Report 2000-027

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ISSN 1389-2355

On double extremes of Gaussian stationary processes

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September 7, 2000

Abstract

We consider a Gaussian stationary process with Pickands' conditions and evaluate an exact asymptotic behavior of probability of two high extremes on two disjoint intervals.

1 Introduction. Main results.

Let $X(t)$, $t \in \mathbb{R}$, be a zero mean stationary Gaussian process with unit variance and covariance function $r(t)$. An object of our interest is the asymptotic behaviour of the probability

$$P_d(u; [T_1, T_2], [T_3, T_4]) = \mathbf{P}\left(\max_{t\in[T_1,T_2]} X(t) > u, \max_{t\in[T_3,T_4]} X(t) > u\right)$$

as $u \to \infty$, where $[T_1, T_2]$ and $[T_3, T_4]$ are disjoint intervals. To evaluate the asymptotic behaviour we develop an analogue of Pickands' theory of high extremes of Gaussian processes, see [1] and extensions in [2]. We follow main steps of the theory. First we assume an analogue of the Pickands' conditions.

A1 For some $\alpha \in (0, 2)$,

$$r(t) = 1 - |t|^\alpha + o(|t|^\alpha) \text{ as } t \to 0,$$
$$|r(t)| < 1 \text{ for all } t > 0.$$

Then, we specify covariations between values of the process on intervals $[T_1, T_2]$ and $[T_3, T_4]$. We assume that there is an only domination point of correlation between the values. This makes some similarity with Pirabarg&Prisyazhn'uck's extension of the Pickands' theory to non-stationary Gaussian processes.

A2 In the interval $S = [T_3 - T_2, T_4 - T_1]$ there exists only point $t_m = \arg\max_{t\in S} r(t) \in (T_3 - T_2, T_4 - T_1)$, $r(t)$ is twice differentiable in a neighbourhood of t_m with $r''(t_m) \neq 0$.

As an alternative of assumption **A2** one can suppose that the point of maximum of $r(t)$ is one of the end points of S, $T_3 - T_2$ is more natural candidate.

*Supported in parts by RFFI Grant of Russian Federation 98-01-00524, by DFG - RFFI grant "Statistik in Funktionsräumen: asymptotische Theorie" and by EURANDOM. E-mail piter@mech.math.msu.su

A3 $r(t)$ is continuously differentiable in a neighbourhood of the point $t_m = T_3 - T_2$, $r'(t_m) < 0$ and $r(t_m) > r(t)$ for all $t \in (T_3 - T_2, T_4 - T_1]$.

A3' $r(t)$ is continuously differentiable in a neighbourhood of the point $t_m = T_4 - T_1$, $r'(t_m) > 0$ and $r(t_m) > r(t)$ for all $t \in [T_3 - T_2, T_4 - T_1)$.

Denote by $B_\alpha(t)$, $t \in \mathbb{R}$, a normed fractional Brownian motion with the Hurst parameter $\alpha/2$, that is a Gaussian process with a.s. continuous trajectories, $B_\alpha(0) = 0$ a.s., $\mathbf{E}B_\alpha(t) \equiv 0$, and $\mathbf{E}(B_\alpha(t) - B_\alpha(s))^2 = 2|t-s|^\alpha$. For any set $T \subset \mathbb{R}$ we denote

$$H_\alpha(T) = \mathbf{E}\exp\left(\sup_{t\in T} B_\alpha(t) - |t|^\alpha\right).$$

It is known, [1], [2], that there exists a positive and finite limit

$$H_\alpha := \lim_{T\to\infty} \frac{1}{T} H_\alpha([0,T]), \tag{1}$$

the Pickands' constant. Further, for a number c denote

$$H_1^c(T) = \mathbf{E}\exp\left(\sup_{t\in T} B_1(t) - |t| - ct\right).$$

It is known, [2], that for any positive c, the limit $H_1^c := \lim_{T\to\infty} H_1^c([0,T])$ exists and is positive. We stand $a \vee b$ for $\max(a,b)$ and $a \wedge b$ for $\min(a,b)$. Denote

$$p_2(u,r) = \frac{(1+r)^2}{2\pi u^2\sqrt{1-r^2}} e^{-\frac{u^2}{1+r}}$$

and notice that for a Gaussian vector (ξ, η) where the components are standard Gaussian and correlation between them is r, $\mathbf{P}(\xi > u, \eta > u) = p_2(u,r)(1+o(1))$ as $u \to \infty$.

Theorem 1 *Let $X(t)$, $t \in \mathbb{R}$, be a Gaussian centred stationary process with a.s. continuous trajectories. Let assumptions* **A1** *and* **A2** *be fulfilled for its covariance function $r(t)$. Then*

$$\begin{aligned} &P_d(u;[T_1,T_2],[T_3,T_4]) \\ &= K\sqrt{\pi A^{-1}}(1+r(t_m))^{-4/\alpha} H_\alpha^2 u^{-3+4/\alpha} p_2(u, r(t_m))(1+o(1)) \end{aligned}$$

as $u \to \infty$, where $K = T_2 \wedge (T_4 - t_m) - T_1 \vee (T_3 - t_m) > 0$,

$$A = -\frac{1}{2}\frac{r''(t_m)}{(1+r(t_m))^2}.$$

Theorem 2 *Let $X(t)$, $t \in \mathbb{R}$, be a Gaussian centred stationary process with a.s. continuous trajectories. Let assumptions* **A1** *and* **A3** *or* **A3'** *be fulfilled for its covariance function $r(t)$. Then,*

(i)for $\alpha > 1$,

$$P_d(u;[T_1,T_2],[T_3,T_4]) = p_2(u, r(t_m))(1+o(1))$$

as $u \to \infty$.

(ii)For $\alpha = 1$,

$$P_d(u; [T_1, T_2], [T_3, T_4]) = \left(H_1^{|r'(t_m)|}\right)^2 p_2(u, r(t_m))(1 + o(1))$$

as $u \to \infty$.

(iii) For $\alpha < 1$,

$$P_d(u; [T_1, T_2], [T_3, T_4]) = B^{-2}(1 + r(t_m))^{-4/\alpha} H_\alpha^2 u^{-6+4/\alpha} p_2(u, r(t_m))(1 + o(1))$$

as $u \to \infty$, *where*

$$B = \frac{r'(t_m)}{(1 + r(t_m))^2}.$$

2 Lemmas

For a set $A \subset \mathbb{R}$ and a number a we write $aA = \{ax : x \in A\}$ and $a + A = \{a + x : x \in A\}$.

Lemma 1 *Let $X(t)$ be a Gaussian process with mean zero and covariance function $r(t)$ satisfying assumptions* **A1, A2**. *Let a time moment $\tau = \tau(u)$ tends to t_m as $u \to \infty$ in such a way that $|\tau - t_m| \le C\sqrt{\log u}/u$, for some positive C. Let T_1 and T_2 be closures of two bounded open subsets of $\mathbb{R}$. Then*

$$\mathbf{P}\left(\max_{t \in u^{-2/\alpha}T_1} X(t) > u, \max_{t \in \tau + u^{-2/\alpha}T_2} X(t) > u\right) =$$
$$= \frac{(1 + r(\theta))^2}{2\pi u^2\sqrt{1 - r^2(\theta)}} e^{-\frac{u^2}{1+r(\tau)}} H_\alpha\left(\frac{T_1}{(1 + r(\theta))^{2/\alpha}}\right) H_\alpha\left(\frac{T_2}{(1 + r(\theta))^{2/\alpha}}\right)(1 + o(1)), \qquad (2)$$

as $u \to \infty$, *where* $\theta = t_m$.

Lemma 2 *Let $X(t)$ be a Gaussian process with mean zero and covariance function $r(t)$ satisfying assumptions* **A1, A2** *with $\alpha < 1$. Let T_1 and T_2 be closures of two bounded open subsets of $\mathbb{R}$. Then, for any (fixed) $\tau > 0$ the asymptotic relation of Lemma 1 holds true with $\theta = \tau$.*

Lemma 3 *Let $X(t)$ be a Gaussian process with mean zero and covariance function $r(t)$ satisfying assumptions* **A1, A2** *with $\alpha = 1$. Let T_1 and T_2 be closures of two bounded open subsets of $\mathbb{R}$. Then*

$$\mathbf{P}\left(\max_{t \in u^{-2}T_1} X(t) > u, \max_{t \in \tau + u^{-2}T_2} X(t) > u\right) =$$
$$= H_1^{r'(\tau)}\left(\frac{T_1}{(1 + r(\tau))^2}\right) H_1^{-r'(\tau)}\left(\frac{T_2}{(1 + r(\tau))^2}\right) p_2(u, r(\tau))(1 + o(1)), \qquad (3)$$

as $u \to \infty$.

Proof of Lemmas 1 - 3. We prove the three lemmas simultaneously, computations of conditional expectation (4) and related evaluations are performed in parallel, separately for each lemma. We have for $u > 0$,

$$\mathbf{P} = \mathbf{P}\left(\max_{t\in u^{-2/\alpha}T_1} X(t) > u, \max_{t\in\tau+u^{-2/\alpha}T_2} X(t) > u\right) =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{P}\left(\max_{t\in u^{-2/\alpha}T_1} X(t) > u, \max_{t\in\tau+u^{-2/\alpha}T_2} X(t) > u \,\middle|\, X(0) = a, X(\tau) = b\right) \mathbf{P}_{0\tau}(a,b)\, dadb,$$

where

$$\mathbf{P}_{0\tau}(a,b) = \frac{1}{2\pi\sqrt{1-r^2(\tau)}} \exp\left(-\frac{1}{2}\cdot\frac{a^2 - 2r(\tau)ab + b^2}{1-r^2(\tau)}\right).$$

Now we change variables, $a = u - x/u$, $b = u - y/u$,

$$\mathbf{P}_{0\tau}(x,y) = \frac{1}{2\pi\sqrt{1-r^2(\tau)}} \times$$

$$\times \exp\left(-\frac{1}{2}\cdot\frac{(u-x/u)^2 - 2r(\tau)(u-x/u)(u-y/u) + (u-y/u)^2}{1-r^2(\tau)}\right)$$

$$= \frac{1}{2\pi\sqrt{1-r^2(\tau)}} \exp\left(-\frac{u^2}{1+r(\tau)}\right) \times$$

$$\times \exp\left(-\frac{1}{2}\cdot\frac{\frac{x^2+y^2}{u^2} - 2x - 2y + 2r(\tau)(x+y) - 2r(\tau)\frac{xy}{u^2}}{1-r^2(\tau)}\right)$$

$$= \frac{1}{2\pi\sqrt{1-r^2(\tau)}} \exp\left(-\frac{u^2}{1+r(\tau)}\right) \cdot \tilde{\mathbf{P}}(u,x,y).$$

Hence,

$$\mathbf{P} = \frac{1}{2\pi\sqrt{1-r^2(\tau)}}\frac{1}{u^2} \exp\left(-\frac{u^2}{1+r(\tau)}\right) \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{P}\left(\max_{t\in u^{-2/\alpha}T_1} X(t) > u,\right.$$

$$\left.\max_{t\in\tau+u^{-2/\alpha}T_2} X(t) > u \,\middle|\, X(0) = u - x/u, X(\tau) = u - y/u\right) \tilde{\mathbf{P}}(u,x,y)\, dxdy.$$

Consider the following families of random processes,

$$\xi_u(t) = u\left(X(u^{-2/\alpha}t) - u\right) + x,\ t \in T_1,$$

$$\eta_u(t) = u\left(X(\tau + u^{-2/\alpha}t) - u\right) + y,\ t \in T_2.$$

We have,

$$\mathbf{P} = \frac{1}{2\pi\sqrt{1-r^2(\tau)}}\frac{1}{u^2} \exp\left(-\frac{u^2}{1+r(\tau)}\right) \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{P}\left(\max_{t\in T_1}\xi_u(t) > x,\right.$$

$$\left.\max_{t\in T_2}\eta_u(t) > y \,\middle|\, X(0) = u - x/u, X(\tau) = u - y/u\right) \tilde{\mathbf{P}}(u,x,y)\, dxdy.$$

Compute first two conditional moments of Gaussian random vector process $(\xi_u(t), \eta_u(t))^\top$. We have

$$\mathbf{E}\begin{pmatrix}\xi_u(t) \\ \eta_u(t)\end{pmatrix}\!\!\left|\begin{matrix}X(0)\\X(\tau)\end{matrix}\right) = \mathbf{E}\begin{pmatrix}\xi_u(t)\\ \eta_u(t)\end{pmatrix} + \mathbf{A}\begin{pmatrix}X(0)\\X(\tau)\end{pmatrix},$$

where

$$\mathbf{A} = \mathbf{cov}\left(\begin{pmatrix}\xi_u(t)\\ \eta_u(t)\end{pmatrix}, \begin{pmatrix}X(0)\\X(\tau)\end{pmatrix}\right)\left[\mathbf{E}\left(\begin{pmatrix}X(0)\\X(\tau)\end{pmatrix}\begin{pmatrix}X(0)\\X(\tau)\end{pmatrix}^\top\right)\right]^{-1},$$

or

$$\mathbf{A} = \frac{u}{1-r^2(\tau)}\begin{pmatrix} r(u^{-2/\alpha}t) - r(\tau)r(\tau - u^{-2/\alpha}t) & r(\tau - u^{-2/\alpha}t) - r(\tau)r(u^{-2/\alpha}t) \\ r(\tau + u^{-2/\alpha}t) - r(\tau)r(u^{-2/\alpha}t) & r(u^{-2/\alpha}t) - r(\tau)r(\tau + u^{-2/\alpha}t)\end{pmatrix}.$$

We denote $\mathbf{cov}X$, the matrix of covariances of a vector X and $\mathbf{cov}(X,Y)$, the matrix of cross-covariances between components of X and Y. Substituting the values $X(0) = u - x/u$, $X(\tau) = u - y/u$, of the conditions, we get from here that

$$\mathbf{E}\left(\begin{matrix}\xi_u(t)\\ \eta_u(t)\end{matrix}\right|\left.\begin{matrix}X(0) = u - x/u\\ X(\tau) = u - y/u\end{matrix}\right) = \begin{pmatrix} \frac{1}{1-r^2(\tau)}\left(r(u^{-2/\alpha}t)\left(u^2 - x - r(\tau)(u^2-y)\right) + \right. \\ \left. + r(\tau - u^{-2/\alpha}t)\left(u^2 - y - r(\tau)(u^2 - x)\right)\right) - u^2 + x \\ \\ \frac{1}{1-r^2(\tau)}\left(r(u^{-2/\alpha}t)\left(u^2 - y - r(\tau)(u^2-x)\right) + \right. \\ \left. + r(\tau + u^{-2/\alpha}t)\left(u^2 - x - r(\tau)(u^2 - y)\right)\right) - u^2 + y\end{pmatrix}. \tag{4}$$

In conditions of every lemma 1-3 we have

$$\mathbf{E}\left(\begin{matrix}\xi_u(t)\\ \eta_u(t)\end{matrix}\right|\left.\begin{matrix}X(0) = u - x/u\\ X(\tau) = u - y/u\end{matrix}\right) = \tag{5}$$

$$\begin{pmatrix} -\frac{1}{1+r(\tau)}|t|^\alpha + o(1) + u^2\frac{r(\tau - u^{-2/\alpha}t) - r(\tau)}{1+r(\tau)} + (y - xr(\tau))\frac{r(\tau) - r(\tau - u^{-2/\alpha}t)}{1-r^2(\tau)} \\ -\frac{1}{1+r(\tau)}|t|^\alpha + o(1) + u^2\frac{r(\tau + u^{-2/\alpha}t) - r(\tau)}{1+r(\tau)} + (y - xr(\tau))\frac{r(\tau) - r(\tau + u^{-2/\alpha}t)}{1-r^2(\tau)}\end{pmatrix}$$

as $u \to \infty$.

Now, let conditions of the Lemma 1 be fulfilled. Since $\alpha < 2$ and $r'(\tau) = O(\sqrt{\log u}/u)$ uniformly in $|\tau - t_m| \le C\sqrt{\log u}/u$, we have,

$$\left|u^2\frac{r(\tau - u^{-2/\alpha}t) - r(\tau)}{1 + r(\tau)}\right| \le \max_{|\tau - t_m| \le C\sqrt{\log u}/u}\left|u^2(-u^{-2/\alpha}t)\frac{r'(\tau)}{1 + r(\tau)}\right| = o(1). \tag{6}$$

Thus

$$\mathbf{E}\left(\begin{matrix}\xi_u(t)\\ \eta_u(t)\end{matrix}\right|\left.\begin{matrix}X(0) = u - x/u\\ X(\tau) = u - y/u\end{matrix}\right) = \begin{pmatrix} -\frac{1}{1+r(t_m)}|t|^\alpha + o(1) \\ -\frac{1}{1+r(t_m)}|t|^\alpha + o(1).\end{pmatrix} \tag{7}$$

as $u \to \infty$.

Let now the conditions of Lemma 2 be fulfilled, that is $\alpha < 1$. In this situation even for fixed τ, by Taylor, the third terms in the column array of right-hand part of (5) tend to zero as $u \to \infty$, hence (7) takes place, with $\theta = \tau$.

Next, let $\alpha = 1$, by differentiability of r,

$$u^2(r(\tau - u^{-2}t) - r(\tau)) \to -tr'(\tau) \text{ and } u^2(r(\tau + u^{-2}t) - r(\tau)) \to tr'(\tau)$$

as $u \to \infty$, therefore in conditions of Lemma 3,

$$\mathbf{E}\begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix}\!\!\left|\begin{matrix} X(0) = u - x/u \\ X(\tau) = u - y/u \end{matrix}\right. = \begin{pmatrix} -\frac{|t|+tr'(\tau)}{1+r(\tau)} + o(1) \\ -\frac{|t|-tr'(\tau)}{1+r(\tau)} + o(1). \end{pmatrix} \tag{8}$$

It is clear that

$$\mathbf{E}\begin{pmatrix} \xi_u(0) \\ \eta_u(0) \end{pmatrix}\!\!\left|\begin{matrix} X(0) = u - x/u \\ X(\tau) = u - y/u \end{matrix}\right. = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{9}$$

$$\mathbf{E}\begin{pmatrix} \xi_u^2(0) \\ \eta_u^2(0) \end{pmatrix}\!\!\left|\begin{matrix} X(0) = u - x/u \\ X(\tau) = u - y/u \end{matrix}\right. = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Computing conditional covariance matrix, we have,

$$\mathbf{cov}\left(\begin{pmatrix} \xi_u(t) - \xi_u(s) \\ \eta_u(t) - \eta_u(s) \end{pmatrix}\middle| \begin{matrix} X(0) \\ X(\tau) \end{matrix}\right) = \mathbf{cov}\begin{pmatrix} \xi_u(t) - \xi_u(s) \\ \eta_u(t) - \eta_u(s) \end{pmatrix} - \mathbf{B}\mathbf{cov}\begin{pmatrix} X(0) \\ X(\tau) \end{pmatrix}\mathbf{B}^\top,$$

where

$$\mathbf{B} = \mathbf{cov}\left(\begin{pmatrix} \xi_u(t) - \xi_u(s) \\ \eta_u(t) - \eta_u(s) \end{pmatrix}, \begin{pmatrix} X(0) \\ X(\tau) \end{pmatrix}\right)\left[\mathbf{E}\left(\begin{pmatrix} X(0) \\ X(\tau) \end{pmatrix}\begin{pmatrix} X(0) \\ X(\tau) \end{pmatrix}^\top\right)\right]^{-1}.$$

Using expressions for $\xi_u(t)$ and $\eta_u(t)$,

$$\mathbf{B} = \frac{u}{1 - r^2(\tau)}\begin{pmatrix} r(u^{-2/\alpha}t) - r(\tau)r(\tau - u^{-2/\alpha}t) - & r(\tau - u^{-2/\alpha}t) - r(\tau)r(u^{-2/\alpha}t) - \\ -r(u^{-2/\alpha}s) + r(\tau)r(\tau - u^{-2/\alpha}s) & -r(\tau - u^{-2/\alpha}s) + r(\tau)r(u^{-2/\alpha}s) \\ & \\ r(\tau + u^{-2/\alpha}t) - r(\tau)r(u^{-2/\alpha}t) - & r(u^{-2/\alpha}t) - r(\tau)r(\tau + u^{-2/\alpha}t) - \\ -r(\tau + u^{-2/\alpha}s) + r(\tau)r(u^{-2/\alpha}s) & r(u^{-2/\alpha}s) + r(\tau)r(\tau + u^{-2/\alpha}s) \end{pmatrix}.$$

Letting now $u \to \infty$, we get

$$\mathbf{cov}\begin{pmatrix} \xi_u(t) - \xi_u(s) \\ \eta_u(t) - \eta_u(s) \end{pmatrix}\!\!\left|\begin{matrix} X(0) = u - \xi/u \\ X(\tau) = u - \eta/u \end{matrix}\right. = \begin{pmatrix} 2|t-s|^\alpha(1+o(1)) & o(1) \\ o(1) & 2|t-s|^\alpha(1+o(1)) \end{pmatrix}, \tag{10}$$

where $o(1)$s are uniform of x and y, moreover they do not depends of values of conditions $X(0)$ and $X(\tau)$. Note that (10) holds true for all $\alpha \in (0, 2)$. From (10) it also followed that for some $C > 0$ all t, s and all sufficiently large u,

$$\mathbf{var}\left(\xi_u(t) - \xi_u(s)|\,(X(0), X(\tau)) = (u - x/u, u - y/u)\right) \le C|t-s|^\alpha, \tag{11}$$

$$\mathbf{var}\left(\eta_u(t) - \eta_u(s)|\,(X(0), X(\tau)) = (u - x/u, u - y/u)\right) \le C|t-s|^\alpha. \tag{12}$$

Thus from (7-11) it follows that the family of conditional Gaussian distributions

$$\mathbf{P}\begin{pmatrix} \xi_u(\cdot) \\ \eta_u(\cdot) \end{pmatrix}\!\!\left|\begin{matrix} X(0) = u - x/u \\ X(\tau) = u - y/u \end{matrix}\right., \tag{13}$$

is weakly compact in $C(T_1) \times C(T_2)$ and converges weakly, under conditions of Lemmas 1 and 2, to the distribution of the random vector process

$$(\xi(t), \eta(t))^\top = (B_\alpha(t) - |t|^\alpha/(1 + r(\tau)), \tilde{B}_\alpha(t) - |t|^\alpha/(1 + r(\tau)))^\top,$$

$t \in \mathbb{R}$, where $\tilde{B}$ is an independent copy of B. If the conditions of Lemma 3 are fulfilled, the family of Gaussian conditional distributions converges to the distribution of

$$(\xi(t), \eta(t))^\top = (B_1(t) - (|t| + tr'(\tau))/(1+r(\tau)), \tilde{B}_1(t) - (|t| - tr'(\tau))/(1+r(\tau)))^\top.$$

Thus

$$\lim_{u\to\infty} \mathbf{P}\left(\max_{t\in T_1} \xi_u(t) > x, \max_{t\in T_2} \eta_u(t) > y \,\middle|\, X(0) = u - x/u, X(\tau) = u - y/u\right)$$
$$= \mathbf{P}\left(\max_{t\in T_1} \xi(t) > x, \max_{t\in T_2} \eta(t) > y\right).$$

In order to prove a convergence of the integral

$$\mathbf{I}(T_1, T_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{P}\left(\max_{t\in T_1} \xi_u(t) > x,\right.$$
$$\left. \max_{t\in T_2} \eta_u(t) > y \,\middle|\, X(0) = u - x/u, X(\tau) = u - y/u\right) \tilde{\mathbf{P}}(u, x, y)\, dxdy$$

as $u \to \infty$, we construct an integrable dominating function, which have different representation in different quadrants of the plane.

1. For the quadrant $(x < 0, y < 0)$ we bound the probability by 1, and the $\tilde{\mathbf{P}}(u,x,y)$ by $\exp(\frac{x+y}{1+r(t_m)})$, using relations $|r(t)| \le 1$ and $x^2 + y^2 \ge 2xy$. The last function is integrable in the considered quadrant, so it is a desirable dominating function.

2. Within the quadrant $(x > 0, y < 0)$ we bound the probability by

$$\mathbf{P}\left(\max_{t\in T_1} \xi_u(t) > x, \,|\, X(0) = u - x/u, X(\tau) = u - y/u\right)$$

and, using arguments similar the above, we bound $\tilde{\mathbf{P}}(u,x,y)$ by

$$\exp\left(\frac{y}{1+r(t_m)} + \frac{x}{0.9 + r(t_m)}\right),$$

for sufficiently large u. The function $p(x)$ can be bounded by a function of type $C\exp(-\epsilon x^2)$, ϵ is positive, using, for example the Borel inequality with relations (7 - 10). Similar arguments one can find in [2].

3. Considerations in the quarter-plane $(x < 0, y > 0)$ are similar, the dominating function is

$$C\exp(-\epsilon y^2)\exp\left(\frac{x}{1+r(t_m)} + \frac{y}{0.9 + r(t_m)}\right).$$

4. In the quarter-plane $(x > 0, y > 0)$ we bound $\tilde{\mathbf{P}}$ by

$$\exp\left(\frac{x}{0.9+r(t_m)} + \frac{y}{0.9 + r(t_m)}\right)$$

and the probability by

$$\mathbf{P}\left(\max_{(t,s)\in T_1\times T_2}\xi_u(t)+\eta_u(s)>x+y\mid X(0)=u-x/u, X(\tau)=u-y/u\right).$$

Again, for the probability we can apply the Borel inequality, just in the same way, to get the bound $C\exp(-\epsilon(x+y)^2)$, for a positive ϵ.

Thus we have the desirable domination on the hole plane and therefore we have,

$$\lim_{u\to\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{P}\left(\max_{t\in T_1}\xi_u(t)>x,\right.$$

$$\left.\max_{t\in T_2}\eta_u(t)>y\,\middle|\,X(0)=u-x/u, X(\tau)=u-y/u\right)\tilde{\mathbf{P}}(u,x,y)\,dxdy$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{\frac{x+y}{1+r(t_m)}}\mathbf{P}\left(\max_{t\in T_1}\xi(t)>x,\max_{t\in T_2}\eta(t)>y\right)dxdy$$

$$=\int_{-\infty}^{+\infty}e^{\frac{x}{1+r(t_m)}}\mathbf{P}\left(\max_{t\in T_1}\xi(t)>x\right)dx\int_{-\infty}^{+\infty}e^{\frac{y}{1+r(\tau)}}\mathbf{P}\left(\max_{t\in T_2}\eta(t)>y\right)dy.$$

Then we proceed,

$$\int_{-\infty}^{+\infty}e^{\frac{x}{1+r(\theta)}}\mathbf{P}\left(\max_{t\in T_1}\xi(t)>x\right)dx=$$

$$=(1+r(\theta))\mathbf{E}\exp\left[\frac{\max_{T_1}\xi(t)}{1+r(\theta)}\right]=(1+r(\theta))\mathbf{E}\exp\left[\frac{\max_{T_1}B_\alpha(t)-\frac{|t|^\alpha}{1+r(\theta)}}{1+r(\theta)}\right]=$$

$$=(1+r(\theta))\mathbf{E}\exp\left[\max_{T_1}B_\alpha\left(\frac{t}{(1+r(\theta))^{2/\alpha}}\right)-\left(\frac{t}{(1+r(\theta))^{2/\alpha}}\right)^\alpha\right]=$$

$$=(1+r(\theta))\mathbf{E}\exp\left[\max_{T_1/(1+r(\theta))^{2/\alpha}}B_\alpha(s)-s^\alpha\right]=(1+r(\theta))H_\alpha\left(\frac{T_1}{(1+r(\theta))^{2/\alpha}}\right),$$

where we use self-similarity properties of Fractional Brownian Motion. Similarly for $\eta(t)$, $t\in T_2$. Similarly for $H_1^{\pm r'(\tau)}$. Thus Lemmas follow.

The following lemma is proved in [2] in multidimensional case. We formulate it here for one-dimensional time.

Lemma 4 *Suppose that $X(t)$ is a Gaussian stationary zero mean process with covariance function $r(t)$ satisfying assumption* **A1**. *Let ε, $\frac{1}{2}>\varepsilon>0$ be such that*

$$1-\frac{1}{2}|t|^\alpha\geq r(t)\geq 1-2|t|^\alpha$$

for all $t\in[0,\varepsilon]$. Then there exists an absolute constant F such that the inequality

$$\mathbf{P}\left(\max_{t\in[0,Tu^{-2/\alpha}]}X(t)>u,\max_{t\in[t_0u^{-2/\alpha},(t_0+T)u^{-2/\alpha}]}X(t)>u\right)\leq FT^2u^{-1}e^{-\frac{1}{2}u^2-\frac{1}{8}(t_0-T)^\alpha}$$

holds for any T, $t_0 > T$ and for any $u \ge (4(T+t_0)/\varepsilon)^{\alpha/2}$.

The following two lemmas are straightforward consequences of Lemma 6.1, [2].

Lemma 5 *Suppose that $X(t)$ is a Gaussian stationary zero mean process with covariance function $r(t)$ satisfying assumption* **A1**. *Then*

$$\mathbf{P}\left(\max_{t\in[0,Tu^{-2/\alpha}]\cup[t_0u^{-2/\alpha},(t_0+T)u^{-2/\alpha}]} X(t) > u\right) = H_\alpha([0,T]\cup[t_0,t_0+T])\frac{1}{\sqrt{2\pi}u}e^{-\frac{1}{2}u^2}(1+o(1))$$

as $u \to \infty$, where

$$H_\alpha([0,T]\cup[t_0,t_0+T]) = \mathbf{E}\exp\left(\max_{t\in[0,T]\cup[t_0,t_0+T]}(B_\alpha(t)-|t|^\alpha)\right).$$

Lemma 6 *Suppose that $X(t)$ is a Gaussian stationary zero mean process with covariance function $r(t)$ satisfying assumption* **A1**. *Then*

$$\mathbf{P}\left(\max_{t\in[0,Tu^{-2/\alpha}]} X(t) > u, \max_{t\in[t_0u^{-2/\alpha},(t_0+T)u^{-2/\alpha}]} X(t) > u\right)$$
$$= H_\alpha([0,T],[t_0,t_0+T])\frac{1}{\sqrt{2\pi}u}e^{-\frac{1}{2}u^2}(1+o(1))$$

as $u \to \infty$, where

$$H_\alpha([0,T],[t_0,t_0+T]) = \int_{-\infty}^{\infty} e^s\mathbf{P}\left(\max_{t\in[0,T]} B_\alpha(t)-|t|^\alpha > s, \max_{t\in[t_0,t_0+T]} B_\alpha(t)-|t|^\alpha > s\right)ds.$$

Proof. Write

$$\mathbf{P}\left(\max_{t\in[0,Tu^{-2/\alpha}]} X(t) > u, \max_{t\in[t_0u^{-2/\alpha},(t_0+T)u^{-2/\alpha}]} X(t) > u\right)$$
$$= \mathbf{P}\left(\max_{t\in[0,Tu^{-2/\alpha}]} X(t) > u\right) + \mathbf{P}\left(\max_{t\in[t_0u^{-2/\alpha},(t_0+T)u^{-2/\alpha}]} X(t) > u\right)$$
$$-\mathbf{P}\left(\max_{t\in[0,Tu^{-2/\alpha}]\cup[t_0u^{-2/\alpha},(t_0+T)u^{-2/\alpha}]} X(t) > u\right)$$

and apply Lemma 6.1, [2] and Lemma 3 to the right-hand part.

From Lemmas 4 and 2 we get,

Lemma 7 *For any $t_0 > T$,*

$$H_\alpha([0,T],[t_0,t_0+T]) \le F\sqrt{2\pi}T^2e^{-\frac{1}{8}(t_0-T)^\alpha}.$$

When $t_0 = T$ the Lemma holds true, but the bound is trivial. A non-trivial bound for $H_\alpha([0,T],[T,2T])$ one can get from the proof of Lemma 7.1, [2], see page 107, inequalities (7.5) and the previous one. These inequalities, Lemma 6.8, [2] and Lemma 5 give the following,

Lemma 8 *There exists a constant F_1 such that for all $T \geq 1$,*

$$H_\alpha([0,T],[T,2T]) \leq F_1 \left(\sqrt{T} + T^2 e^{-\frac{1}{8}T^{\alpha/2}}\right).$$

Applying Lemma 1 to the sets $T_1 = [0,T] \cup [t_0, t_0 + T]$, $T_2 = [0,T] \cup [t_1, t_1 + T]$ and combining probabilities similarly as in the proof of Lemma 4, we get,

Lemma 9 *Let $X(t)$ be a Gaussian process with mean zero and covariance function $r(t)$ satisfying conditions of Theorem 1. Let $\tau = \tau(u)$ tends to t_m as $u \to \infty$ in such a way that $|\tau - t_m| \leq C\sqrt{\log u}/u$, for some positive C. Then for all $T > 0$, $t_0 \geq T$, $t_1 \geq T$*

$$\begin{aligned}
&\mathbf{P}\Bigg(\max_{t\in[0,u^{-2/\alpha}T]} X(t) > u, \max_{t\in[u^{-2/\alpha}t_0, u^{-2/\alpha}(t_0+T)]} X(t) > u,\\
&\qquad \max_{t\in[\tau,\tau+u^{-2/\alpha}T]} X(t) > u, \max_{t\in[\tau+u^{-2/\alpha}t_1, \tau+u^{-2/\alpha}(t_1+T)]} X(t) > u\Bigg)\\
&= \frac{(1+r(t_m))^2}{2\pi\sqrt{1-r^2(t_m)}} \cdot \frac{1}{u^2} e^{-\frac{u^2}{1+r(\tau)}}\\
&\times H_\alpha\left(\left[0, \frac{T}{(1+r(t_m))^{2/\alpha}}\right], \left[\frac{t_0}{(1+r(t_m))^{2/\alpha}}, \frac{t_0+T}{(1+r(t_m))^{2/\alpha}}\right]\right)\\
&\times H_\alpha\left(\left[0, \frac{T}{(1+r(t_m))^{2/\alpha}}\right], \left[\frac{t_1}{(1+r(t_m))^{2/\alpha}}, \frac{t_1+T}{(1+r(t_m))^{2/\alpha}}\right]\right)(1+o(1)),
\end{aligned}$$

as $u \to \infty$.

3 Proofs

3.1 Proof of Theorem 1

We denote $\Pi = [T_1, T_2] \times [T_3, T_4]$, $\delta = \delta(u) = C\sqrt{\log u}/u$, the value of the positive C we specify later on. $D = \{(t,s) \in \Pi : \ |t - s - t_m| \leq \delta\}$. We have,

$$\begin{aligned}
&\mathbf{P}\left(\max_{t\in[T_1,T_2]} X(t) > u, \max_{t\in[T_3,T_4]} X(t) > u\right) = \mathbf{P}\left(\bigcup_{(s,t)\in\Pi} \{X(t) > u\} \cap \{X(s) > u\}\right)\\
&= \mathbf{P}\left(\left\{\bigcup_{(s,t)\in D} \{X(t) > u\} \cap \{X(s) > u\}\right\} \bigcup \left\{\bigcup_{(s,t)\in\Pi\setminus D} \{X(t) > u\} \cap \{X(s) > u\}\right\}\right)\\
&\leq \mathbf{P}\left(\bigcup_{(s,t)\in D} \{X(t) > u\} \cap \{X(s) > u\}\right) + \mathbf{P}\left(\bigcup_{(s,t)\in\Pi\setminus D} \{X(t) > u\} \cap \{X(s) > u\}\right). \quad (14)
\end{aligned}$$

From the other hand,

$$\mathbf{P}\left(\max_{t\in[T_1,T_2]} X(t) > u, \max_{t\in[T_3,T_4]} X(t) > u\right) = \mathbf{P}\left(\bigcup_{(s,t)\in\Pi} \{X(t) > u\} \cap \{X(s) > u\}\right)$$

$$= \mathbf{P}\left(\left\{\bigcup_{(s,t)\in D} \{X(t) > u\} \cap \{X(s) > u\}\right\} \bigcup \left\{\bigcup_{(s,t)\in\Pi\setminus D} \{X(t) > u\} \cap \{X(s) > u\}\right\}\right)$$

$$\geq \mathbf{P}\left(\bigcup_{(s,t)\in D} \{X(t) > u\} \cap \{X(s) > u\}\right). \tag{15}$$

The second term in the right-hand part of (14) we estimate as following,

$$\mathbf{P}\left(\bigcup_{(s,t)\in\Pi\setminus D} \{X(t) > u\} \cap \{X(s) > u\}\right) \leq \mathbf{P}\left(\max_{(s,t)\in\Pi\setminus D} X(t) + X(s) > 2u\right). \tag{16}$$

Making use of Theorem 8.1, [2], we get that the last probability does not ecceed

$$const \cdot u^{-1+2/\alpha} \exp\left(-\frac{u^2}{1 + \max_{(t,s)\in\Pi\setminus D} r(t-s)}\right). \tag{17}$$

Further, for $\epsilon = 1/6$ and all sufficiently large u,

$$\max_{(t,s)\in\Pi\setminus D} r(t-s) \leq r(t_m) + (\frac{1}{2} - \epsilon) r''(t_m)\delta^2 = r(t_m) + \frac{1}{3} C^2 r''(t_m) \log u/u.$$

Hence,

$$\mathbf{P}\left(\bigcup_{(s,t)\in\Pi\setminus D} \{X(t) > u\} \cap \{X(s) > u\}\right) \leq const \cdot u^{-1+2/\alpha} \exp\left(-\frac{u^2}{1 + r(t_m)}\right) u^{-G}, \tag{18}$$

where

$$G = \frac{-2C^2 r''(t_m)}{3(1 + r(t_m))^2}.$$

Now we deal with the first probability in the right-hand part of (14). It is equal to the probability in right-hand part of (15). We are hence in a position to bound the probability from above and from below getting equal orders for the bounds. Denote $\Delta = Tu^{-2/\alpha}$, $T > 0$, and define the intervals

$$\Delta_k = [T_1 + k\Delta, T_1 + (k+1)\Delta],\ 0 \leq k \leq N_k,\ N_k = [(T_2 - T_1)/\Delta],$$
$$\Delta_l = [T_3 + l\Delta, T_3 + (l+1)\Delta],\ 0 \leq l \leq N_l,\ N_l = [(T_4 - T_3)/\Delta],$$

where $[\cdot]$ stands for the integer part of a number. In virtue of Lemma 1,

$$\mathbf{P}\left(\bigcup_{(s,t)\in D}\{X(t)>u\}\cap\{X(s)>u\}\right)$$

$$\le \mathbf{P}\left(\bigcup_{(k,l):\,\Delta_k\cap D\neq\emptyset,\,\Delta_l\cap D\neq\emptyset}\ \bigcup_{t\in\Delta_k,s\in\Delta_l}\{X(t)>u\}\cap\{X(s)>u\}\right)$$

$$\le \sum_{(k,l):\,\Delta_k\cap D\neq\emptyset,\,\Delta_l\cap D\neq\emptyset}\mathbf{P}\left(\max_{t\in\Delta_k}X(t)>u,\ \max_{t\in\Delta_l}X(t)>u\right)$$

$$\le \frac{(1+\gamma(u))}{2\pi u^2\sqrt{1-r^2(t_m)}}H_\alpha^2\left(\frac{T}{(1+r(t_m))^{2/\alpha}}\right)\sum_{(k,l):\,\Delta_k\cap D\neq\emptyset,\,\Delta_l\cap D\neq\emptyset}\exp\left(-\frac{u^2}{1+r(\tau_{k,l})}\right), \quad (19)$$

where $\gamma(u)\downarrow 0$ as $u\to\infty$ and $\tau_{k,l}=T_3-T_1+(l-k)\Delta$. For the last sum we get,

$$S=\sum_{(k,l):\,\Delta_k\cap D\neq\emptyset,\,\Delta_l\cap D\neq\emptyset}\exp\left(-\frac{u^2}{1+r(\tau_{k,l})}\right)$$

$$=\exp\left(-\frac{u^2}{1+r(t_m)}\right)\sum_{(k,l):\,\Delta_k\cap D\neq\emptyset,\,\Delta_l\cap D\neq\emptyset}\exp\left(-u^2\frac{r(t_m)-r(\tau_{k,l})}{(1+r(\tau_{k,l}))(1+r(t_m)}\right).$$

Define θ by $t_m=T_3-T_1+\Delta\theta$, we obtain,

$$\frac{r(t_m)-r(\tau_{k,l})}{(1+r(\tau_{k,l}))(1+r(t_m)}\le(\ge)\frac{-\frac{1}{2}r''(t_m)(\tau_{k,l}-t_m)^2}{(1+r(t_m))^2}(1+(-)\gamma_1(u))$$

$$=-A((k-l)\Delta-\theta\Delta)^2(1+(-)\gamma_1(u)),$$

where $\gamma_1(u)\downarrow 0$ as $u\to\infty$. In the last sum, index k variates between $(T_{\min}+O(\delta(u)))/\Delta$ and $(T_{\max}+O(\delta(u)))/\Delta$, as $u\to\infty$, where $T_{\min}=T_1\vee(T_3-t_m)$ and $T_{\max}=T_2\wedge(T_4-t_m)$. Indeed, for the co-ordinate x of the left end of a segment of length t_m which variates having left end inside $[T_1,T_2]$ and right end inside $[T_3,T_4]$, we have the restrictions $T_1<x<T_2$, and $T_3<x+t_m<T_4$, so that $x\in(T_{\min},T_{\max})$. The index $m=k-l-\theta$ variates thus between $-\delta(u)/\Delta+O(\Delta)$ and $\delta(u)/\Delta+O(\Delta)$ as $u\to\infty$. Note that $u\Delta\to 0$ as $u\to\infty$. Using this, we continue,

$$S=(1+o(1))\exp\left(-\frac{u^2}{1+r(t_m)}\right)\frac{T_{\max}-T_{\min}}{\Delta}\sum_{m=-\delta(u)/\Delta+O(\Delta)}^{\delta(u)/\Delta+O(\Delta)}\exp\left(-A(mu\Delta)^2\right)$$

$$=(1+o(1))\exp\left(-\frac{u^2}{1+r(t_m)}\right)\frac{T_{\max}-T_{\min}}{u\Delta^2}\sum_{mu\Delta=-u\delta(u)+O(u\Delta^2)}^{u\delta(u)+O(u\Delta^2)}\exp\left(-A(mu\Delta)^2\right)u\Delta$$

$$=(1+o(1))\exp\left(-\frac{u^2}{1+r(t_m)}\right)\frac{T_{\max}-T_{\min}}{u\Delta^2}\int_{-\infty}^{\infty}e^{-Ax^2}dx.$$

Compute the integral and substitute this in right-hand part of (19), we get,

$$\mathbf{P}\left(\bigcup_{(s,t)\in D}\{X(t)>u\}\cap\{X(s)>u\}\right) \tag{20}$$
$$\leq \frac{(1+r(t_m))^2(1+\gamma_2(u))(T_{\max}-T_{\min})u^{-3+4/\alpha}}{2\sqrt{A\pi}(1-r^2(t_m)}\frac{1}{T^2}H_\alpha^2\left(\frac{T}{(1+r(t_m))^{2/\alpha}}\right)\exp\left(-\frac{u^2}{1+r(t_m)}\right),$$

where $\gamma_2(u)\downarrow 0$ as $u\to\infty$.

Now we bound from below the probability in the right-hand part of (15). We have

$$\mathbf{P}\left(\bigcup_{(s,t)\in D}\{X(t)>u\}\cap\{X(s)>u\}\right)$$
$$\geq \mathbf{P}\left(\bigcup_{(k,l):\,\Delta_k\subset D,\Delta_l\subset D}\ \bigcup_{t\in\Delta_k,s\in\Delta_l}\{X(t)>u\}\cap\{X(s)>u\}\right)$$
$$\geq \sum_{(k,l):\,\Delta_k\subset D,\Delta_l\subset D}\mathbf{P}\left(\max_{t\in\Delta_k}X(t)>u,\ \max_{t\in\Delta_l}X(t)>u\right)$$
$$-\sum\sum\mathbf{P}\left(\max_{t\in\Delta_k}X(t)>u,\ \max_{t\in\Delta_l}X(t)>u,\ \max_{t\in\Delta_{k'}}X(t)>u,\ \max_{t\in\Delta_{l'}}X(t)>u\right), \tag{21}$$

where the double-sum is taken over the set

$$\{(k,l,k',l'):\,(k',l')\neq(k,l),\ \Delta_k\cap D\neq\emptyset,\ \Delta_l\cap D\neq\emptyset,\ \Delta_{k'}\cap D\neq\emptyset,\ \Delta_{l'}\cap D\neq\emptyset\}.$$

The first sum in the right-hand part of (21) can be bounded from below exactly by the same way as the previous sum, thus we have,

$$\sum_{(k,l):\,\Delta_k\subset D,\,\Delta_l\subset D}\mathbf{P}\left(\max_{t\in\Delta_k}X(t)>u,\ \max_{t\in\Delta_l}X(t)>u\right) \tag{22}$$
$$\geq \frac{(1+r(t_m))^2(1-\gamma_2(u))(T_{\max}-T_{\min})u^{-3+4/\alpha}}{2\sqrt{A\pi}(1-r^2(t_m)}\frac{1}{T^2}H_\alpha^2\left(\frac{T}{(1+r(t_m))^{2/\alpha}}\right)\exp\left(-\frac{u^2}{1+r(t_m)}\right),$$

where $\gamma_2(u)\downarrow 0$ as $u\to\infty$. We are now able to select the constant C. We take it as large as $G>2-2/\alpha$ to get that left-hand part of (18) is infinitely smaller then left-hand part of (22) as $u\to\infty$.

Consider the second sum (the double-sum) in the right-hand part of (21). For sakes of simplicity we denote

$$H(m)=H_\alpha\left(\left[0,\frac{T}{(1+r(t_m))^{2/\alpha}}\right],\left[\frac{mT}{(1+r(t_m))^{2/\alpha}},\frac{(m+1)T}{(1+r(t_m))^{2/\alpha}}\right]\right)$$

and notice that

$$H(0)=H_\alpha\left(\left[0,\frac{T}{(1+r(t_m))^{2/\alpha}}\right]\right).$$

In virtue of Lemma 9 we have for the double-sum in (21), taking into account only different (k,l) and (k',l'),

$$\begin{aligned}
\Sigma_2 &:= \sum\sum \mathbf{P}\left(\max_{t\in\Delta_k} X(t) > u,\ \max_{t\in\Delta_l} X(t) > u,\ \max_{t\in\Delta_{k'}} X(t) > u,\ \max_{t\in\Delta_{l'}} X(t) > u\right) \\
&\le \frac{(1+r(t_m))^2(1+\Gamma(u))}{2\pi u^2\sqrt{1-r^2(t_m)}} \sum\sum H(|k-k'|)H(|l-l'|)\exp\left(-\frac{u^2}{1+r(\tau_{k,l})}\right) \\
&= \frac{2(1+r(t_m))^2(1+\Gamma(u))}{2\pi u^2\sqrt{1-r^2(t_m)}} \sum_{n=1}^{\infty} H(n)\left(H(0)+2\sum_{m=1}^{\infty} H(m)\right) \\
&\quad\times \sum_{(k,l):\ \Delta_k\cap D\neq\emptyset,\ \Delta_l\cap D\neq\emptyset} \exp\left(-\frac{u^2}{1+r(\tau_{k,l})}\right),
\end{aligned}$$

where $\Gamma(u)\downarrow 0$ as $u\to\infty$. The last sum is already bounded from above, therefore by (19) and (20) we have,

$$\begin{aligned}
\Sigma_2 &\le \frac{2}{T^2}\sum_{n=1}^{\infty} H(n)\left(H(0)+2\sum_{m=1}^{\infty} H(m)\right) \\
&\quad\times \frac{(1+r(t_m))^2(1+\Gamma_2(u))(T_{\max}-T_{\min})u^{-3+4/\alpha}}{2\sqrt{A\pi(1-r^2(t_m)}}\exp\left(-\frac{u^2}{1+r(t_m)}\right).
\end{aligned}$$

By Lemmas 6.8, [2], 7 and 8 we get that $H(0)\le const\cdot T$, $H(1)\le const\cdot\sqrt{T}$ and for $m>1$,

$$H(m)\le const\cdot e^{-\frac{1}{8}m^{\alpha/2}T^{\alpha/2}},$$

hence

$$\sum_{n=1}^{\infty} H(n)\left(H(0)+2\sum_{m=1}^{\infty} H(m)\right)\le const\cdot T^{3/2}.$$

Thus

$$\Sigma_2\le const\cdot T^{-1/2}u^{-3+4/\alpha}\exp\left(-\frac{u^2}{1+r(t_m)}\right). \tag{23}$$

Now since by (1),

$$\lim_{T\to\infty}\frac{1}{T}H_\alpha\left(\frac{T}{(1+r(t_m))^{2/\alpha}}\right)=(1+r(t_m))^{-2/\alpha}H_\alpha,$$

we get that the double sum can be made infinitely smaller by choosing large T. Thus Theorem 1 follows.

3.2 Proof of Theorem 2.

We prove the theorem for the case $t_m = T_3 - T_2$, another case can be considered similarly. First, as in the proof of Theorem 1 put $D=\{(t,s)\in\Pi:\ |t-s-t_m|\le\delta\}$, but with

$\delta = \delta(u) = C\sqrt{\log u}/u^2$, for sufficiently large C. The evaluations (14), (16) and (17) still hold true. Further we have for $\epsilon = 1/6$ and all sufficiently large u,

$$\max_{(t,s)\in\Pi\setminus D} r(t-s) \le r(t_m) + (\frac{1}{2}-\epsilon)r'(t_m)\delta = r(t_m) + \frac{1}{3}C^2 r'(t_m)\log u/u^2.$$

Hence, (18) holds true with

$$G = \frac{-2C^2 r'(t_m)}{3(1+r(t_m))^2}.$$

Let now $\alpha > 1$. For any positive arbitrarily small ϵ we have for all sufficiently large u that, $\epsilon u^{-2/\alpha} > \delta(u)$, hence for such values of u,

$$\begin{aligned}&\mathbf{P}\left(\bigcup_{(s,t)\in D}\{X(t)>u\}\cap\{X(s)>u\}\right)\\ &\quad\le \mathbf{P}\left(\max_{t\in[T_2-\epsilon u^{-2/\alpha},T_2]} X(t)>u,\ \max_{t\in[T_3,T_3+\epsilon u^{-2/\alpha}]} X(t)>u\right).\end{aligned}\tag{24}$$

We wish to apply Lemma 1 to the last probability for the intervals $[-\epsilon, 0]$ and $[t_m, t_m+\epsilon]$. To this end we turn to (5). Since for a sufficiently small ϵ, $r'(t_m) < 0$, we have that

$$\frac{r(\tau - u^{-2/\alpha}t) - r(\tau)}{1+r(\tau)} < 0 \text{ for all } t\in[-\epsilon,0]$$

and

$$\frac{r(\tau + u^{-2/\alpha}t) - r(\tau)}{1+r(\tau)} < 0 \text{ for all } t\in[t_m,t_m+\epsilon],$$

hence

$$\limsup_{u\to\infty}\mathbf{E}\left(\xi_u(t)|\,X(0)=u-x/u, X(\tau)=u-y/u\right) \le -\frac{1}{1+r(t_m)}|t|^\alpha,$$

for all $t\in[-\epsilon,0]$, and

$$\limsup_{u\to\infty}\mathbf{E}\left(\eta_u(t)|\,X(0)=u-x/u, X(\tau)=u-y/u\right) \le -\frac{1}{1+r(t_m)}|t|^\alpha,$$

for all $t\in[t_m,t_m+\epsilon]$. All other arguments in the proof of Lemma 1 still hold true, therefore, using time-symmetry of the fractional Brownian motion, we have,

$$\begin{aligned}&\limsup_{u\to\infty} u^2 e^{\frac{u^2}{1+r(t_m)}}\mathbf{P}\left(\max_{t\in[T_2-\epsilon u^{-2/\alpha},T_2]} X(t)>u,\ \max_{t\in[T_3,T_3+\epsilon u^{-2/\alpha}]} X(t)>u\right)\\ \le\ & \frac{(1+r(t_m))^2}{2\pi\sqrt{1-r^2(t_m)}}H_\alpha^2\left(\frac{[0,\epsilon]}{(1+r(t_m))^{2/\alpha}}\right)\end{aligned}\tag{25}$$

Using Fatou monotone convergence we have $\lim_{\epsilon\downarrow 0} H_\alpha(\epsilon) = 1$, therefore

$$\begin{aligned}&\limsup_{u\to\infty} u^2 e^{\frac{u^2}{1+r(t_m)}}\mathbf{P}\left(\max_{t\in[T_2-\epsilon u^{-2/\alpha},T_2]} X(t)>u,\ \max_{t\in[T_3,T_3+\epsilon u^{-2/\alpha}]} X(t)>u\right)\\ \le\ & \frac{(1+r(t_m))^2}{2\pi\sqrt{1-r^2(t_m)}}\end{aligned}\tag{26}$$

But

$$P_d(u;[T_1,T_2],[T_3,T_4]) \geq \mathbf{P}\left(X(T_2)>u, X(T_3)>u\right) = \frac{(1+r(t_m))^2}{2\pi u^2\sqrt{1-r^2(t_m)}} e^{-\frac{u^2}{1+r(t_m)}}(1+o(1))$$

as $u \to \infty$. Thus (i) follows.

Let now $\alpha = 1$. From now on, we redefine Δ_k and Δ_l, by

$$\Delta_k = [T_2-(k+1)\Delta, T_2-k\Delta],\ 0\leq k\leq N_k,\ N_k=[(T_2-T_1)/\Delta],$$
$$\Delta_l = [T_3+l\Delta, T_3+(l+1)\Delta],\ 0\leq l\leq N_l,\ N_l=[(T_4-T_3)/\Delta],$$

for the case of Δ_k, $k=0$, we denote $\Delta_0=\Delta_{-0}$, indicating difference with Δ_0 for the case Δ_l, $l=0$. Recall that now $\Delta = Tu^{-2/\alpha} = Tu^{-2}$. We have for sufficiently large u,

$$\mathbf{P}\left(\bigcup_{(s,t)\in D}\{X(t)>u\}\cap\{X(s)>u\}\right) \geq \mathbf{P}\left(\max_{t\in\Delta_{-0}} X(t)>u,\ \max_{t\in\Delta_0} X(t)>u\right), \tag{27}$$

and

$$\mathbf{P}\left(\bigcup_{(s,t)\in D}\{X(t)>u\}\cap\{X(s)>u\}\right) \leq \mathbf{P}\left(\max_{t\in\Delta_{-0}} X(t)>u,\ \max_{t\in\Delta_0} X(t)>u\right)+$$

$$+\sum_{k=0,l=0,\,k+l>0}^{[\log u/T]+1}\mathbf{P}\left(\max_{t\in\Delta_k} X(t)>u,\ \max_{t\in\Delta_l} X(t)>u\right). \tag{28}$$

First probability in right-hand parts of the inequalities is already considered by Lemma 3. We set $\tau=t_m=T_3-T_2$, $T_1=[-T,0]$, $T_2=[0,T]$, by time-symmetry of Brownian motion, we have that

$$H_1^{r'(\tau)}([-T,0]) = H_1^{-r'(\tau)}([0,T]). \tag{29}$$

In order to estimate the sum, we observe, that for all sufficiently large u and all $t\in[T_3,T_3+\delta(u)]$, $s\in[T_2-\delta(u),T_2]$,

$$r(t-s)\leq r(t_m)+\frac{1}{3}r'(t_m)(t-s-t_m)\ \text{ and } r(t-s)\geq r(t_m)+\frac{2}{3}r'(t_m)(t-s-t_m). \tag{30}$$

Hence

$$\begin{aligned}\frac{-u^2}{1+r(t_m+(k+l)\Delta)} &\leq \frac{-u^2}{1+r(t_m)+\frac{1}{3}r'(t_m)(k+l)Tu^{-2}}\\ &\leq \frac{-u^2}{1+r(t_m)}+\frac{r'(t_m)(k+l)T}{6(1+r(t_m))^2} = \frac{-u^2}{1+r(t_m)}-a(k+l)T,\end{aligned}$$

where $a>0$. Now, in Lemma 3 let $\tau=t_m+(k+l)\Delta$, $T_1=[-T,0]$, $T_2=[0,T]$, using the above mentioned property of the constants $H_1^c(T)$, we get, that for all sufficiently large u and T,

$$\mathbf{P}\left(\max_{t\in\Delta_k} X(t)>u,\ \max_{t\in\Delta_l} X(t)>u\right)\leq Cp_2(u,r(\tau_m))e^{-a(k+l)T},$$

From here we get,

$$\sum_{k=0,l=0,\,k+l>0}^{[\log u/T]+1} \mathbf{P}\left(\max_{t\in\Delta_k} X(t) > u,\, \max_{t\in\Delta_l} X(t) > u\right) \le Cp_2(u, r(\tau_m))e^{-a(k+l)T},$$

Applying now Lemma 3 to first summands in right-part hands of (27, 28) and letting $T \to \infty$, we get the assertion (ii) of Theorem.

Let now $\alpha < 1$. Proof of the Theorem in this case is similar to the proof of Theorem 1. We have to consider a sum of small almost equal probabilities and a double sum. Using the more recent definition of Δ_k and Δ_l, we have by Lemma 2,

$$\begin{aligned}
&\mathbf{P}\left(\bigcup_{(s,t)\in D} \{X(t) > u\} \cap \{X(s) > u\}\right) \\
&\le \mathbf{P}\left(\bigcup_{(k,l):\,\Delta_k\cap D\neq\emptyset,\,\Delta_l\cap D\neq\emptyset}\ \bigcup_{t\in\Delta_k, s\in\Delta_l} \{X(t) > u\} \cap \{X(s) > u\}\right) \\
&\le \sum_{(k,l):\,\Delta_k\cap D\neq\emptyset,\,\Delta_l\cap D\neq\emptyset} \mathbf{P}\left(\max_{t\in\Delta_k} X(t) > u,\, \max_{t\in\Delta_l} X(t) > u\right) \\
&\le \frac{(1+r(t_m))^2(1+\gamma(u))}{2\pi u^2\sqrt{1-r^2(t_m)}} H_\alpha^2\left(\frac{T}{(1+r(t_m))^{2/\alpha}}\right) \sum_{(k,l):\,\Delta_k\cap D\neq\emptyset,\,\Delta_l\cap D\neq\emptyset} \exp\left(-\frac{u^2}{1+r(\tau_{k,l})}\right),
\end{aligned} \tag{31}$$

where $\gamma(u) \downarrow 0$ as $u \to \infty$ and now $\tau_{k,l} = T_3 - T_2 + (l+k)\Delta$. For the last sum we get,

$$S = \sum_{(k,l):\,\Delta_k\cap D\neq\emptyset,\,\Delta_l\cap D\neq\emptyset} \exp\left(-\frac{u^2}{1+r(\tau_{k,l})}\right)$$

$$= \exp\left(-\frac{u^2}{1+r(t_m)}\right) \sum_{(k,l):\,\Delta_k\cap D\neq\emptyset,\,\Delta_l\cap D\neq\emptyset} \exp\left(-u^2\frac{r(t_m)-r(\tau_{k,l})}{(1+r(\tau_{k,l}))(1+r(t_m))}\right).$$

Next,

$$\begin{aligned}
\frac{r(t_m)-r(\tau_{k,l})}{(1+r(\tau_{k,l}))(1+r(t_m))} &\le (\ge) \frac{-r'(t_m)(t_m-\tau_{k,l})}{(1+r(t_m))^2}(1+(-)\gamma_1(u)) \\
&= -B(k+l)\Delta(1+(-)\gamma_1(u)),
\end{aligned}$$

where $\gamma_1(u) \downarrow 0$ as $u \to \infty$. Remind that now $u^2\Delta \to 0$ as $u \to \infty$. Using this, and denoting $m = k+l$, we continue,

$$S = (1+o(1))\exp\left(-\frac{u^2}{1+r(t_m)}\right) \sum_{m=0}^{\delta(u)/\Delta+O(\Delta)} m\exp\left(-Bu^2m\Delta\right)$$

$$= (1+o(1))\exp\left(-\frac{u^2}{1+r(t_m)}\right)\frac{1}{(\Delta u^2)^2} \sum_{m=0}^{\delta(u)/\Delta+O(\Delta)} m\Delta u^2\exp\left(-Bm\Delta u^2\right)(\Delta u^2)$$

$$= (1+o(1))\exp\left(-\frac{u^2}{1+r(t_m)}\right)\frac{1}{u^4\Delta^2}\int_0^\infty xe^{-Bx}dx = (1+o(1))\exp\left(-\frac{u^2}{1+r(t_m)}\right)\frac{1}{B^2u^4\Delta^2}.$$

Substitute this in right-hand part of (31), we get,

$$\mathbf{P}\left(\bigcup_{(s,t)\in D}\{X(t)>u\}\cap\{X(s)>u\}\right) \tag{32}$$
$$\leq \frac{(1+r(t_m))^2(1+\gamma_2(u))u^{-6+4/\alpha}}{2\pi B^2\sqrt{(1-r^2(t_m)}}\frac{1}{T^2}H_\alpha^2\left(\frac{T}{(1+r(t_m))^{2/\alpha}}\right)\exp\left(-\frac{u^2}{1+r(t_m)}\right),$$

where $\gamma_2(u)\downarrow 0$ as $u\to\infty$.

Estimation the probability from below repeats the corresponding steps in the proof of Theorem 1, see (21) and followed. Thus Theorem 2 follows.

References

[1] J. Pickands III (1969). Upcrossing probabilities for stationary Gaussian processes. *Trans. Amer. Math. Soc.* **145** 51-73.

[2] V. I. Piterbarg (1996) *Asymptotic Methods in the Theory of Gaussian Processes and Fields.* AMS, MMONO, **148**.