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A. Ladneva, V. Piterbarg
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Anna Ladneva and Vladimir Piterbarg
Faculty of Mechanics and Mathematics
Moscow Lomonosov state university

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Abstract

We consider a Gaussian stationary process with Pickands’ conditions and evaluate an exact asymptotic behavior of probability of two high extremes on two disjoint intervals.

1 Introduction. Main results.

Let $X(t)$, $t \in \mathbb{R}$, be a zero mean stationary Gaussian process with unit variance and covariance function $r(t)$. An object of our interest is the asymptotic behaviour of the probability

$$P_d(u; [T_1, T_2], [T_3, T_4]) = \mathbb{P}\left( \max_{t \in [T_1, T_2]} X(t) > u, \max_{t \in [T_3, T_4]} X(t) > u \right)$$

as $u \to \infty$, where $[T_1, T_2]$ and $[T_3, T_4]$ are disjoint intervals. To evaluate the asymptotic behaviour we develop an analogue of Pickands’ theory of high extremes of Gaussian processes, see [1] and extensions in [2]. We follow main steps of the theory. First we assume an analogue of the Pickands’ conditions.

A1 For some $\alpha \in (0, 2)$,

$$r(t) = 1 - |t|^\alpha + o(|t|^\alpha) \quad \text{as} \quad t \to 0,$$

$$|r(t)| < 1 \quad \text{for all} \quad t > 0.$$

Then, we specify covariances between values of the process on intervals $[T_1, T_2]$ and $[T_3, T_4]$. We assume that there is an only domination point of correlation between the values. This makes some similarity with Prirabarg & Prisyazhn'uck’s extension of the Pickands’ theory to non-stationary Gaussian processes.

A2 In the interval $S = [T_3 - T_2, T_4 - T_1]$ there exists only point $t_m = \arg \max_{t \in S} r(t) \in (T_3 - T_2, T_4 - T_1)$, $r(t)$ is twice differentiable in a neighbourhood of $t_m$ with $r''(t_m) \neq 0$.

As an alternative of assumption A2 one can suppose that the point of maximum of $r(t)$ is one of the end points of $S$, $T_3 - T_2$ is more natural candidate.

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\textbf{A3} \( r(t) \) is continuously differentiable in a neighbourhood of the point \( t_m = T_3 - T_2 \), \( r'(t_m) < 0 \) and \( r(t_m) > r(t) \) for all \( t \in (T_3 - T_2, T_4 - T_1) \).

\textbf{A3'} \( r(t) \) is continuously differentiable in a neighbourhood of the point \( t_m = T_4 - T_1 \), \( r'(t_m) > 0 \) and \( r(t_m) > r(t) \) for all \( t \in (T_3 - T_2, T_4 - T_1) \).

Denote by \( B_\alpha(t), t \in \mathbb{R} \), a normed fractional Brownian motion with the Hurst parameter \( \alpha/2 \), that is a Gaussian process with a.s. continuous trajectories, \( B_\alpha(0) = 0 \) a.s., \( \mathbb{E}B_\alpha(t) = 0 \), and \( \mathbb{E}(B_\alpha(t) - B_\alpha(s))^2 = 2|t-s|^\alpha \). For any set \( T \subset \mathbb{R} \) we denote

\[ H_\alpha(T) = \mathbb{E} \exp \left( \sup_{t \in T} B_\alpha(t) - |t|^\alpha \right). \]

It is known, [1], [2], that there exists a positive and finite limit

\[ H_\alpha := \lim_{T \to \infty} \frac{1}{T} H_\alpha([0,T]), \tag{1} \]

the Pickands' constant. Further, for a number \( c \) denote

\[ H^c_\alpha(T) = \mathbb{E} \exp \left( \sup_{t \in T} B_1(t) - |t| - ct \right). \]

It is known, [2], that for any positive \( c \), the limit \( H^c_\alpha := \lim_{T \to \infty} H^c_\alpha([0,T]) \) exists and is positive. We stand \( a \lor b \) for \( \max(a,b) \) and \( a \land b \) for \( \min(a,b) \). Denote

\[ p_2(u,r) = \frac{(1 + r)^2}{2\pi u^2 \sqrt{1 - r^2}} e^{-\frac{u^2}{4(1 + r)}} \]

and notice that for a Gaussian vector \((\xi, \eta)\) where the components are standard Gaussian and correlation between them is \( r \), \( \mathbb{P}(\xi > u, \eta > u) = p_2(u,r)(1 + o(1)) \) as \( u \to \infty \).

\textbf{Theorem 1} Let \( X(t), t \in \mathbb{R} \), be a Gaussian centred stationary process with a.s. continuous trajectories. Let assumptions \textbf{A1} and \textbf{A2} be fulfilled for its covariance function \( r(t) \). Then

\[ P_d(u;[T_1,T_2],[T_3,T_4]) = K \sqrt{\pi A^{-1}} (1 + r(t_m))^{-4/\alpha} H_\alpha^2 u^{-3 + 4/\alpha} p_2(u,r(t_m))(1 + o(1)) \]

as \( u \to \infty \), where \( K = T_2 \land (T_4 - t_m) - T_1 \lor (T_3 - t_m) > 0 \),

\[ A = -\frac{1}{2} \frac{r''(t_m)}{(1 + r(t_m))^2}. \]

\textbf{Theorem 2} Let \( X(t), t \in \mathbb{R} \), be a Gaussian centred stationary process with a.s. continuous trajectories. Let assumptions \textbf{A1} and \textbf{A3} or \textbf{A3'} be fulfilled for its covariance function \( r(t) \). Then,

(i) for \( \alpha > 1 \),

\[ P_d(u;[T_1,T_2],[T_3,T_4]) = p_2(u,r(t_m))(1 + o(1)) \]
as $u \to \infty$.

(ii) For $\alpha = 1$,

$$P_d(u; [T_1, T_2], [T_3, T_4]) = \left(H_{1}^{r'(t_m)}\right)^2 p_2(u, r(t_m))(1 + o(1))$$

as $u \to \infty$.

(iii) For $\alpha < 1$,

$$P_d(u; [T_1, T_2], [T_3, T_4]) = B^{-2}(1 + r(t_m))^{-4/\alpha}H_\alpha^2 u^{-6+4/\alpha}p_2(u, r(t_m))(1 + o(1))$$

as $u \to \infty$, where

$$B = \frac{r'(t_m)}{(1 + r(t_m))^2}.$$

2 Lemmas

For a set $A \subset \mathbb{R}$ and a number $a$ we write $aA = \{ax : x \in A\}$ and $a + A = \{a + x : x \in A\}$.

Lemma 1 Let $X(t)$ be a Gaussian process with mean zero and covariance function $r(t)$ satisfying assumptions A1, A2. Let a time moment $\tau = \tau(u)$ tends to $t_m$ as $u \to \infty$ in such a way that $|\tau - t_m| \leq C \sqrt{\log u}/u$, for some positive $C$. Let $T_1$ and $T_2$ be closures of two bounded open subsets of $\mathbb{R}$. Then

$$P\left(\max_{t \in u^{-2/\alpha}T_1} X(t) > u, \max_{t \in \tau + u^{-2/\alpha}T_2} X(t) > u\right) = \frac{(1 + r(\theta))^2}{2\pi u^2 \sqrt{1 - r^2(\theta)}} \frac{u^2}{H_\alpha\left(\frac{T_1}{(1 + r(\theta))^{2/\alpha}}\right)H_\alpha\left(\frac{T_2}{(1 + r(\theta))^{2/\alpha}}\right)}(1 + o(1)), \quad (2)$$

as $u \to \infty$, where $\theta = t_m$.

Lemma 2 Let $X(t)$ be a Gaussian process with mean zero and covariance function $r(t)$ satisfying assumptions A1, A2 with $\alpha < 1$. Let $T_1$ and $T_2$ be closures of two bounded open subsets of $\mathbb{R}$. Then, for any (fixed) $\tau > 0$ the asymptotic relation of Lemma 1 holds true with $\theta = \tau$.

Lemma 3 Let $X(t)$ be a Gaussian process with mean zero and covariance function $r(t)$ satisfying assumptions A1, A2 with $\alpha = 1$. Let $T_1$ and $T_2$ be closures of two bounded open subsets of $\mathbb{R}$. Then

$$P\left(\max_{t \in u^{-2}T_1} X(t) > u, \max_{t \in \tau + u^{-2}T_2} X(t) > u\right) = H_{1}^{r'(\tau)} \left(\frac{T_1}{(1 + r(\tau))^2}\right)H_{1}^{-r'(\tau)} \left(\frac{T_2}{(1 + r(\tau))^2}\right)p_2(u, r(\tau))(1 + o(1)), \quad (3)$$

as $u \to \infty$. 

3
Proof of Lemmas 1 - 3. We prove the three lemmas simultaneously, computations of conditional expectation (4) and related evaluations are performed in parallel, separately for each lemma. We have for $u > 0$,

$$P = \mathbb{P} \left( \max_{t \in \mathbb{R}^2 / \mathcal{Q}_1} X(t) > u, \max_{t \in \mathbb{R}^2 / \mathcal{Q}_2} X(t) > u \right) = \int \int P \left( \max_{t \in \mathbb{R}^2 / \mathcal{Q}_1} X(t) > u, \max_{t \in \mathbb{R}^2 / \mathcal{Q}_2} X(t) > u \mid X(0) = a, X(\tau) = b \right) P_0 r(a, b) dadb,$$

where

$$P_0 r(a, b) = \frac{1}{2\pi \sqrt{1 - r^2(\tau)}} \exp \left( -\frac{1}{2} \cdot \frac{a^2 - 2r(\tau)ab + b^2}{1 - r^2(\tau)} \right).$$

Now we change variables, $a = u - x/u$, $b = u - y/u$,

$$P_0 r(x, y) = \frac{1}{2\pi \sqrt{1 - r^2(\tau)}} \frac{1}{u^2} \exp \left( -\frac{1}{2} \cdot \frac{(u - x/u)^2 - 2r(\tau)(u - x/u)(u - y/u) + (u - y/u)^2}{1 - r^2(\tau)} \right) \times$$

$$\times \exp \left( -\frac{1}{2} \cdot \frac{x^2 + y^2 - 2x - 2y + 2r(\tau)(x + y) - 2r(\tau) y^2}{1 - r^2(\tau)} \right) \times \exp \left( -\frac{1}{2} \cdot \frac{u^2}{1 + r(\tau)} \right) \cdot \tilde{P}(u, x, y).$$

Hence,

$$P = \frac{1}{2\pi \sqrt{1 - r^2(\tau)}} \frac{1}{u^2} \exp \left( -\frac{u^2}{1 + r(\tau)} \right) \int \int P \left( \max_{t \in \mathbb{R}^2 / \mathcal{Q}_1} X(t) > u, \max_{t \in \mathbb{R}^2 / \mathcal{Q}_2} X(t) > u \mid X(0) = u - x/u, X(\tau) = u - y/u \right) \tilde{P}(u, x, y) dx dy.$$

Consider the following families of random processes,

$$\xi_u(t) = u \left( X(u^{-2/\alpha t}) - u \right) + x, \quad t \in T_1,$$

$$\eta_u(t) = u \left( X(\tau + u^{-2/\alpha t}) - u \right) + y, \quad t \in T_2.$$

We have,

$$P = \frac{1}{2\pi \sqrt{1 - r^2(\tau)}} \frac{1}{u^2} \exp \left( -\frac{u^2}{1 + r(\tau)} \right) \int \int P \left( \max_{t \in \mathcal{Q}_1} \xi_u(t) > x, \max_{t \in \mathcal{T}_2} \eta_u(t) > y \mid X(0) = u - x/u, X(\tau) = u - y/u \right) \tilde{P}(u, x, y) dx dy.$$
Compute first two conditional moments of Gaussian random vector process $(\xi_u(t), \eta_u(t))^T$. We have

\[
E \left( \begin{array}{c}
\xi_u(t) \\
\eta_u(t)
\end{array} \mid X(0), X(\tau) \right) = E \left( \begin{array}{c}
\xi_u(t) \\
\eta_u(t)
\end{array} \right) + A \left( \begin{array}{c}
X(0) \\
X(\tau)
\end{array} \right),
\]

where

\[
A = \text{cov} \left( \begin{array}{c}
(\xi_u(t)) \\
(X(0))
\end{array} \right) \left[ E \left( \begin{array}{c}
X(0) \\
X(\tau)
\end{array} \right) \left( \begin{array}{c}
X(0) \\
X(\tau)
\end{array} \right)^T \right]^{-1},
\]

or

\[
A = \frac{u}{1 - r^2(\tau)} \begin{pmatrix}
r(u^{-2/\alpha t}) - r(\tau)r(\tau - u^{-2/\alpha t}) & r(\tau - u^{-2/\alpha t}) - r(\tau)r(u^{-2/\alpha t}) \\
r(\tau + u^{-2/\alpha t}) - r(\tau)r(u^{-2/\alpha t}) & r(u^{-2/\alpha t}) - r(\tau)r(\tau + u^{-2/\alpha t})
\end{pmatrix}.
\]

We denote $\text{cov} X$, the matrix of covariances of a vector $X$ and $\text{cov}(X, Y)$, the matrix of cross-covariances between components of $X$ and $Y$. Substituting the values $X(0) = u - x/u$, $X(\tau) = u - y/u$, of the conditions, we get from here that

\[
E \left( \begin{array}{c}
\xi_u(t) \\
\eta_u(t)
\end{array} \mid X(0) = u - x/u, X(\tau) = u - y/u \right) = \begin{pmatrix}
\frac{1}{1 - r^2(\tau)} (r(u^{-2/\alpha t}) (u^2 - x - r(\tau)(u^2 - y)) + r(\tau - u^{-2/\alpha t}) (u^2 - y - r(\tau)(u^2 - x))) - u^2 + x \\
\frac{1}{1 - r^2(\tau)} (r(u^{-2/\alpha t}) (u^2 - y - r(\tau)(u^2 - x)) + r(\tau + u^{-2/\alpha t}) (u^2 - x - r(\tau)(u^2 - y))) - u^2 + y
\end{pmatrix}.
\]

(4)

In conditions of every lemma 1-3 we have

\[
E \left( \begin{array}{c}
\xi_u(t) \\
\eta_u(t)
\end{array} \mid X(0) = u - x/u, X(\tau) = u - y/u \right) = \begin{pmatrix}
\frac{1}{1 + r(\tau)} |t|^\alpha + o(1) + u^2 r(\tau - u^{-2/\alpha t}) - r(\tau) + (y - x r(\tau)) r(\tau - u^{-2/\alpha t}) \frac{1}{1 + r(\tau)} \\
\frac{1}{1 + r(\tau)} |t|^\alpha + o(1) + u^2 r(\tau + u^{-2/\alpha t}) - r(\tau) + (y - x r(\tau)) r(\tau + u^{-2/\alpha t}) \frac{1}{1 + r(\tau)}
\end{pmatrix}
\]

(5)

as $u \to \infty$.

Now, let conditions of the Lemma 1 be fulfilled. Since $\alpha < 2$ and $r'(\tau) = O(\sqrt{\log u}/u)$ uniformly in $|\tau - t_m| \leq C\sqrt{\log u}/u$, we have,

\[
\left| u^2 r(\tau - u^{-2/\alpha t}) - r(\tau) \right| \leq \max_{|\tau - t_m| \leq C\sqrt{\log u}/u} \left| u^2 (-u^{-2/\alpha t}) + r'(\tau) \right| = o(1).
\]

(6)

Thus

\[
E \left( \begin{array}{c}
\xi_u(t) \\
\eta_u(t)
\end{array} \mid X(0) = u - x/u, X(\tau) = u - y/u \right) = \begin{pmatrix}
\frac{1}{1 + r(\tau)} |t|^\alpha + o(1) \\
\frac{1}{1 + r(\tau)} |t|^\alpha + o(1)
\end{pmatrix}
\]

(7)

as $u \to \infty$.

Let now the conditions of Lemma 2 be fulfilled, that is $\alpha < 1$. In this situation even for fixed $\tau$, by Taylor, the third terms in the column array of right-hand part of (5) tend to zero as $u \to \infty$, hence (7) takes place, with $\theta = \tau$.

Next, let $\alpha = 1$, by differentiability of $r$,

\[
u^2 (r(\tau - u^{-2}t) - r(\tau)) \to -tr'(\tau) \quad \text{and} \quad u^2 (r(\tau + u^{-2}t) - r(\tau)) \to tr'(\tau)
\]
as \( u \to \infty \), therefore in conditions of Lemma 3,
\[
\mathbb{E} \left( \frac{\xi_u(t)}{\eta_u(t)} \big| X(0) = u - x/u, X(\tau) = u - y/u \right) = \begin{pmatrix}
-\frac{|t| + t^\alpha(\tau)}{1 + t(\tau)} + o(1) \\
-\frac{|t| - t^\alpha(\tau)}{1 + t(\tau)} + o(1)
\end{pmatrix}.
\] (8)

It is clear that
\[
\begin{align*}
\mathbb{E} \left( \frac{\xi_u(0)}{\eta_u(0)} \big| X(0) = u - x/u, X(\tau) = u - y/u \right) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
\mathbb{E} \left( \frac{\xi_u^2(0)}{\eta_u^2(0)} \big| X(0) = u - x/u, X(\tau) = u - y/u \right) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{align*}
\] (9)

Computing conditional covariance matrix, we have,
\[
\text{cov} \left( \frac{\xi_u(t) - \xi_u(s)}{\eta_u(t) - \eta_u(s)} \big| X(0), X(\tau) \right) = \text{cov} \left( \frac{\xi_u(t) - \xi_u(s)}{\eta_u(t) - \eta_u(s)} \right) - \text{Bcov} \left( \frac{X(0)}{X(\tau)} \right) \text{B}^T,
\]
where
\[
\text{B} = \text{cov} \left( \frac{\xi_u(t) - \xi_u(s)}{\eta_u(t) - \eta_u(s)}, \left( \frac{X(0)}{X(\tau)} \right) \right) \left[ \mathbb{E} \left( \left( \frac{X(0)}{X(\tau)} \right)^T \right) \mathbb{E} \left( \frac{X(0)}{X(\tau)} \right) \right]^{-1}.
\]

Using expressions for \( \xi_u(t) \) and \( \eta_u(t) \),
\[
\text{B} = \frac{u}{1 - t^2(\tau)} \begin{pmatrix}
 r(u^{-2/\alpha}t) - r(\tau)r(u^{-2/\alpha}t) - r(\tau)r(u^{-2/\alpha}t) - r(\tau)r(u^{-2/\alpha}t) \\
 -r(u^{-2/\alpha}s) + r(\tau)r(u^{-2/\alpha}s) + r(\tau)r(u^{-2/\alpha}s) + r(\tau)r(u^{-2/\alpha}s)
\end{pmatrix}.
\]

Letting now \( u \to \infty \), we get
\[
\text{cov} \left( \frac{\xi_u(t) - \xi_u(s)}{\eta_u(t) - \eta_u(s)} \big| X(0) = u - \xi/u, X(\tau) = u - \eta/u \right) = \begin{pmatrix}
 2|t - s|^\alpha(1 + o(1)) & o(1) \\
 o(1) & 2|t - s|^\alpha(1 + o(1))
\end{pmatrix},
\] (10)

where \( o(1) \)'s are uniform of \( x \) and \( y \), moreover they do not depends of values of conditions \( X(0) \) and \( X(\tau) \). Note that (10) holds true for all \( \alpha \in (0,2) \). From (10) it also followed that for some \( C > 0 \) all \( t, s \) and all sufficiently large \( u \),
\[
\text{var} \left( \frac{\xi_u(t) - \xi_u(s)}{\eta_u(t) - \eta_u(s)} \big| (X(0), X(\tau)) = (u - x/u, u - y/u) \right) \leq C|t - s|^\alpha, \quad \text{(11)}
\]
\[
\text{var} \left( \frac{\xi_u(t) - \xi_u(s)}{\eta_u(t) - \eta_u(s)} \big| (X(0), X(\tau)) = (u - x/u, u - y/u) \right) \leq C|t - s|^\alpha. \quad \text{(12)}
\]

Thus from (7-11) it follows that the family of conditional Gaussian distributions
\[
\mathbb{P} \left( \frac{\xi_u(t)}{\eta_u(t)} \big| X(0) = u - x/u, X(\tau) = u - y/u \right) = \begin{pmatrix}
 (B_\alpha(t) - |t|^{\alpha/(1 + \tau(\tau)), \tilde{B}_\alpha(t) - |t|^{\alpha/(1 + \tau(\tau))})^T
\end{pmatrix},
\] (13)
is weakly compact in \( C(T_1) \times C(T_2) \) and converges weakly, under conditions of Lemmas 1 and 2, to the distribution of the random vector process
\[
(\xi(t), \eta(t))^T = (B_\alpha(t) - |t|^{\alpha/(1 + \tau(\tau)), \tilde{B}_\alpha(t) - |t|^{\alpha/(1 + \tau(\tau))})^T,
\]

$t \in \mathbb{R}$, where $\tilde{B}$ is an independent copy of $B$. If the conditions of Lemma 3 are fulfilled, the family of Gaussian conditional distributions converges to the distribution of

$$(\xi(t), \eta(t))^T = (B_1(t) - (|t| + tr'(\tau))/(1 + r(\tau)), \tilde{B}_1(t) - (|t| - tr'(\tau))/(1 + r(\tau)))^T.$$ 

Thus

$$\lim_{u \to \infty} P \left( \max_{t \in T_1} \xi_u(t) > x, \max_{t \in T_2} \eta_u(t) > y \mid X(0) = u - x/u, X(\tau) = u - y/u \right) = P \left( \max_{t \in T_1} \xi(t) > x, \max_{t \in T_2} \eta(t) > y \right).$$

In order to prove a convergence of the integral

$$I(T_1, T_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P \left( \max_{t \in T_1} \xi_u(t) > x, \max_{t \in T_2} \eta_u(t) > y \mid X(0) = u - x/u, X(\tau) = u - y/u \right) \tilde{P}(u, x, y) \, dx \, dy$$

as $u \to \infty$, we construct an integrable dominating function, which have different representation in different quadrants of the plane.

1. For the quadrant $(x < 0, y < 0)$ we bound the probability by 1, and the $\tilde{P}(u, x, y)$ by $\exp(\frac{x+y}{1+r(t_m)})$, using relations $|r(t)| \leq 1$ and $x^2 + y^2 \geq 2xy$. The last function is integrable in the considered quadrant, so it is a desirable dominating function.

2. Within the quadrant $(x > 0, y < 0)$ we bound the probability by

$$P \left( \max_{t \in T_1} \xi_u(t) > x, \mid X(0) = u - x/u, X(\tau) = u - y/u \right)$$

and, using arguments similar the above, we bound $\tilde{P}(u, x, y)$ by

$$\exp \left( \frac{y}{1+r(t_m)} + \frac{x}{0.9 + r(t_m)} \right),$$

for sufficiently large $u$. The function $p(x)$ can be bounded by a function of type $C \exp(-\epsilon x^2)$, $\epsilon$ is positive, using, for example the Borel inequality with relations (7 - 10). Similar arguments one can find in [2].

3. Considerations in the quarter-plane $(x < 0, y > 0)$ are similar, the dominating function is

$$C \exp(-\epsilon y^2) \exp \left( \frac{x}{1+r(t_m)} + \frac{y}{0.9 + r(t_m)} \right).$$

4. In the quarter-plane $(x > 0, y > 0)$ we bound $\tilde{P}$ by

$$\exp \left( \frac{x}{0.9 + r(t_m)} + \frac{y}{0.9 + r(t_m)} \right)$$
and the probability by
\[ P \left( \max_{(t,s) \in T_1 \times T_2} \xi_u(t) + \eta_u(s) > x + y \mid X(0) = u - x/u, X(r) = u - y/u \right). \]

Again, for the probability we can apply the Borel inequality, just in the same way, to get the bound \( C \exp(-\epsilon(x+y)^2) \), for a positive \( \epsilon \).

Thus we have the desirable domination on the hole plane and therefore we have,
\[
\begin{align*}
\lim_{u \to \infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P \left( \max_{t \in T_1} \xi_u(t) > x, \max_{t \in T_2} \eta_u(t) > y \mid X(0) = u - x/u, X(r) = u - y/u \right) P(u, x, y) \, dx \, dy \\
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{\frac{x+y}{1+\tau(r)}} P \left( \max_{t \in T_1} \xi(t) > x, \max_{t \in T_2} \eta(t) > y \right) \, dx \, dy \\
= \int_{-\infty}^{+\infty} e^{\frac{x}{1+\tau(r)}} P \left( \max_{t \in T_1} \xi(t) > x \right) \, dx \int_{-\infty}^{+\infty} e^{\frac{y}{1+\tau(r)}} P \left( \max_{t \in T_2} \eta(t) > y \right) \, dy.
\end{align*}
\]

Then we proceed,
\[
\begin{align*}
\int_{-\infty}^{+\infty} e^{\frac{x}{1+\tau(r)}} P \left( \max_{t \in T_1} \xi(t) > x \right) \, dx &= \\
= (1 + r(\theta)) \exp \left[ \frac{\max_{T_1} \xi(t)}{1 + r(\theta)} \right] &= (1 + r(\theta)) \exp \left[ \frac{\max_{T_1} B_\alpha(t) - \frac{|t|^\alpha}{1 + r(\theta)}}{1 + r(\theta)} \right] \\
= (1 + r(\theta)) \exp \left[ \frac{\max_{T_1} B_\alpha \left( \frac{t}{(1 + r(\theta))^{2/\alpha}} \right) - \left( \frac{t}{(1 + r(\theta))^{2/\alpha}} \right)^\alpha}{1 + r(\theta)} \right] \\
= (1 + r(\theta)) \exp \left[ \frac{\max_{T_1/(1 + r(\theta))^{2/\alpha}} B_\alpha(s) - s^\alpha}{1 + r(\theta)} \right] = (1 + r(\theta)) H_\alpha \left( \frac{T_1}{(1 + r(\theta))^{2/\alpha}} \right),
\end{align*}
\]

where we use self-similarity properties of Fractional Brownian Motion. Similarly for \( \eta(t), t \in T_2 \). Similarly for \( H_1^{1 + r'(r)} \). Thus Lemmas follow.

The following lemma is proved in [2] in multidimensional case. We formulate it here for one-dimensional time.

**Lemma 4** Suppose that \( X(t) \) is a Gaussian stationary zero mean process with covariance function \( r(t) \) satisfying assumption A1. Let \( \epsilon, \frac{1}{2} > \epsilon > 0 \) be such that
\[
1 - \frac{1}{2} |t|^\alpha \geq r(t) \geq 1 - 2|t|^\alpha
\]
for all \( t \in [0, \varepsilon] \). Then there exists an absolute constant \( F \) such that the inequality
\[
P \left( \max_{t \in [0, T^{1-2/\alpha}]} X(t) > u, \max_{t \in [t_0 - 2\alpha, t_0 + 2\alpha]} X(t) > u \right) \leq F T^2 u^{-1} e^{-\frac{1}{2} u^2 - \frac{1}{2} (t_0 - T)^2}
\]

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holds for any $T$, $t_0 > T$ and for any $u \geq (4(T + t_0)/\varepsilon)^{\alpha/2}$.

The following two lemmas are straightforward consequences of Lemma 6.1, [2].

**Lemma 5** Suppose that $X(t)$ is a Gaussian stationary zero mean process with covariance function $r(t)$ satisfying assumption A1. Then

$$
P \left( \max_{t \in [0,T_u^{-2/\alpha} \cup [t_0u^{-2/\alpha}, (t_0 + T)_u^{-2/\alpha}]} X(t) > u \right) = H_\alpha([0,T] \cup [t_0, t_0 + T]) \frac{1}{\sqrt{2\pi u}} e^{-\frac{1}{2}u^2(1 + o(1))}
$$

as $u \to \infty$, where

$$H_\alpha([0,T] \cup [t_0, t_0 + T]) = \mathbb{E} \exp \left( \max_{t \in [0,T] \cup [t_0, t_0 + T]} (B_\alpha(t) - |t|^{\alpha}) \right).$$

**Lemma 6** Suppose that $X(t)$ is a Gaussian stationary zero mean process with covariance function $r(t)$ satisfying assumption A1. Then

$$
P \left( \max_{t \in [0,T_u^{-2/\alpha}]} X(t) > u, \max_{t \in [t_0u^{-2/\alpha}, (t_0 + T)_u^{-2/\alpha}]} X(t) > u \right)
$$

$$= H_\alpha([0,T], [t_0, t_0 + T]) \frac{1}{\sqrt{2\pi u}} e^{-\frac{1}{2}u^2(1 + o(1))}
$$

as $u \to \infty$, where

$$H_\alpha([0,T], [t_0, t_0 + T]) = \int_{-\infty}^{\infty} e^s P \left( \max_{t \in [0,T]} B_\alpha(t) - |t|^{\alpha} > s, \max_{t \in [t_0, t_0 + T]} B_\alpha(t) - |t|^{\alpha} > s \right) ds.$$

**Proof.** Write

$$P \left( \max_{t \in [0,T_u^{-2/\alpha}]} X(t) > u, \max_{t \in [t_0u^{-2/\alpha}, (t_0 + T)_u^{-2/\alpha}]} X(t) > u \right)
$$

$$= P \left( \max_{t \in [0,T_u^{-2/\alpha}]} X(t) > u \right) + P \left( \max_{t \in [t_0u^{-2/\alpha}, (t_0 + T)_u^{-2/\alpha}]} X(t) > u \right)
$$

$$- P \left( \max_{t \in [0,T_u^{-2/\alpha} \cup [t_0u^{-2/\alpha}, (t_0 + T)_u^{-2/\alpha}]} X(t) > u \right)
$$

and apply Lemma 6.1, [2] and Lemma 3 to the right-hand part.

From Lemmas 4 and 2 we get,

**Lemma 7** For any $t_0 > T$,

$$H_\alpha([0,T], [t_0, t_0 + T]) \leq F\sqrt{2\pi T^2} e^{-\frac{1}{2}(t_0 - T)^{\alpha}}.$$

When $t_0 = T$ the Lemma holds true, but the bound is trivial. A non-trivial bound for $H_\alpha([0,T], [T, 2T])$ one can get from the proof of Lemma 7.1, [2], see page 107, inequalities (7.5) and the previous one. These inequalities, Lemma 6.8, [2] and Lemma 5 give the following,
Lemma 8 There exists a constant $F_1$ such that for all $T \geq 1$,

$$H_\alpha([0,T], [T, 2T]) \leq F_1 \left( \sqrt{T} + T^2 e^{-\frac{T^\alpha}{8}} \right).$$

Applying Lemma 1 to the sets $T_1 = [0, T] \cup [t_0, t_0 + T]$, $T_2 = [0, T] \cup [t_1, t_1 + T]$ and combining probabilities similarly as in the proof of Lemma 4, we get,

Lemma 9 Let $X(t)$ be a Gaussian process with mean zero and covariance function $r(t)$ satisfying conditions of Theorem 1. Let $\tau = \tau(u)$ tends to $t_m$ as $u \to \infty$ in such a way that $|\tau - t_m| \leq C \sqrt{\log u}/u$, for some positive $C$. Then for all $T > 0$, $t_0 \geq T$, $t_1 \geq T$

$$P \left( \max_{t \in [0, u^{-2/\alpha} T]} X(t) > u, \max_{t \in [u^{-2/\alpha} t_0, u^{-2/\alpha} (t_0 + T)]} X(t) > u, \max_{t \in [r, r + u^{-2/\alpha} T]} X(t) > u, \max_{t \in [r + u^{-2/\alpha} t_1, r + u^{-2/\alpha} (t_1 + T)]} X(t) > u \right)$$

$$= \frac{(1 + r(t_m))^2}{2\pi \sqrt{1 - r^2(t_m)}} \cdot \frac{1}{u^2} e^{-\frac{t_0^2}{1 + r(r)}} \times H_\alpha \left( \begin{bmatrix} 0, T \\ 1 + r(t_m) \end{bmatrix}^{2/\alpha} \right) \times H_\alpha \left( \begin{bmatrix} 0, T \\ 1 + r(t_m) \end{bmatrix}^{2/\alpha} \right) (1 + o(1)),$$

as $u \to \infty$.

3 Proofs

3.1 Proof of Theorem 1

We denote $\Pi = [T_1, T_3] \times [T_3, T_4]$, $\delta = \delta(u) = C \sqrt{\log u}/u$, the value of the positive $C$ we specify later on. $D = \{(t,s) \in \Pi : |t - s - t_m| \leq \delta\}$. We have,

$$P \left( \max_{t \in [T_1, T_3]} X(t) > u, \max_{t \in [T_3, T_4]} X(t) > u \right) = P \left( \bigcup_{(s,t) \in \Pi} \{X(t) > u \cap X(s) > u\} \right)$$

$$= P \left( \bigcup_{(s,t) \in D} \{X(t) > u \cap X(s) > u\} \right) \bigcup \left( \bigcup_{(s,t) \in \Pi \setminus D} \{X(t) > u \cap X(s) > u\} \right)$$

$$\leq P \left( \bigcup_{(s,t) \in D} \{X(t) > u \cap X(s) > u\} \right) + P \left( \bigcup_{(s,t) \in \Pi \setminus D} \{X(t) > u \cap X(s) > u\} \right). \quad (14)$$
From the other hand,

\[
P \left( \max_{t \in [T_1,T_2]} X(t) > u, \max_{t \in [T_3,T_4]} X(t) > u \right) = P \left( \bigcup_{(s,t) \in D} (X(t) > u) \cap (X(s) > u) \right)
\]

\[
= P \left( \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right) \cup \left( \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right)
\]

\[
\geq P \left( \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right). \tag{15}
\]

The second term in the right-hand part of (14) we estimate as following,

\[
P \left( \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right) \leq P \left( \max_{(s,t) \in D} X(t) + X(s) > 2u \right). \tag{16}
\]

Making use of Theorem 8.1, [2], we get that the last probability does not exceed

\[
\text{const} \cdot u^{-1+2/\alpha} \exp \left( -\frac{u^2}{1 + \max_{(t,s) \in D} r(t-s)} \right). \tag{17}
\]

Further, for \( \varepsilon = 1/6 \) and all sufficiently large \( u \),

\[
\max_{(t,s) \in D} r(t-s) \leq r(t_m) + \left( \frac{1}{2} - \varepsilon \right) r''(t_m) \delta^2 = r(t_m) + \frac{1}{3} C^2 r''(t_m) \log u/u.
\]

Hence,

\[
P \left( \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right) \leq \text{const} \cdot u^{-1+2/\alpha} \exp \left( -\frac{u^2}{1 + r(t_m)} \right) u^{-G}, \tag{18}
\]

where

\[
G = \frac{-2C^2 r''(t_m)}{3(1 + r(t_m))^2}.
\]

Now we deal with the first probability in the right-hand part of (14). It is equal to the probability in right-hand part of (15). We are hence in a position to bound the probability from above and from below getting equal orders for the bounds. Denote \( \Delta = Tu^{-2/\alpha}, T > 0 \), and define the intervals

\[
\Delta_k = [T_1 + k\Delta, T_1 + (k+1)\Delta], \quad 0 \leq k \leq N_k, \; N_k = [(T_2 - T_1)/\Delta],
\]

\[
\Delta_l = [T_3 + l\Delta, T_3 + (l+1)\Delta], \quad 0 \leq l \leq N_l, \; N_l = [(T_4 - T_3)/\Delta],
\]

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where \([\cdot]\) stands for the integer part of a number. In virtue of Lemma 1,

\[
P \left( \bigcup_{(s,t) \in D} \{ X(t) > u \} \cap \{ X(s) > u \} \right) \\
\leq P \left( \bigcup_{(k,l)} \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset, t \in \Delta_k, \delta \in \Delta_l \right) \\
= \sum_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} P \left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u \right) \\
\leq \frac{(1 + \gamma(u))}{2\pi u^2 \sqrt{1 - r^2(t_m)}} H_\alpha \left( \frac{T}{(1 + r(t_m))^{2/\alpha}} \right) \sum_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} \exp \left( -\frac{u^2}{1 + r(\tau_{k,l})} \right), \quad (19)
\]
where \(\gamma(u) \downarrow 0\) as \(u \to \infty\) and \(\tau_{k,l} = T_3 - T_1 + (l - k)\Delta\). For the last sum we get,

\[
S = \sum_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} \exp \left( -\frac{u^2}{1 + r(\tau_{k,l})} \right) \\
= \exp \left( -\frac{u^2}{1 + r(t_m)} \right) \sum_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} \exp \left( -\frac{r(t_m) - r(\tau_{k,l})}{1 + r(\tau_{k,l})} \right) \\
\]
Define \(\theta\) by \(t_m = T_3 - T_1 + \Delta \theta\), we obtain,

\[
\frac{r(t_m) - r(\tau_{k,l})}{1 + r(\tau_{k,l})}(1 + r(t_m)) \leq (1 - \frac{1}{2}r''(t_m)(\tau_{k,l} - t_m)^2 \leq (1 + (-)\gamma_1(u)) \\
= -A((k - l)\Delta - \theta \Delta)^2 (1 + (-)\gamma_1(u)),
\]
where \(\gamma_1(u) \downarrow 0\) as \(u \to \infty\). In the last sum, index \(k\) varies between \((T_{\min} + O(\delta(u))) / \Delta\) and \((T_{\max} + O(\delta(u))) / \Delta\), as \(u \to \infty\), where \(T_{\min} = T_1 \vee (T_3 - t_m)\) and \(T_{\max} = T_2 \wedge (T_4 - t_m)\). Indeed, for the co-ordinate \(x\) of the left end of a segment of length \(t_m\), which varies having left end inside \([T_1, T_2]\) and right end inside \([T_3, T_4]\), we have the restrictions \(T_1 < x < T_2\), and \(T_3 < x + t_m < T_4\), so that \(x \in (T_{\min}, T_{\max})\). The index \(m = k - l - \theta\) varies thus between \(-\delta(u) / \Delta + O(\Delta)\) and \(\delta(u) / \Delta + O(\Delta)\) as \(u \to \infty\). Note that \(u \Delta \to 0\) as \(u \to \infty\). Using this, we continue,

\[
S = (1 + o(1)) \exp \left( -\frac{u^2}{1 + r(t_m)} \right) \frac{T_{\max} - T_{\min}}{\Delta} \sum_{m = -\delta(u)/\Delta + O(\Delta)}^{\delta(u)/\Delta + O(\Delta)} \exp \left( -A(mu \Delta)^2 \right) \\
= (1 + o(1)) \exp \left( -\frac{u^2}{1 + r(t_m)} \right) \frac{T_{\max} - T_{\min}}{u \Delta^2} \sum_{m = -u \delta(u)/\Delta}^{u \delta(u)/\Delta} \exp \left( -A(mu \Delta)^2 \right) u \Delta \\
= (1 + o(1)) \exp \left( -\frac{u^2}{1 + r(t_m)} \right) \frac{T_{\max} - T_{\min}}{u \Delta^2} \int_{-\infty}^{\infty} e^{-Ax^2} dx.
\]
Compute the integral and substitute this in right-hand part of (19), we get,

\[ P \left( \bigcup_{(s,t) \in D} \{ X(t) > u \} \cap \{ X(s) > u \} \right) \leq \frac{(1 + r(t_m))^2(1 + \gamma_2(u))(T_{\max} - T_{\min})u^{-3+4/\alpha}}{2\sqrt{4\pi(1 - r^2(t_m))}} \frac{1}{T^2} H_\alpha^2 \left( \frac{T}{(1 + r(t_m))^{2/\alpha}} \right) \exp \left( -\frac{u^2}{1 + r(t_m)} \right), \]

where \( \gamma_2(u) \downarrow 0 \) as \( u \to \infty \).

Now we bound from below the probability in the right-hand part of (15). We have

\[
P \left( \bigcup_{(s,t) \in D} \{ X(t) > u \} \cap \{ X(s) > u \} \right) \geq P \left( \bigcup_{(k,l) : \Delta_k \subset D, \Delta_l \subset D} \bigcup_{t \in \Delta_k, s \in \Delta_l} \{ X(t) > u \} \cap \{ X(s) > u \} \right) \geq \sum_{(k,l) : \Delta_k \subset D, \Delta_l \subset D} P \left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u \right) - \sum \sum P \left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_{k'}} X(t) > u, \max_{t \in \Delta_l} X(t) > u, \max_{t \in \Delta_{l'}} X(t) > u \right),
\]

where the double-sum is taken over the set

\[ \{(k,l, k', l') : (k', l') \neq (k, l), \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset, \Delta_{k'} \cap D \neq \emptyset, \Delta_{l'} \cap D \neq \emptyset\}. \]

The first sum in the right-hand part of (21) can be bounded from below exactly by the same way as the previous sum, thus we have,

\[
\sum_{(k,l) : \Delta_k \subset D, \Delta_l \subset D} P \left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u \right) \geq \frac{(1 + r(t_m))^2(1 - \gamma_2(u))(T_{\max} - T_{\min})u^{-3+4/\alpha}}{2\sqrt{4\pi(1 - r^2(t_m))}} \frac{1}{T^2} H_\alpha^2 \left( \frac{T}{(1 + r(t_m))^{2/\alpha}} \right) \exp \left( -\frac{u^2}{1 + r(t_m)} \right),
\]

where \( \gamma_2(u) \downarrow 0 \) as \( u \to \infty \). We are now able to select the constant \( C \). We take it as large as \( G > 2 - 2/\alpha \) to get that left-hand part of (18) is infinitely smaller then left-hand part of (22) as \( u \to \infty \).

Consider the second sum (the double-sum) in the right-hand part of (21). For sake of simplicity we denote

\[ H(m) = H_\alpha \left( \left[ 0, \frac{T}{(1 + r(t_m))^{2/\alpha}} \right], \left[ \frac{mT}{1 + r(t_m))^{2/\alpha}}, \frac{(m + 1)T}{(1 + r(t_m))^{2/\alpha}} \right] \right), \]

and notice that

\[ H(0) = H_\alpha \left( \left[ 0, \frac{T}{(1 + r(t_m))^{2/\alpha}} \right] \right). \]
In virtue of Lemma 9 we have for the double-sum in (21), taking into account only different 
\((k, l)\) and \((k', l')\),
\[
\Sigma_2 := \sum_{t \in \Delta_k} \sum_{t \in \Delta_l} P\left(\max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u, \max_{t \in \Delta_k^c} X(t) > u, \max_{t \in \Delta_l^c} X(t) > u\right)
\leq \frac{(1 + r(t_m))^2 (1 + \Gamma(u))}{2\pi u^2 \sqrt{1 - r^2(t_m)}} \sum_{n=1}^{\infty} H(n) \left(\frac{H(0) + 2}{1 + \Gamma(t_m)} + \sum_{m=1}^{\infty} H(m)\right) \exp \left(-\frac{u^2}{1 + r(t_m)}\right)
\times \sum_{(k,l) : \Delta_k \cap \Delta_l \neq \emptyset, \Delta_k \cap \Delta_l \neq \emptyset} \exp \left(-\frac{u^2}{1 + r(t_{k,l})}\right),
\]
where \(\Gamma(u) \downarrow 0\) as \(u \to \infty\). The last sum is already bounded from above, therefore by (19) and (20) we have,
\[
\Sigma_2 \leq \frac{2}{T^2} \sum_{n=1}^{\infty} H(n) \left(\frac{H(0) + 2}{1 + \Gamma(t_m)} + \sum_{m=1}^{\infty} H(m)\right) \exp \left(-\frac{u^2}{1 + r(t_m)}\right).
\]
By Lemmas 6.8, [2], 7 and 8 we get that \(H(0) \leq \text{const} \cdot T\), \(H(1) \leq \text{const} \cdot \sqrt{T}\) and for \(m > 1\),
\[
H(m) \leq \text{const} \cdot e^{-\frac{1}{2}m^{a/2}T^{a/2}},
\]
hence
\[
\sum_{n=1}^{\infty} H(n) \left(\frac{H(0) + 2}{1 + \Gamma(t_m)} + \sum_{m=1}^{\infty} H(m)\right) \leq \text{const} \cdot T^{3/2}.
\]
Thus
\[
\Sigma_2 \leq \text{const} \cdot T^{-1/2} u^{-3+4/\alpha} \exp \left(-\frac{u^2}{1 + r(t_m)}\right). \tag{23}
\]
Now since by (1),
\[
\lim_{T \to \infty} \frac{1}{T} H_x \left(\frac{T}{(1 + r(t_m))^{2/\alpha}}\right) = (1 + r(t_m))^{-2/\alpha} H_x,
\]
we get that the double sum can be made infinitely smaller by choosing large \(T\). Thus Theorem 1 follows.

3.2 Proof of Theorem 2.

We prove the theorem for the case \(t_m = T_3 - T_2\), another case can be considered similarly. First, as in the proof of Theorem 1 put \(D = \{(t,s) \in \Pi : |t - s - t_m| \leq \delta\}\), but with
\[ \delta = \delta(u) = C \sqrt{\log u/u^2}, \text{ for sufficiently large } C. \] The evaluations (14), (16) and (17) still hold true. Further we have for \( \epsilon = 1/6 \) and all sufficiently large \( u \),

\[
\max_{(t,s) \in I \setminus D} r^2(t-s) \leq r(t_m) + \frac{1}{2} - \epsilon r'(t_m) \delta = r(t_m) + \frac{1}{3} C^2 r'(t_m) \log u/u^2.
\]

Hence, (18) holds true with

\[
G = \frac{-2C^2 r'(t_m)}{3(1 + r(t_m))^2}.
\]

Let now \( \alpha > 1 \). For any positive arbitrarily small \( \epsilon \) we have for all sufficiently large \( u \) that, \( \epsilon u^{-2/\alpha} > \delta(u) \), hence for such values of \( u \),

\[
P \left( \bigcup_{(t,s) \in D} \{ X(t) > u \} \cap \{ X(s) > u \} \right) \leq P \left( \max_{t \in [T_2 - \epsilon u^{-2/\alpha}, T_3]} X(t) > u, \max_{t \in [T_3, T_3 + \epsilon u^{-2/\alpha}]} X(t) > u \right). \quad (24)
\]

We wish to apply Lemma 1 to the last probability for the intervals \([-\epsilon, 0]\) and \([t_m, t_m + \epsilon]\). To this end we turn to (5). Since for a sufficiently small \( \epsilon \), \( r'(t_m) < 0 \), we have that

\[
\frac{r(\tau - u^{-2/\alpha} t) - r(\tau)}{1 + r(\tau)} < 0 \text{ for all } \tau \in [-\epsilon, 0]
\]

and

\[
\frac{r(\tau + u^{-2/\alpha} t) - r(\tau)}{1 + r(\tau)} < 0 \text{ for all } \tau \in [t_m, t_m + \epsilon],
\]

hence

\[
\limsup_{u \to \infty} \mathbb{E} (\xi_u(t) | X(0) = u - x/u, X(\tau) = u - y/u) \leq -\frac{1}{1 + r(t_m)} |t|^\alpha,
\]

for all \( t \in [-\epsilon, 0] \), and

\[
\limsup_{u \to \infty} \mathbb{E} (\eta_u(t) | X(0) = u - x/u, X(\tau) = u - y/u) \leq -\frac{1}{1 + r(t_m)} |t|^\alpha,
\]

for all \( t \in [t_m, t_m + \epsilon] \). All other arguments in the proof of Lemma 1 still hold true, therefore, using time-symmetry of the fractional Brownian motion, we have,

\[
\limsup_{u \to \infty} u^2 e^{1+r(t_m) \frac{\Delta}{2\pi}} \mathbb{P} \left( \max_{t \in [T_2 - u^{-2/\alpha}, T_3]} X(t) > u, \max_{t \in [T_3, T_3 + u^{-2/\alpha}]} X(t) > u \right) \leq \frac{(1 + r(t_m))^2}{2\pi \sqrt{1 - r^2(t_m)}} H_\alpha \left( \frac{|0, \epsilon|}{(1 + r(t_m))^2/\alpha} \right). \quad (25)
\]

Using Fatou monotone convergence we have \( \lim_{\epsilon \to 0} H_\alpha(\epsilon) = 1 \), therefore

\[
\limsup_{u \to \infty} u^2 e^{1+r(t_m) \frac{\Delta}{2\pi}} \mathbb{P} \left( \max_{t \in [T_2 - u^{-2/\alpha}, T_3]} X(t) > u, \max_{t \in [T_3, T_3 + u^{-2/\alpha}]} X(t) > u \right) \leq \frac{(1 + r(t_m))^2}{2\pi \sqrt{1 - r^2(t_m)}} \quad (26)
\]
But
\[
P_d(u; [T_1, T_2], [T_3, T_4]) \geq P(X(T_2) > u, X(T_3) > u) = \frac{(1 + r(t_m))^2}{2\pi u^2 \sqrt{1 - r^2(t_m)}} e^{-\frac{u^2}{2(1 + r(t_m))}} (1 + o(1))
\]
as \(u \to \infty\). Thus (i) follows.

Let now \(\alpha = 1\). From now on, we redefine \(\Delta_k\) and \(\Delta_t\), by
\[
\begin{align*}
\Delta_k &= [T_2 - (k + 1)\Delta, T_2 - k\Delta], \quad 0 \leq k \leq N_k, \quad N_k = \lfloor (T_2 - T_1)/\Delta \rfloor, \\
\Delta_t &= [T_3 + l\Delta, T_3 + (l + 1)\Delta], \quad 0 \leq l \leq N_l, \quad N_l = \lfloor (T_4 - T_3)/\Delta \rfloor,
\end{align*}
\]
for the case of \(\Delta_k\), \(k = 0\), we denote \(\Delta_0 = \Delta_{-0}\), indicating difference with \(\Delta_0\) for the case \(\Delta_t\), \(l = 0\). Recall that now \(\Delta = Tu^{-2/\alpha} = Tu^{-2}\). We have for sufficiently large \(u\),
\[
P \left( \bigcup_{(s,t) \in D} \{ X(t) > u \} \cap \{ X(s) > u \} \right) \geq P \left( \max_{t \in \Delta_{-0}} X(t) > u, \max_{t \in \Delta_0} X(t) > u \right), \quad (27)
\]
and
\[
P \left( \bigcup_{(s,t) \in D} \{ X(t) > u \} \cap \{ X(s) > u \} \right) \leq P \left( \max_{t \in \Delta_{-0}} X(t) > u, \max_{t \in \Delta_0} X(t) > u \right) +
\]
\[
+ \sum_{k=0, l=0, k+l>0}^{[\log u/T]+1} P \left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_t} X(t) > u \right). \quad (28)
\]
First probability in right-hand parts of the inequalities is already considered by Lemma 3. We
set \(r = t_m = T_3 - T_2, \quad T_1 = [-T, 0], \quad T_2 = [0, T]\), by time-symmetry of Brownian motion, we have that
\[
H_1^r([T, 0]) = H_{-r}^{-r}(0, T). \quad (29)
\]
In order to estimate the sum, we observe, that for all sufficiently large \(u\) and all \(t \in [T_3, T_3 + \delta(u)], \quad s \in [T_2 - \delta(u), T_2], \)
\[
r(t - s) \leq r(t_m) + \frac{1}{3}r'(t_m)(t - s - t_m) \quad \text{and} \quad r(t - s) \geq r(t_m) + \frac{2}{3}r'(t_m)(t - s - t_m). \quad (30)
\]
Hence
\[
\frac{-u^2}{1 + r(t_m) + (k + l)\Delta} \leq \frac{-u^2}{1 + r(t_m) + \frac{1}{3}r'(t_m)(k + l)Tu^{-2}} \leq \frac{-u^2}{1 + r(t_m) + \frac{r'(t_m)(k + l)}{6(1 + r(t_m))}T} = \frac{-u^2}{1 + r(t_m) - a(k + l)T},
\]
where \(a > 0\). Now, in Lemma 3 let \(r = t_m + (k + l)\Delta, \quad T_1 = [-T, 0], \quad T_2 = [0, T]\), using the above mentioned property of the constants \(H_1^r(T)\), we get, that for all sufficiently large \(u\) and \(T\),
\[
P \left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_t} X(t) > u \right) \leq Cp_d(u, r(t_m))e^{-a(k+l)T},
\]

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From here we get,
\[
\sum_{k=0, l=0, k+l>0} \mathbb{P}\left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u \right) \leq C \rho_2(u, r(\tau_m)) e^{-a(k+l)T},
\]

Applying now Lemma 3 to first summands in right-hand sides of (27, 28) and letting \( T \to \infty \), we get the assertion (ii) of Theorem.

Let now \( \alpha < 1 \). Proof of the Theorem in this case is similar to the proof of Theorem 1. We have to consider a sum of small almost equal probabilities and a double sum. Using the more recent definition of \( \Delta_k \) and \( \Delta_l \), we have by Lemma 2,
\[
P\left( \bigcup_{(s,t) \in D} \{ X(t) > u \} \cap \{ X(s) > u \} \right) \leq \sum_{k \in \Delta_k} \mathbb{P}\left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u \right) \leq \sum_{(k,l): \Delta_k \cap \Delta_l \neq \emptyset, \Delta_l \cap \Delta_l \neq \emptyset} \exp \left( -\frac{u^2}{1 + r(\tau_{k,l})} \right),
\]

where \( \gamma(u) \downarrow 0 \) as \( u \to \infty \) and now \( \tau_{k,l} = T_3 - T_2 + (l + k)\Delta \). For the last sum we get,
\[
S = \sum_{(k,l): \Delta_k \cap \Delta_l \neq \emptyset, \Delta_l \cap \Delta_l \neq \emptyset} \exp \left( -\frac{u^2}{1 + r(\tau_{k,l})} \right) = \exp \left( -\frac{u^2}{1 + r(t_m)} \right) \sum_{(k,l): \Delta_k \cap \Delta_l \neq \emptyset, \Delta_l \cap \Delta_l \neq \emptyset} \exp \left( -u^2 \frac{r(t_m) - r(\tau_{k,l})}{1 + r(\tau_{k,l})(1 + r(t_m))} \right).
\]

Next,
\[
\frac{r(t_m) - r(\tau_{k,l})}{(1 + r(\tau_{k,l}))(1 + r(t_m))} \leq \left( \frac{\gamma_1(u)}{1 + r(\tau_{k,l})} \right) = -B(k+l)\Delta(1 + (-)\gamma_1(u)),
\]

where \( \gamma_1(u) \downarrow 0 \) as \( u \to \infty \). Remind that now \( u^2\Delta \to 0 \) as \( u \to \infty \). Using this, and denoting \( m = k + l \), we continue,
\[
S = (1 + o(1)) \exp \left( -\frac{u^2}{1 + r(t_m)} \right) \sum_{m=0}^{\delta(u)/\Delta + O(\Delta)} m \exp (-Bu^2\Delta)
\]
\[
= (1 + o(1)) \exp \left( -\frac{u^2}{1 + r(t_m)} \right) \frac{1}{(\Delta u^2)^2} \sum_{m=0}^{\delta(u)/\Delta + O(\Delta)} m\Delta u^2 \exp (-Bm\Delta u^2) (\Delta u^2)
\]

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\[ = (1 + o(1)) \exp \left( -\frac{u^2}{1 + r(t_m)} \right) \frac{1}{u^4 \Delta^2} \int_0^\infty xe^{-B^2x} dx = (1 + o(1)) \exp \left( -\frac{u^2}{1 + r(t_m)} \right) \frac{1}{B^2 u^4 \Delta^2}. \]

Substitute this in right-hand part of (31), we get,

\[
\mathbb{P} \left( \bigcup_{(s,t) \in D} \{ X(t) > u \} \cap \{ X(s) > u \} \right) \leq \frac{(1 + r(t_m))^2 (1 + \gamma_2(u)) u^{-\delta+4/\alpha}}{2 \pi B^2 \sqrt{1 - r^2(t_m)}} \frac{1}{T^2 \beta^2} \frac{T}{(1 + r(t_m))^{2/\alpha}} \exp \left( -\frac{u^2}{1 + r(t_m)} \right),
\]

where \( \gamma_2(u) \downarrow 0 \) as \( u \to \infty \).

Estimation the probability from below repeats the corresponding steps in the proof of Theorem 1, see (21) and followed. Thus Theorem 2 follows.

References
