) dt = \frac{1}{v^{k+1}} \int_0^{nv} u^k V^r(y + u) du.$$

For $u \le nv = o(y)$ we have

$$V^r(y + u) = V^r(y) e^{-\frac{ur}{z(y)}(1 + o(1))}.$$

Since

$$\int_0^A t^k e^{-t} dt \le \min\left(k!, \frac{A^{k+1}}{k+1}\right),$$

it follows that

$$\int_0^{nv} u^k V^r(y + u) \, du \le V^r(y) \left(\frac{z(y)}{r}\right)^{k+1} \int_0^{\frac{nvr}{z(y)}} t^k e^{-t(1 + o(1))} \, dt$$

$$\le cV^r(y) \left(\frac{z(y)}{r}\right)^{k+1} \min\left[k!, \left(\frac{nvr}{z(y)}\right)^{k+1} \frac{1}{k+1}\right].$$

Clearly, this estimate, while proving (5.31), persists for arbitrary nv. The proof of the lemma is over.

PROOF OF THEOREM 5.2. For $n \le z(y)$ we have

$$\sigma(y) = nw(y) \le z(y) y^{-2} l(y) \sim \frac{1}{\alpha y}.$$

Therefore, Theorem 5.1 yields

$$P(a, v) \le \mathbf{P}\left(\overline{S}_n > x, \ \bigcap_{j=1}^{n} \{X_j < y + vn\}\right) \le c[nV(y_1)]^r,$$

where $y_1 = y + vn < y + vz(y) \sim y$. This implies that

$$P(a, v) \le c[nV(y)]^r.$$

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Now, let n be arbitrary. We first evaluate

$$\mathbf{P}(S_n - an > x; \ B(v)) \le \mathbf{P}\left(S_n > x + an; \ \bigcap_{j=1}^{n} \{X_j \le y + vn\}\right).$$

Apply Theorem 5.1, taking $x_1 = x + an$ as x and taking as y and r the respective quantities $y_1 = y + vn$ and $r_1 = \frac{x_1}{y_1}$ so that

$$r_1 \ge r \frac{x + an}{x + a(1 - \delta)n} \quad \text{for } v \le \frac{a(1 - \delta)}{r}. \tag{5.32}$$

By Theorem 5.1

$$\mathbf{P}(S_n - an > x; \ B(v)) \le c[nV(y_1)]^{r_1 - \frac{h\sigma_1}{4}},$$

where $\sigma_1 = nw(y_1) = ny_1^{\alpha - 2} L(y_1) = o\left(\frac{n}{x}\right)$ for $y \ge \varepsilon x$, $x \to \infty$. On the other hand,

$$r_1 \ge r \left(1 + f\left(\frac{n}{x}\right)\right),$$

where by (5.32)

$$f(t) = \frac{1 + at}{1 + at(1 - \delta)} - 1 = \frac{at\delta}{1 + at(1 - \delta)} \ge c \min(1, t).$$

Therefore, for all x large enough

$$r_1 - \frac{h\sigma_1}{2} \ge r, \quad \mathbf{P}(S_n - an > x; \ B(v)) \le c[nV(y + vn)]^r.$$

This allows us to evaluate

$$P(a, v) \le \sum_{k=1}^{n} \mathbf{P}(S_k - ak > x; \ B(v)) \le c \sum_{k=1}^{n} k^r V^r(y + vk) \le c_1 [\min(z(y), n)]^{r+1} V^r(y). \tag{5.33}$$

In the last inequality we have used Lemma 5.2. The proof of Theorem 5.2 is over.

PROOF OF THEOREM 5.3. For $n \le z$ the claim of the theorem follows from Corollary 5.1. Indeed, in this case

$$\sigma l(x) \le z l^2(x) x^{-2} \sim \frac{l(x)}{\alpha x} \to 0$$

and hence the conditions of the third assertion in item 1 of Corollary 5.1 are satisfied. Therefore,

$$\mathbf{P}(\overline{S}_n(a) > x) \le \mathbf{P}(\overline{S}_n > x) \le nV(x)(1 + o(1)).$$

For $n \ge z$ we use the results of [12] which imply that for $1 - F(t) = V(t)$

$$\mathbf{P}(\overline{S}_\infty(a) > x) = \frac{1}{a} \int_0^\infty V(x + t) dt (1 + o(1)).$$

But we only increase $\overline{S}_\infty(a)$ in distribution if instead of $[M^+]$ we assume that $1 - F(t) = V(t)$. Therefore (see Lemma 5.1),

$$\mathbf{P}(\overline{S}_n(a) > x) \le \mathbf{P}(\overline{S}_\infty(a) > x) \le czV(x),$$

which completes the proof of the theorem.

The claim of Theorem 5.3 can also be derived as a corollary to the results of [11], wherein relation (3.10) was established for the so-called strongly subexponential distributions. As was communicated to me by D. A. Korshunov, sufficient conditions for the membership in the class of subexponential distributions are satisfied whenever $[R]$ and $[D]$ are valid.

§6. Lower Estimates. Some Consequences for Regular Tails

In Sections §2–§5 we gave estimates for the distributions of $\overline{S}_n$ and $\overline{S}_n(a)$ from above. We now obtain estimates for the distributions of S_n from below. They are essentially simpler and more general.

6.1. A general lower estimate. Lower estimates in the case of $\mathbf{E}X_j^2 = \infty$. Here we do not need the assumptions of existence of regularly varying majorants or minorants. We put

$$\overline{F}(t) = 1 - F(t).$$

Theorem 6.1. Let $K(n)$ be an arbitrary sequence and let $Q_n(t) = \mathbf{P}\left(\frac{S_n}{K(n)} \leq -t\right)$. Then for $y = x + tK(n-1)$

$$\mathbf{P}(S_n > x) \geq n\overline{F}(y)\left(1 - Q_{n-1}(t) - \frac{n-1}{2}\overline{F}(y)\right).$$

PROOF. Put $G_n = \{S_n > x\}$ and $B_j = \{X_j \leq y\}$. Then

$$\mathbf{P}(S_n > x) \geq \mathbf{P}\left(G_n; \bigcup_{j=1}^{n} \overline{B}_j\right) \geq \sum_{j=1}^{n} \mathbf{P}(G_n\overline{B}_j) - \sum_{i<j} \mathbf{P}(G_n\overline{B}_i\overline{B}_j)$$

$$\geq \sum_{j=1}^{n} \mathbf{P}(G_n\overline{B}_j) - \frac{n(n-1)}{2}(\overline{F}(y))^2.$$

Here for $y = x + tK(n-1)$

$$\mathbf{P}(G_n\overline{B}_j) = \int_{y}^{\infty} dF(u)\mathbf{P}(S_{n-1} > x - u) \geq \mathbf{P}(S_{n-1} > x - y)\overline{F}(y) = \overline{F}(y)(1 - Q_{n-1}(t)).$$

The proof of the theorem is over.

Now, we find out conditions guaranteeing explicit estimates for $K(n)$ and $Q_n(t)$ in the case of $\mathbf{E}X_j^2 = \infty$.

We say that *condition* $[M_+^+]$ is satisfied if for some $c \geq 1$

$$V(t) \leq 1 - F(t) \leq cV(t), \tag{6.1}$$

where $V(t)$ is defined by (1.3). (If $c = 1$ then $[M_+^+]$ coincides with $[R]$.)

Also, we will use the condition

$[R_\rho]$. *Condition $[R]$ is satisfied for $\beta < 2$; moreover,*

$$\lim_{t\to\infty} \frac{F(-t)}{V(t)} = \rho, \quad 0 \leq \rho < \infty.$$

For $\rho = 0$ we assume condition $[M^-]$ satisfied.

When condition $[R_\rho]$ is satisfied, the normalized sums $\frac{S_n}{N(n)}$, $N(n) = V^{(-1)}\left(\frac{1}{n}\right)$, converge in distribution to the stable law F_β with parameter β (see [13]; recall that we assume that $\mathbf{E}X_j = 0$ for $\beta > 1$):

$$\mathbf{P}\left(\frac{S_n}{N(t)} < t\right) \Rightarrow F_\beta(t). \tag{6.2}$$

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Theorem 6.2. 1. *Assume that condition* $[M^-]$ *is satisfied with* $\alpha < 1$ *and that* $N_W(n) = W^{(-1)}\left(\frac{1}{n}\right)$. *Then for* $y = x + tN_W(n-1)$

$$\mathbf{P}(S_n > x) \geq n\overline{F}(y)\left(1 - ct^{-\alpha+\delta} - \frac{n-1}{2}n\overline{F}(y)\right) \tag{6.3}$$

for every fixed $\delta > 0$ *and a suitable* $c < \infty$.

Additionally assume satisfied conditions $[M_+^+]$. *Then for* $W(t) \leq c_1 V(t)$ *and* $x = sN(n) \to \infty$ $(N(n) = V^{(-1)}\left(\frac{1}{n}\right))$ *we have*

$$\mathbf{P}(S_n > x) \geq nV(x)(1 - \varphi(s)), \tag{6.4}$$

where $\varphi \downarrow 0$ *as* $s \uparrow \infty$.

2. *Assume that conditions* $[M^\pm]$ *are satisfied for* $\alpha \in (1, 2)$, $\mathbf{E}X_j = 0$, *and* $W(x) \leq c_1 V(x)$. *Then* (6.3) *holds for* $y = x + tN_W(n-1)$, $t \geq (\alpha + \delta)\frac{N(n-1)}{N_W(n-1)}$, *and every fixed* $\delta > 0$. *If moreover condition* $[M_+^+]$ *is satisfied then for* $x = sN(n)$

$$\mathbf{P}(S_n \geq x) \geq nV\left(x\left(1 + \theta\frac{\ln n}{s}\right)\right)(1 - \varphi(s)), \tag{6.5}$$

where $\theta = \frac{(\alpha+\delta)(\alpha-\beta)}{\alpha\beta}$. *For* $\alpha = \beta$ *the summand* $\theta \ln n$ *should be replaced with* $o(\ln n)$.

3. *If condition* $[R_\rho]$ *is satisfied with* $\rho > 0$ *and* $\beta < 2$ *then* (6.4) *holds. If condition* $[R_\rho]$ *is satisfied with* $\rho = 0$ *and* $\beta \in (1, 2)$, *and* $\mathbf{E}X_j = 0$ *then for* $x = sN(n)$, $n \to \infty$,

$$\mathbf{P}(S_n > x) \geq nV(x)[1 - F_\beta(0)(1 + o(1)) - \varphi(s)], \tag{6.6}$$

where $F_\beta(0) < 1$.

PROOF. We first assume that $\alpha < 1$. Put in Theorem 6.1 $K(n) = N_W(n)$. Then by Corollary 2.1 applied to the sums $-S_n$, we obtain

$$Q_n(t) = \mathbf{P}\left(-S_n \geq tN_W(n)\right) \leq c_1 nW(tN_W(n)) \leq ct^{-\alpha+\delta} \tag{6.7}$$

for every fixed $\delta > 0$ and all $t \geq 1$. This proves (6.3).

Now, suppose additionally that condition $[M_+^+]$ is satisfied, $W(t) \leq c_1 V(t)$, and $x = sN(n)$. Then $N_W(n) \leq c_2 N(n)$. Hence, for $t = s^{1-\delta}$, $\delta > 0$, we have $y = sN(n) + s^{1-\delta}N_W(n-1) \leq x\left(1 + c_2 s^{-\delta}\right)$, $\overline{F}(y) \geq V(x)(1 + \varphi(s))$, and $\varphi(s) \downarrow 0$ as $s \uparrow \infty$.

Choosing δ so as to have $(-\alpha + \delta)(1 - \delta) \leq -\frac{\alpha}{2}$ and using (6.1) for $\beta \leq \alpha$, from (6.3) we deduce (6.4).

Now, suppose that conditions $[M^\pm]$ are satisfied with $\alpha \in (1, 2)$, $\mathbf{E}X_j = 0$, and $W(x) < c_1 V(x)$. In view of Corollary 3.1, (6.3) then holds only for t such that

$$nV\left(\frac{tN_W(n)}{|\ln nW(tN_W(n))|}\right) \leq 1. \tag{6.8}$$

Since $nW(tN_W(n)) > t^{-\alpha-\delta}$, $\delta > 0$, as $tN_W(n) \to \infty$; it follows that $|\ln nW(tN_W(n))| < (\alpha + \delta)\ln t$ and (6.8) is satisfied if

$$\frac{t}{\ln t} = (\alpha + \delta)\frac{N(n)}{N_W(n)}.$$

This proves (6.3).

Observe that the last equality implies

$$\ln t \sim \frac{\alpha - \beta}{\alpha\beta}\ln n$$

(for $\alpha = \beta$ we understand this relation to be $\ln t = o(\ln n)$).

Now, we derive relations like (6.4). First, assume that $\alpha \neq \beta$. Then

$$y = sN(n) + tN_W(n-1) \leq sN(n) + (\alpha + \delta)N(n)\ln t \leq sN(n)\left(1 + \frac{(\alpha + \delta)(\alpha - \beta)}{\alpha\beta}\frac{\ln n}{s}\right).$$

This proves (6.5).

For $\alpha = \beta$, by the above $\theta \ln n$ should be replaced with $o(\ln n)$.

Now, consider the third assertion of the theorem when condition $[R_\rho]$ is satisfied.

If $\rho > 0$ then $W(t) \sim \rho V(t)$, $N_W(n) \sim \rho^\alpha N(n)$, and by Corollaries 2.1 and 3.1 relations (6.7), (6.3), and (6.4) hold again.

If $\rho = 0$ and $\beta > 1$ then

$$Q_n(t) \leq \mathbf{P}(S_n \leq 0) \to F_\beta(0)$$

as $n \to \infty$, where $F_\beta(0) < 1$ for $\beta > 1$, since the mean of the distribution F_β in this case equals 0.

The proof of the theorem is over.

We now derive some corollaries for regular tails.

Corollary 6.1. 1. *Assume that condition $[R_\rho]$ is satisfied; moreover, either $\rho > 0$ or $\alpha < 1$. Then for $x = sN(n)$ and $\Pi = \Pi(x) = nV(x)$*

$$\inf_{x:s>t} \frac{\mathbf{P}(S_n > x)}{\Pi} \geq 1 - \varphi(t), \tag{6.9}$$

$\varphi(t) \downarrow 0$ *as* $t \uparrow 0$.

2. *Assume that condition $[R_\rho]$ is satisfied with $\rho = 0$, $\alpha \in (1,2)$, and $\mathbf{E}X_j = 0$. Then*

$$\inf_{x:s>t} \frac{\mathbf{P}(S_n > x)}{\Pi} \geq 1 - \varphi\left(\frac{t}{\ln n}\right). \tag{6.10}$$

This corollary is obvious from Theorem 6.2.

Corollary 6.2. *Suppose that the conditions of item 1 of Corollary 6.1 are satisfied. Then there exists a function $\varphi(t) \downarrow 0$, $t \uparrow \infty$, such that*

$$\sup_{x:s>t} \left|\frac{\mathbf{P}(S_n > x)}{\Pi} - 1\right| \leq \varphi(t), \tag{6.11}$$

$$\sup_{x:s>t} \left|\frac{\mathbf{P}(\overline{S}_n > x)}{\Pi} - 1\right| \leq \varphi(t). \tag{6.12}$$

If the conditions of item 2 of Corollary 6.1 are satisfied then we cannot derive the convergence $\frac{\mathbf{P}(S_n > x)}{\Pi} \to 1$ as $s \to \infty$, $s \leq c\ln n$, from the obtained inequalities, since in this case the right-hand side of (6.10) does not converge to 1 in general.

PROOF. The claim of Corollary 6.2 ensues from Corollaries 2.1, 3.1, and 6.1.

The equivalence relation $\mathbf{P}(S_n > x) \sim nV(x)$ for $x = t_nN(n)$, $t_n \to \infty$, and under the condition $[R_\rho]$ of convergence of $\frac{S_n}{N(n)}$ in distribution to a stable law was obtained in [14–16]. A similar assertion for $\overline{S}_n$ follows from [17] but under more stringent assumptions on F (under the condition $F \in \mathscr{L}$, where $\mathscr{L}$ is defined in the next section).

6.2. Lower estimates in the case of $\mathbf{E}X_j^2 < \infty$. Corollaries for regular tails with power $\beta > 2$ and for semiexponential tails.

Theorem 6.3. *Assume that* $\mathbf{E}X_j = 0$ *and* $\mathbf{E}X_j^2 = 1$. *Then for* $y = x + u\sqrt{n-1}$

$$\mathbf{P}(S_n > x) \geq n\overline{F}(y) \left[1 - u^{-2} - \frac{n-1}{2}\overline{F}(y) \right]. \tag{6.13}$$

PROOF. The claim follows from Theorem 6.1 on putting $K(n) = \sqrt{n}$ and using the Chebyshev inequality which implies

$$Q_n(u) \leq u^{-2}.$$

Corollary 6.3. *Assume that the regularity condition* $[R]$ *is satisfied with* $\beta > 2$, $\mathbf{E}X_j^2 < \infty$, *and* $x = sN(n)$, $N(n) = \sqrt{(2-\beta)n\ln n}$. *Then there exists a function* $\varphi(t) \downarrow 0$, $t \uparrow \infty$, *such that*

$$\sup_{x:s\geq t} \left| \frac{\mathbf{P}(S_n > x)}{nV(x)} - 1 \right| \leq \varphi(t), \tag{6.14}$$

$$\sup_{x:s\geq t} \left| \frac{\mathbf{P}(\overline{S}_n > x)}{nV(x)} - 1 \right| \leq \varphi(t). \tag{6.15}$$

PROOF. Put in Theorem 6.3 $y = x + u\sqrt{n}$, $u = \sqrt{s}$. Then for every $\delta > 0$ and x large enough we have

$$y = x\left(1 + \frac{u}{s\sqrt{(\beta-2)\ln n}} \right),$$

$$\frac{V(y)}{V(x)} \geq \left(1 + \frac{uc_1}{s\sqrt{\ln n}} \right)^{-\beta-\delta} \geq 1 - \frac{c_2 u}{s\sqrt{\ln n}} = 1 - \frac{c_2}{\sqrt{s\ln n}}.$$

Furthermore,

$$nV(y) \leq nV(x) = nV(s\sqrt{(2-\beta)n\ln n}) \leq cns^{-\beta+\delta}(n\ln n)^{-\frac{\beta-\delta}{2}}.$$

Choosing $\delta < \beta - 2$, by Theorem 6.3 we obtain

$$\mathbf{P}(S_n > x) \geq nV(x)\left(1 - \frac{c_1}{\sqrt{s}} \right)\left(1 - \frac{c_2}{s} \right).$$

It remains to use Corollary 4.2 (see also Remark 4.1). The proof of the corollary is over.

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$$\mathbf{P}(S_n > x) \sim nV(x), \quad \mathbf{P}(\overline{S}_n > x) \sim nV(x)$$

under the conditions $[R]$ and $\beta > 2$ can be found, for instance, in [9, 18–20].

An analog of Corollary 6.3 is also valid for *semiexponential tails*.

Corollary 6.4. *Assume that condition* $[R]$ *is satisfied for semiexponential functions* $V(t)$ *of the form* (1.4) *and that* $\mathbf{E}X_j^2 < \infty$. *Let* $w_2^{(-1)}(\cdot)$ *be the inverse function of* $w_2(t) = t^{2\alpha-2}L^2(t)$ *and* $N_2(n) = w_2^{(-1)}\left(\frac{1}{n}\right)$ *(see* (5.9), (5.10)*). Put* $x = sN_2(n)$. *Then there exists a function* $\varphi(t) \downarrow 0$, $t \uparrow \infty$, *such that relations* (6.14) *and* (6.15) *hold.*

The equivalence relation $\mathbf{P}(S_n > x) \sim nV(x)$ for $x \gg N_2(n)$ was earlier established in [21].

Corollaries 6.2–6.4 establish *uniform* convergence in the corresponding limit theorems in the domain of all values n and x such that $x \geq tN(n)$, where $t \to \infty$ is an arbitrary fixed sequence tending to ∞ and $N(n)$ is an appropriate function defined above in each concrete case.

Proof of Corollary 6.4 is perfectly analogous to that of Corollary 6.3. We have to use Theorem 6.3 and Corollary 5.1 (also see Remark 5.1).

6.3. Lower estimates for the distribution of $\overline{S}_n(a)$. Consider the case in which

$$d^b = \mathbf{E}|X_1|^b < \infty, \quad 1 < b \leq 2, \quad \mathbf{E}X_1 = 0. \tag{6.16}$$

Put

$$Z_b(x,t) = \sum_{j=1}^{n} \overline{F}(x + aj + td(j-1)^{1/b}).$$

Obviously, $Z_b(x,t) \to 0$ as $x \to \infty$.

Theorem 6.4. *For all n, x, and t*

$$\mathbf{P}(\overline{S}_n(a) > x) \geq Z_b(x,t)[1 - (1 + \varphi(t))t^{-b} - Z_b(x,t)], \tag{6.17}$$

where $\varphi(t) \equiv 0$ for $b = 2$ and $\varphi(t) \to 0$ as $t \to \infty$ for $b < 2$.

Put

$$I_b(x,t) = \int_{1}^{n+1} F(x + au + td(u-1)^{1/b}) \leq Z_b(x,t).$$

It is easy to indicate values $t_0 > 1$ and $z_0 = z_0(t_0) > 0$ such that for $Z_b(x,t) < z_0$ and $t > t_0$

$$\mathbf{P}(\overline{S}_n(a) > x) \geq I_b(x,t)[1 - (1 + \varphi(t))t^{-b} - I_b(x,t)]. \tag{6.18}$$

Indeed, the function $g(z) = z(1 - ct^{-b} - z)$ for $t > t_0 > 1$ and $c = \max_t(1 + \varphi(t))$ is monotone increasing on $[0, z_0]$, where $z_0 = z_0(t_0) > 0$. Hence, for $Z_b(x,t) < z_0$ the right-hand side of (6.17) is greater than

$$I_b(x,t)(1 - (1 + \varphi(t))t^{-b} - I_b(x,t)).$$

The values t_0 and z_0 can be evaluated explicitly. For example, for $b = 2$ we can take $t_0 = 2$ and $z_0 = 3/8$.

Corollary 6.5. *Assume that (6.16) is satisfied together with condition $[M_+]$, where the function V is of the form (1.3) or (1.4) and l satisfies $[D]$. Then*

$$\mathbf{P}(\overline{S}_n(a) > x) \geq \frac{1}{a} \int_{x}^{x+an} V(u)\, du(1 + o(1)) \tag{6.19}$$

as $x \to \infty$.

PROOF OF COROLLARY 6.5. Put in (6.18) $t = \ln x$. Then

$$I_b(x,t) \geq \int_{1}^{n+1} V(x + au + d(u-1)^{1/b} \ln x)\, du = \frac{1}{a} \int_{x}^{x+an} V(u)\, du(1 + o(1))$$

as $x \to \infty$. Since $I_b(x,t) \to 0$ as $x \to \infty$, (6.18) implies (6.19).

PROOF OF THEOREM 6.4. Put

$$G_n = \{\overline{S}_n(a) > x\}, \quad B_j = \{X_j \leq x + aj + td(j-1)^{1/b}\}.$$

Using the same arguments as in Theorem 6.1, we then obtain

$$\mathbf{P}(G_n) \geq \sum_{j=1}^{n} \mathbf{P}(G_n \overline{B}_j) - \left(\sum_{j=1}^{n} \mathbf{P}(\overline{B}_j)\right)^2. \tag{6.20}$$

Here

$$\mathbf{P}(G_n\overline{B}_j) > \mathbf{P}(S_{j-1} > -t(d(j-1))^{1/b}; \overline{B}_j)$$
$$= \overline{F}(x + aj + td(j-1)^{1/b})[1 - \mathbf{P}(S_{j-1} \leq -td(j-1)^{1/b})]. \tag{6.21}$$

If $b = 2$ then Chebyshev's inequality yields

$$\mathbf{P}(S_{j-1} < -td(j-1)^{1/2}) \leq t^{-2}.$$

If $b < 2$ then conditions $[M^{\pm}]$ are satisfied with $\alpha = \beta = b$ and $V(t) = W(t) = d^b t^{-b}$. Therefore, by Corollary 3.1

$$\mathbf{P}(S_{j-1} < -td(j-1)^{1/b}) \leq (1 + \varphi(t))(j-1)W(td(j-1)^{1/b}) = (1 + \varphi(t))t^{-b},$$

where $\varphi(t) \to 0$ as $t \to \infty$. From (6.20), (6.21), and what was said above, we infer that

$$\sum_{j=1}^{n}\mathbf{P}(G_n\overline{B}_j) \geq Z_b(x,t)\big(1 - (1 + \varphi(t))t^{-b}\big),$$

$$\sum_{j=1}^{n}\mathbf{P}(\overline{B}_j) = Z_b(x,t), \quad \mathbf{P}(G_n) \geq Z_b(x,t)(1 - (1 + \varphi(t))t^{-b} - Z_b(x,t)),$$

completing the proof of the theorem.

Corollary 6.5 can also be derived from [11]. As mentioned, the results of [11] imply that if condition $[R]$ is satisfied and the functions V are of the form (1.3) or (1.4), where l satisfies $[D]$, then the asymptotic representation (3.10) is valid and implies (6.19).

§7. Uniform Relative Convergence to a Stable Law. The Law of the Iterated Logarithm in the Case of $\mathbf{E}X_j^2 = \infty$

In this section we give some consequences of the estimates of §2, §3, and §6.

7.1. Uniform relative convergence to a stable law. Denote by $\mathscr{L}$ the class of distributions F satisfying $[R_\rho]$ and such that $L(t) \to L = \mathrm{const}$ as $t \to \infty$. For the distributions in $\mathscr{L}$, the inverse function $V^{(-1)}$ has a simple explicit asymptotics:

$$V^{(-1)}\left(\frac{1}{n}\right) = N(n) \sim (Ln)^{1/\beta}. \tag{7.1}$$

Clearly, the stable distribution F_β in (6.1) also belongs to $\mathscr{L}$; moreover, for every $F \in \mathscr{L}$

$$nV(vN(n)) \sim nv^{-\beta}(Ln)^{-1}L = v^{-\beta}. \tag{7.2}$$

The class $\mathscr{L}$ is nothing but the *domain of normal attraction of the stable law* F_β (see [13]).

Property (7.2) enables us to obtain the following assertion about *uniform relative convergence to a stable law.*

Theorem 7.1. *Assume that* $[R_\rho]$ *is satisfied; moreover,* $\rho > 0$ *or* $\alpha < 1$. *In this case* $F \in \mathscr{L}$ *if and only if*

$$\sup_{t \geq 0}\left|\frac{\mathbf{P}\big(\frac{S_n}{N(n)} > t\big)}{1 - F_\beta(t)} - 1\right| \to 0 \tag{7.3}$$

as $n \to \infty$

The claim of the theorem means that for $F \in \mathscr{L}$ the problem of large deviations for $\mathbf{P}(S_n > x)$ is in a sense absent: the limit law $1 - F_\beta(t)$ guarantees the good approximation

$$\mathbf{P}(S_n > x) \sim 1 - F_\beta(x/N(n))$$

uniformly in all $x \geq 0$. This is possible in the central limit theorem on convergence to a normal law only if X_j have exactly a normal distribution.

An assertion like (7.3) (with an estimate for the convergence rate) follows also from the results of [22], but under the considerably more stringent assumption of existence of the pseudomoments

$$\int |t|^\gamma |F - F_\beta|(dt) < \infty$$

of order $\gamma > \beta$, which necessitates a high rate for the convergence of $F(t) - F_\beta(t)$ to 0.

PROOF OF THEOREM 7.1. *Sufficiency*: Suppose that $F \in \mathscr{L}$. Corollary 6.2 implies (see (6.11)) that for every sequence $t \to \infty$ and $x = sN(n)$

$$\sup_{s>t} \left| \frac{\mathbf{P}(S_n > x)}{nV(x)} - 1 \right| \to 0. \tag{7.4}$$

If $F = F_\beta$ then by (6.2) for every fixed s

$$1 - F_\beta(s) \sim \mathbf{P}(S_n > sN(n)), \quad n \to \infty,$$

where by (7.2) and (7.4) the right-hand side is close to $s^{-\beta}$ for large s. This implies that $1 - F_\beta(s) \sim s^{-\beta}$ as $s \to \infty$, and (7.4) can also be written down as

$$\sup_{s>t} \left| \frac{\mathbf{P}(S_n > sN(n))}{1 - F_\beta(s)} - 1 \right| \to 0 \tag{7.5}$$

as $n \to \infty$, $t = t_n \to \infty$. On the other hand, from the weak convergence (6.2) and continuity of F_β it follows that for every $t > 0$

$$\sup_{s \leq t} \left| \frac{\mathbf{P}(S_n > sN(n))}{1 - F_\beta(s)} - 1 \right| \to 0. \tag{7.6}$$

This means that there exists an increasing sequence $t_n \to \infty$ of sufficiently slow growth such that (7.6) remains valid after the replacement of t with t_n. Together with (7.5), this proves (7.3).

Necessity: From (7.3) and (7.4) we have

$$nV(tN(n)) \sim ct^{-\beta}$$

or, which is the same,

$$V(tN) \sim t^{-\beta}V(N), \quad L(tN) \sim L(N) \tag{7.7}$$

for arbitrary sequences of t and N. But this is possible only when $L(N) \to L = $ const. If we assume the contrary, for example, assume that $L(N) \to \infty$ as $N \to \infty$, then we can choose a sequence N' such that

$$L(N') > L^2(N). \tag{7.8}$$

On putting in (7.7) $t = \frac{N'}{N}$, we obtain $L(N') \sim L(N)$, which contradicts (7.8). The proof of the theorem is over.

REMARK 7.1. From the proof of the theorem and Corollary 6.1 we see that in the case of $\rho = 0$, $\alpha \in (1, 2)$, and $\mathbf{E}X_j = 0$ relation (7.3) persists if in it we replace $\sup_{t \geq 0}$ with $\sup_{t \in B_n}$, where $B_n = $

$(0, \infty) \setminus (t_n, t_n \ln n)$ and $t_n \to \infty$ sufficiently slowly. Seemingly, convergence in the interval $(t_n, t_n \ln n)$ can be obtained by using estimates for the rate of the convergence of $\mathbf{P}(S_n/N(n) > v)$ to $F_\beta(v)$ (cf. [22]).

An analog of Theorem 7.1 can be obtained for the distribution of $\overline{S}_n$ as well.

First of all, observe that the "invariance principle" in the domain of convergence to stable laws implies that

$$\frac{\overline{S}_n}{N(n)} \Rightarrow \bar{\zeta}(1), \tag{7.9}$$

where $\zeta(u)$ is the stable process corresponding to the distribution F_β ($\zeta(1) \in F_\beta$), $\bar{\zeta}(t) = \sup_{u \le t} \zeta(u)$. Denote the distribution function of $\bar{\zeta}(1)$ by H_β. Then, by analogy to the above, Corollary 6.2, Theorem 7.1, (7.2), and the fact that $\overline{S}_n \ge S_n$ imply that

$$1 - H_\beta(t) \sim t^{-\beta} \quad \text{as } t \to \infty. \tag{7.10}$$

Note that convergence (7.9) can be also deduced from the results of [23]; an explicit form of H_β is found in the same article.

Theorem 7.2. *Suppose that the conditions of Theorem 7.1 are satisfied. In this case $F \in \mathscr{L}$ if and only if*

$$\sup_{t>0} \left| \frac{\mathbf{P}(\overline{S}_n > tN(n))}{1 - H_\beta(t)} - 1 \right| \to 0$$

as $n \to \infty$.

The proof of Theorem 7.2 repeats that of Theorem 7.1. We merely have to replace S_n with $\overline{S}_n$ and F_β with H_β throughout.

7.2. Laws of the iterated logarithm in the case when the second moment is infinite.
The above-established upper and lower estimates for the distributions of $\overline{S}_n$ and S_n allow us to obtain assertions like the law of the iterated logarithm for the sequence $\{S_n\}$ in the case when $\mathbf{E}X_j^2 = \infty$.

Theorem 7.3. 1. *Assume that condition $[M^+]$ is satisfied with $\alpha < 1$. Then for every $\varepsilon > 0$*

$$\limsup_{n\to\infty} \frac{S_n}{N(n)(\ln n)^{\frac{1+\varepsilon}{\beta}}} < 1 \quad a.s. \tag{7.11}$$

2. *Assertion (7.11) persists if the conditions $[M^\pm]$, $\beta > 1$, $\mathbf{E}X_j = 0$, and $W(t) \le c_1 V(t)$ are satisfied.*

3. *Assume that conditions $[M^-]$ with $\alpha < 1$ and $[M_+^+]$ are satisfied and that $W(t) \le c_1 V(t)$. Then for every $\varepsilon > 0$*

$$\limsup_{n\to\infty} \frac{S_n}{N(n)(\ln n)^{\frac{1-\varepsilon}{\beta}}} > 1 \quad a.s. \tag{7.12}$$

4. *Assertion (7.12) persists if the conditions $[R_\rho]$, $\beta > 1$, and $\mathbf{E}X_j = 0$ are satisfied.*

Denote $\ln^+ t = \ln\max(1, t)$. Theorem 7.3 yields

Corollary 7.1. 1. *Assume that the conditions $[M^-]$, $[M_+^+]$ with $\alpha < 1$, and $W(t) \le c_1 V(t)$ are satisfied. Then*

$$\limsup_{n\to\infty} \frac{\ln^+ S_n - \ln N(n)}{\ln \ln n} = \frac{1}{\beta} \quad a.s. \tag{7.13}$$

2. *Assertion (7.13) persists if the conditions $[R_\rho]$, $\beta > 1$, and $\mathbf{E}X_j = 0$ are satisfied.*

Relation (7.13) can be rewritten as

$$\limsup_{n\to\infty} \left(\frac{S_n}{N(n)} \right)^{\frac{1}{\ln \ln n}} = e^{\frac{1}{\beta}} \quad a.s. \tag{7.14}$$

If $V(t) = t^{-\beta}L(t)$, $|\ln L(t)| \ll \ln\ln t$, then the function $L_1(n)$ in the representation $N(n) = n^{1/\beta}L_1(n)$ possesses a similar property and we can replace $N(n)$ in relations (7.11)–(7.14) with $n^{1/\beta}$.

The statement (7.13) to some extent justifies the term "law of the iterated logarithm," since it involves the normalizing factor $\ln\ln n$ (in (2.23) and (2.24) for the *very sums* S_n (rather than for $\ln^+ S_n$) it is absent). There are many articles devoted to the law of the iterated logarithm in the case of $\mathbf{E}X_j^2 = \infty$ (see a bibliography, for example, in [24, 25]; however, for derivation of (7.14) they presume rather rigid conditions on X_j, for example, membership in the domain of normal attraction to a stable law ($F \in \mathscr{L}$) (see [25]). Theorem 7.1 generalizes these results.

PROOF OF THEOREM 7.3. If we follow the classical scheme of the proof of the laws of the repeated logarithm which are based on the use of the Borel–Cantelli lemma (see, for instance, [26]), then the problem is reduced to the following (see Chapter 19 of [26]): to prove (7.11), we have to demonstrate that

$$\sum_k \mathbf{P}(\overline{S}_{n_k} > x_k) < \infty, \tag{7.15}$$

where $n_k = [A^k]$ $(A > 1)$ and $x_k = N(n_k)(\ln n_k)^{\frac{1+\varepsilon}{\beta}}$. To prove (7.12), we have to establish that

$$\sum_k \mathbf{P}(S_{n_k} - S_{n_{k-1}} > y_k) = \infty$$

or, which is the same, that

$$\sum_k \mathbf{P}(S_{m_k} > y_k) = \infty, \tag{7.16}$$

where $m_k = n_k - n_{k-1} = [n_k(1 - A^{-1})] + i$, i takes the values 0 and 1, and $y_k = N(n_k)(\ln n_k)^{\frac{1-\varepsilon}{\beta}}$.

Prove (7.15) and (7.11). By Corollary 2.1, for $x > N(n)$

$$\mathbf{P}(\overline{S}_n > x) \le cnV(x).$$

Putting $x = N(n)(\ln n)^{\frac{1+\varepsilon}{\beta}}$, we obtain

$$\mathbf{P}(\overline{S}_n > x) \le c(\ln n)^{-\frac{1+\varepsilon}{\beta}(\beta-\delta)}$$

for $n \to \infty$ and every fixed $\delta > 0$. Putting $\delta = \varepsilon/3$, for ε small enough we have

$$\frac{1+\varepsilon}{\beta}(\beta - \delta) > 1 + \varepsilon/2, \quad \mathbf{P}(\overline{S}_{n_k} > x_k) \le c_1 k^{-(1+\varepsilon/2)}.$$

This means that the series (7.15) converges and (7.11) holds.

The proof of the second assertion proceeds in exactly the same way but on using Corollary 3.1 whose conditions are satisfied.

Now, prove (7.16) and (7.12). By assertion (6.3) of Theorem 6.2, for $x > N(n)$ and $m = [n(1 - A^{-1})]$ we have

$$\mathbf{P}(S_m > x) \ge cnV(x).$$

Putting $x = N(n)(\ln n)^{\frac{1-\varepsilon}{\beta}}$, we obtain

$$\mathbf{P}(S_m > x) \ge (\ln n)^{-\frac{1-\varepsilon}{\beta}(\beta+\delta)},$$

where $\frac{1-\varepsilon}{\beta}(\beta + \delta) < 1 - \varepsilon/2$ for $\delta = \varepsilon/2$ and ε small enough. This gives

$$\mathbf{P}(S_{m_k} > y_k) \ge c_1 k^{-(1-\varepsilon/2)}$$

which implies convergence of the series (7.16) and validity of (7.12).

The last assertion of the theorem is proved in exactly the same manner on using the third assertion of Theorem 6.2.

The proof of the theorem is over.

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