BSDE's, Clark-Ocone Formula, and Feynman-Kac Formula for Lévy Processes*

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Abstract

In this paper we show the existence and uniqueness of a solution for backward stochastic differential equations driven by a Lévy process with moments of all orders. An application to Clark-Ocone and Feynman-Kac formulas for Lévy processes is presented.

1 Introduction

The first paper concerned with Backward Stochastic Differential Equations (BSDE's) is the paper Bismut (1973), where he introduced a non-linear Ricatti BSDE and showed existence and uniqueness of bounded solutions. Pardoux and Peng (1990) considered general BSDE's and this paper was the starting point for the development of the study of these equations. On the

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other hand, BSDE's have important applications in the theory of mathematical finance, especially, they play a major role in hedging and non-linear pricing theory for imperfect markets (see El Karoui and Quenez (1997)).

One can consider a BSDE driven by a Brownian motion as a nonlinear generalization of the integral representation theorem for square integrable random martingales. Then it is natural to extend these kind of equations to the case of Lévy process, that is, processes with independent and stationary increments. We recall that a Lévy process consists of three stochastically independent parts: a purely deterministic linear part, a Brownian motion and a pure-jump process. In Situ (1997) BSDE's driven by a Brownian motion and a Poisson point process are studied. Ouknine (1998) considers BSDE's driven by a Poisson random measure. In both papers the main ingredient is the integral representation of square integrable random variables in terms of a Poisson random measure (see Jacod (1979)).

In Nualart and Schoutens (2000) a martingale representation theorem for Lévy processes satisfying some exponential moment condition was proved. The purpose of this paper is to use this martingale representation result to establish the existence and uniqueness of solutions for BSDE's driven by a Lévy process of the kind considered in Nualart and Schoutens (2000).

The paper is organized as follows. Section 2 contains some preliminaries on Lévy processes. Section 3 contains the main result on BSDE's driven by Lévy processes. Finally in Section 4 we have included some applications of BSDE's driven by Lévy processes to the Clark-Ocone and the Feynman-Kac formulas for Lévy processes.

2 Preliminaries

Let $X = \{X_t,\ t \geq 0\}$ be a Lévy process defined on a complete probability space $(\Omega, \mathcal{F}, P)$. That is, X is a real-valued process starting from 0 with stationary and independent increments and with càdlàg trajectories. It is known that X_t has a characteristic function of the form

$$E\left(e^{i\theta X_t}\right) = \exp\left[ia\theta t - \frac{1}{2}\sigma^2\theta^2 t + t\int_{\mathbb{R}}\left(e^{i\theta x} - 1 - i\theta x \mathbf{1}_{\{|x|<1\}}\right)\nu(dx)\right],$$

where $a \in \mathbb{R}$, $\sigma > 0$, and ν is a measure on $\mathbb{R}$ with $\int_{\mathbb{R}}(1 \wedge x^2)\nu(dx) < \infty$. We will assume that the Lévy measure ν satisfies for some $\lambda > 0$

$$\int_{(-\epsilon,\epsilon)^c} e^{\lambda|x|}\nu(dx) < \infty,$$

for every $\varepsilon > 0$. This implies that the random variables X_t have moments of all orders. We refer to Sato (2000) or Bertoin (1996) for a detailed account on Lévy processes.

For $t \geq 0$, let $\mathcal{F}_t$ denote the σ-algebra generated by the family of random variables $\{X_s, 0 \leq s \leq t\}$ augmented with the P-null sets of $\mathcal{F}$. Fix a time interval $[0,T]$ and set $L_T^2 = L^2(\Omega, \mathcal{F}_T, P)$. We will denote by $\mathcal{P}$ the predictable sub-σ-field of $\mathcal{F}_T \otimes \mathcal{B}_{[0,T]}$. First we introduce some notation:

- Let H_T^2 denote the space of square integrable and $\mathcal{F}_t-$progressively measurable processes $\phi = \{\phi_t, t \in [0,T]\}$such that

$$||\phi||^2 = E\left[\int_0^T |\phi_t|^2 dt\right] < \infty.$$

- M_T^2 will denote the subspace of H_T^2 formed by predictable processes.

- $H_T^2(l^2)$ and $M_T^2(l^2)$ are the corresponding spaces of l^2-valued processes equipped with the norm

$$||\phi||^2 = E\left[\int_0^T \sum_{i=1}^{\infty} |\phi_t^{(i)}|^2 dt\right].$$

- Set $\mathcal{H}_T^2 = H_T^2 \times M_T^2(l^2)$.

Following Nualart and Schoutens (2000) we define for every $i = 1, 2, ...$ the so-called power-jump processes$\{X_t^{(i)}, t \geq 0\}$ and their compensated version $\{Y_t^{(i)} = X_t^{(i)} - E[X_t^{(i)}], t \geq 0\}$, also called the Teugels martingales, as follows

$$\begin{aligned} X_t^{(1)} &= X_t \text{ and } X_t^{(i)} = \sum_{0<s\leq t} (\Delta X_s)^i \text{ for } i = 2, 3, ... \\ Y_t^{(i)} &= X_t^{(i)} - E[X_t^{(i)}] = X_t^{(i)} - t\ E[X_1^{(i)}] \text{ for } i \geq 1. \end{aligned}$$

An orthonormalization procedure can be applied to the martingales $Y^{(i)}$ in order to obtain a set of pairwise strongly orthonormal martingales $\{H^{(i)}\}_{i=1}^{\infty}$ such that each $H^{(i)}$ is a linear combination of the $Y^{(j)}$, $j = 1, ..., i$:

$$H^{(i)} = c_{i,i} Y^{(i)} + c_{i,i-1} Y^{(i-1)} + ... + c_{i,1} Y^{(1)}.$$

It was shown in Nualart and Schoutens (2000) that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1, x, x^2, ...$ with respect to the measure $\mu(dx) = x^2 v(dx) + \sigma^2 \delta_0(dx)$:

$$q_{i-1}(x) = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + ... + c_{i,1}.$$

Set

$$\begin{aligned} p_i(x) &= xq_{i-1}(x) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + ... + c_{i,1}x \\ \tilde{p}_i(x) &= x(q_{i-1}(x) - q_{i-1}(0)) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + ... + c_{i,2}x^2. \end{aligned}$$

Then

$$\begin{aligned} H_t^{(i)} &= \sum_{0<s\leq t} \left(c_{i,i} \left(\Delta X_s\right)^i \ + ... + c_{i,2} \left(\Delta X_s\right)^2 \right) + c_{i,1} X_t \\ &\quad -tE\left[c_{i,i} X_1^{(i)} + ... + c_{i,2} X_1^{(2)} \right] - tc_{i,1} E\left[X_1\right] \\ &= q_{i-1}(0) X_t + \sum_{0<s\leq t} \tilde{p}_i(\Delta X_s) - tE\left[\sum_{0<s\leq 1} \tilde{p}_i(\Delta X_s) \right] - tq_{i-1}(0) E\left[X_1\right]. \end{aligned}$$

As a consequence, $\Delta H_t^{(i)} = p_i(\Delta X_t)$ for each $i \geq 1$. In the particular case $i = 1$ we obtain

$$H_t^{(1)} = c_{1,1}(X_t - tE[X_1]),$$

where

$$c_{1,1} = \left[\int_{\mathbb{R}} y^2 \nu(dy) + \sigma^2 \right]^{-1/2}$$

and

$$E[X_1] = a + \int_{\{|x|\geq 1\}} z\nu(dz).$$

In the case $\int_{\mathbb{R}} |z| v(dz) < \infty$, assuming $a = \int_{\{|x|<1\}} z\nu(dz)$, we obtain $E[X_1] = \int_{\mathbb{R}} z\nu(dz)$.

The main results in Nualart and Schoutens (2000) is the Predictable Representation Property: Every square integrable random variable $F \in L_T^2$ has a representation of the form

$$F = E[F] + \sum_{i=1}^{\infty} \int_0^T Z_s^{(i)} dH_s^{(i)},$$

where Z_t is a predictable process in the space $M_T^2(l^2)$.

3 BSDE for Lévy Processes

Taking into account the results and notation presented in the previous section, it seems natural to consider the BSDE

$$-dY_t = f(t, Y_{t-}, Z_t)dt - \sum_{i=1}^{\infty} Z_t^{(i)} dH_t^{(i)}, \qquad Y_T = \xi, \tag{1}$$

where:

- $H_t^{(i)}$ is the orthonormalized Teugels martingale of order i associated with the Lévy process X.
- $f : \Omega \times [0,T] \times \mathbb{R} \times M_T^2(l^2) \to \mathbb{R}$ is a measurable function such that $f(.,0,0) \in H_T^2$.
- f is uniformly Lipschitz in the first two components, i.e., there exists $C > 0$ such that $dt \otimes dP$ a.s., for all (y_1, z_1) and (y_2, z_2) in $\mathbb{R} \times l^2$

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C \left(|y_1 - y_2| + \|z_1 - z_2\|_{l^2} \right).$$

- $\xi \in L_T^2$.

If (f, ξ) satisfies the above assumptions, the pair (f, ξ) is said to be **standard data** for the BSDE. A solution of the BSDE is a pair of processes, $\{(Y_t, Z_t), 0 \leq t \leq T\} \in H_T^2 \times M_T^2(l^2)$ such that the following relation holds for all $t \in [0,T]$:

$$Y_t = \xi + \int_t^T f(s, Y_{s-}, Z_s)ds - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} dH_s^{(i)}. \tag{2}$$

Note that the progressive measurability of $\{(Y_t, Z_t), 0 \leq t \leq T\}$ implies that (Y_0, Z_0) is deterministic.

Theorem 1 *Given standard data (f, ξ), there exists a unique solution (Y, Z) which solves the BSDE (2).*

Proof: We define a mapping Φ from $\mathcal{H}_T^2$ into itself such that $(Y, Z) \in \mathcal{H}_T^2$ is a solution of the BSDE if and only if it is a fixed point of Φ. Given $(U, V) \in \mathcal{H}_T^2$, we define $(Y, Z) = \Phi(U, V)$ as follows:

$$Y_t = E\left[\xi + \int_t^T f(s, U_{s-}, V_s)ds | \mathcal{F}_t\right], \ 0 \leq t \leq T,$$

and $\{Z_t, 0 \leq t \leq T\}$ is given by the martingale representation of Nualart and Schoutens (2000) applied to the square integrable random variable

$$\xi + \int_0^T f(s, U_{s-}, V_s) ds,$$

i.e.,

$$\begin{aligned} \xi + \int_0^T f(s, U_{s-}, V_s) ds &= E\left[\xi + \int_0^T f(s, U_{s-}, V_s) ds\right] \\ &\quad + \sum_{i=1}^{\infty} \int_0^T Z_s^{(i)} dH_s^{(i)}. \end{aligned}$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ in the last identity yields

$$Y_t + \int_0^t f(s, U_{s-}, V_s) ds = Y_0 + \sum_{i=1}^{\infty} \int_0^t Z_s^{(i)} dH_s^{(i)},$$

from which we deduce that

$$Y_t = \xi + \int_t^T f(s, U_{s-}, V_s) ds - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} dH_s^{(i)}$$

and we have shown that $(Y, Z) \in \mathcal{H}_T^2$ solves our BSDE if and only if it is a fixed point of Φ.

Next we prove that Φ is a strict contraction on $\mathcal{H}_T^2$ equipped with the norm

$$\|(Y, Z)\|_\beta = \left(E\left[\int_0^T e^{\beta s} \left(|Y_{s-}|^2 + \sum_{i=1}^{\infty} |Z_s^{(i)}|^2 \right) ds \right] \right)^{1/2},$$

for a suitable $\beta > 0$. Let (U, V) and (U', V') be two elements of $\mathcal{H}_T^2$ and set $\Phi(U, V) = (Y, Z)$ and $\Phi(U', V') = (Y', Z')$. Denote $(\overline{U}, \overline{V}) = (U - U', V - V')$ and $(\overline{Y}, \overline{Z}) = (Y - Y', Z - Z')$.

Applying Itô's formula from $s=t$ to $s=T$, to $e^{\beta s}\left(Y_s - Y'_s\right)^2$, it follows that

$$\begin{aligned} e^{\beta t}\left(Y_t - Y'_t\right)^2 &= -\beta \int_t^T e^{\beta s}\left(Y_{s-} - Y'_{s-}\right)^2 ds \\ &\quad -2\int_t^T e^{\beta s}\left(Y_{s-} - Y'_{s-}\right) d(Y_s - Y'_s) \\ &\quad -\int_t^T e^{\beta s} d[Y-Y', Y-Y']_s. \end{aligned} \tag{3}$$

We have

$$\begin{aligned} -d(Y_t - Y'_t) &= (f(t, U_{t-}, V_t) - f(t, U'_{t-}, V'_t))dt \\ &\quad -\sum_{i=1}^{\infty}\left(Z_t^{(i)} - Z_t'^{(i)}\right) dH_t^{(i)}, \\ d\left[Y-Y', Y-Y'\right]_t &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(Z_t^{(i)} - Z_t'^{(i)}\right)\left(Z_t^{(j)} - Z_t'^{(j)}\right) d[H^{(i)}, H^{(j)}]_t, \\ \left\langle H^{(i)}, H^{(j)}\right\rangle_t &= \delta_{ij} t. \end{aligned}$$

Hence, taking expectations in (3), we have

$$\begin{aligned} &E\left[e^{\beta t}\left(Y_t - Y'_t\right)^2\right] + \sum_{i=1}^{\infty} E\left[\int_t^T e^{\beta s}\left(Z_s^{(i)} - Z_s'^{(i)}\right)^2 ds\right] \\ &= -\beta E\left[\int_t^T e^{\beta s}\left(Y_{s-} - Y'_{s-}\right)^2 ds\right] \\ &+2E\left[\int_t^T e^{\beta s}\left(Y_{s-} - Y'_{s-}\right)\left(f(s, U_{s-}, V_s) - f(s, U'_{s-}, V'_s)\right) ds\right]. \end{aligned}$$

Using the fact that f is Lipschitz with constant C yields

$$\begin{aligned} &E\left[e^{\beta t}\left(Y_t - Y'_t\right)^2\right] + \sum_{i=1}^{\infty} E\left[\int_t^T e^{\beta s}(\left(Z_s^{(i)} - Z_s'^{(i)}\right)^2 ds\right] \\ &\leq -\beta E\left[\int_t^T e^{\beta s}(Y_{s-} - Y'_{s-})^2 ds\right] \\ &+2CE\left[\int_t^T e^{\beta s}\left|Y_{s-} - Y'_{s-}\right|\left(|U_{s-} - U'_{s-}| + \sqrt{\sum_{i=1}^{\infty}|V_s^{(i)} - V_s'^{(i)}|^2}\right) ds\right]. \end{aligned}$$

If we now use the fact that for every $c > 0$ and $a, b \in \mathbb{R}$ we have that $2ab \leq ca^2 + \frac{1}{c}b^2$ and $(a+b)^2 \leq 2a^2 + 2b^2$, we obtain

$$
\begin{aligned}
& E\left[e^{\beta t}\left|Y_t - Y'_t\right|^2\right] + \sum_{i=1}^{\infty} E\left[\int_t^T e^{\beta s}\left(Z_s^{(i)} - Z_s'^{(i)}\right)^2 ds\right] \\
\leq \; & (4C^2 - \beta) E\left[\int_t^T e^{\beta s}\left|Y_s - Y'_s\right|^2 ds\right] \\
& + \frac{1}{2} E\left[\int_t^T e^{\beta s}\left(|U_{s-} - U'_{s-}|^2 + \sum_{i=1}^{\infty} |V_s^{(i)} - V_s'^{(i)}|^2\right) ds\right].
\end{aligned}
$$

Taking now $\beta = 4C^2 + 1$, and noting that $e^{\beta t} E[(Y_t - Y'_t)^2] \geq 0$, we finally derive

$$
\begin{aligned}
& E\left[\int_t^T e^{\beta s}\left|Y_s - Y'_s\right|^2 ds\right] + \sum_{i=1}^{\infty} E\left[\int_t^T e^{\beta s}(Z_s^{(i)} - Z_s'^{(i)})^2 ds\right] \\
\leq \; & \frac{1}{2} E\left[\int_t^T e^{\beta s}\left(|U_{s-} - U'_{s-}|^2 + \sum_{i=1}^{\infty} |V_s^{(i)} - V_s'^{(i)}|^2\right) ds\right],
\end{aligned}
$$

that is,

$$
\|(Y, Z)\|_\beta^2 \leq \frac{1}{2} \|(U, V)\|_\beta^2 ,
$$

from which it follows that Φ is a strict contraction on $\mathcal{H}_T^2$ equipped with the norm $\|\cdot\|_\beta$ if $\beta = 4C^2 + 1$. Then Φ has a unique fixed point and the theorem is proved.$\diamond$

We now prove the continuous dependence of the solution on the final data ξ and the function f.

Theorem 2 *Given standard data (f, ξ) and (f', ξ'), let (Y, Z) and $(Y'Z')$ be the unique adapted solutions of the BSDE (2) corresponding to (f, ξ) and (f', ξ'). Then*

$$
\begin{aligned}
& E\left[\int_0^T \left(|Y_{s-} - Y'_{s-}|^2 + \sum_{i=1}^{\infty} |Z_s^{(i)} - Z_s'^{(i)}|^2\right) ds\right] \\
\leq \; & C\left(E[|\xi - \xi'|^2] + E\left[\int_0^T |f(s, Y_{s-}, Z_s) - f'(s, Y_{s-}, Z_s|^2 ds\right]\right).
\end{aligned}
$$

Proof: Applying Itô's formula from $s = t$ to $s = T$, to $(Y_s - Y'_s)^2$, it follows that

$$\left(Y_T - Y'_T\right)^2 - \left(Y_t - Y'_t\right)^2 = 2\int_t^T \left(Y_{s-} - Y'_{s-}\right) d(Y_s - Y'_s) + \int_t^T d[Y - Y', Y - Y']_s.$$

Taking expectations and using the relations

$$\begin{aligned} -d(Y_t - Y'_t) &= f(t, Y_{t-}, Z_t) - f'(t, Y'_{t-}, Z'_t)dt \\ &\quad -\sum_{i=1}^{\infty} \left(Z_t^{(i)} - Z_t'^{(i)}\right) dH_t^{(i)} \\ d[Y - Y', Y - Y']_t &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left(Z_t^{(i)} - Z_t'^{(i)}\right)\left(Z_t^{(j)} - Z_t'^{(j)}\right) d[H^{(i)}, H^{(j)}]_t, \\ \left\langle H^{(i)}, H^{(j)} \right\rangle_t &= \delta_{ij} t, \end{aligned}$$

we have

$$\begin{aligned} & E[\left(Y_t - Y'_t\right)^2] + \sum_{i=1}^{\infty} E\left[\int_t^T \left|Z_s^{(i)} - Z_s'^{(i)}\right|^2 ds\right] \\ = \ & E[\left(\xi - \xi'\right)^2] \\ & +2E\left[\int_t^T \left(Y_{s-} - Y'_{s-}\right)\left(f(s, Y_{s-}, Z_s) - f'(s, Y'_{s-}, Z'_s)\right) ds\right]. \end{aligned}$$

Using the Lipschitz property of f', and computations similar to those of the proof of Theorem 1 we obtain

$$\begin{aligned} & E[\left|Y_t - Y'_t\right|^2] + \frac{1}{2}E\left[\int_t^T \sum_{i=1}^{\infty} |Z_s^{(i)} - Z_s'^{(i)}|^2 ds\right] \\ \leq \ & E[|\xi - \xi'|^2] + (1 + 2C' + 2C'^2)E\left[\int_t^T |Y_{s-} - Y'_{s-}|^2 ds\right] \\ & +E\left[\int_t^T |f(s, Y_{s-}, Z_s) - f'(s, Y_{s-}, Z_s|^2 ds\right]. \end{aligned}$$

Then by Gronwall's inequality the result follows.⋄

4 Applications

Suppose our Lévy process X_t has no Brownian part, i.e. $X_t = at + L_t$, where L_t is pure jump process with Lévy measure $\nu(dx)$.

4.1 Clark-Ocone Formula and Feynman-Kac Formula

We first prove a technical lemma which will be needed later on.

Lemma 3 *Let $h : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be a random function measurable with respect to $\mathcal{P} \otimes \mathcal{B}_{\mathbb{R}}$ such that*

$$|h(s,y)| \le a_s(y^2 \wedge |y|) \quad a.s., \tag{4}$$

where $\{a_s, 0 \le s \le T\}$ is a nonnegative predictable process such that $E[\int_0^T a_s ds] < \infty$. Then for each $t \in [0,T]$ we have

$$\sum_{t<s\le T} h(s, \Delta X_s) = \sum_{i=1}^{\infty} \int_t^T \langle h(s,\cdot), p_i \rangle_{L^2(\nu)} \, dH_s^{(i)} + \int_t^T \int_{\mathbb{R}} h(s,y)\nu(dy)ds.$$

Proof: Because (4) implies that $E[\int_0^t \int_{\mathbb{R}} |h(s,y)|^2 \nu(dy)ds] < \infty$, we have that

$$M_t = \sum_{0<s\le t} h(s,\Delta X_s) - \int_0^t \int_{\mathbb{R}} h(s,y)\nu(dy)ds.$$

is a square integrable martingale. By the Predictable Representation Theorem, there exists a process ϕ in the space $M_T^2(l^2)$ such that

$$M_t = \sum_{i=1}^{\infty} \int_0^t \phi_s^{(i)} dH_s^{(i)}.$$

Taking into account that $\langle H^{(i)}, H^{(j)} \rangle_t = t\delta_{ji}$, we have

$$\left\langle M, H^{(i)} \right\rangle_t = \int_0^t \phi_s^{(i)} ds. \tag{5}$$

On the other hand, using that $\Delta M_s \Delta H_s^{(i)} = h(s,\Delta X_s)p_i(\Delta X_s)$ we obtain

$$\left\langle M, H^{(i)} \right\rangle_t = \int_0^t \int_{\mathbb{R}} h(s,y)p_i(y)\nu(dy)ds. \tag{6}$$

Consequently, (5) and (6) imply

$$\phi_s^{(i)} = \int_{\mathbb{R}} h(s,y) p_i(y) \nu(dy),$$

and the result follows. $\diamondsuit$

Let us consider the simple case of a BSDE where $f = 0$, and the terminal random variable ξ is a function of X_T, that is,

$$-dY_t = -\sum_{i=1}^{\infty} Z_t^{(i)} dH_t^{(i)}; \qquad Y_T = g(X_T)$$

or equivalently

$$Y_t = g(X_T) - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} dH_s^{(i)}, \tag{7}$$

where $E(g(X_T)^2) < \infty$. Let $\theta = \theta(t,x)$ be the solution of the following PDIE (Partial Differential Integral Equation) with terminal value g:

$$\frac{\partial \theta}{\partial t}(t,x) + \int_{\mathbb{R}} (\theta(t,x+y) - \theta(t,x) - \frac{\partial \theta}{\partial x}(s,x)y) \ \nu(dy) + a' \frac{\partial \theta}{\partial x}(t,x) = 0,$$

$$\theta(T,x) = g(x), \tag{8}$$

where $a' = a + \int_{\{|y| \geq 1\}} y\nu(dy)$. Set

$$\theta^{(1)}(t,x,y) = \theta(t,x+y) - \theta(t,x) - \frac{\partial \theta}{\partial x}(s,x)y. \tag{9}$$

Suppose that θ is a $C^{1,2}$ function such that $\frac{\partial \theta}{\partial x}$ and $\frac{\partial^2 \theta}{\partial x^2}$ is bounded by a polynomial function of x. Under these hypotheses the function $\theta^{(1)}(t,x,y)$ satisfy the hypotheses in Lemma 3 imposed on h due to the mean value theorem, when we take $x = X_{t-}$.

The following result is a version of the Clark-Ocone formula for functions of a Lévy process.

Proposition 4 *Under the above assumptions, the unique adapted solution of (7) is given by*

$$\begin{aligned} Y_t &= \theta(t, X_t) \\ Z_t^{(i)} &= \int_{\mathbb{R}} \theta^{(1)}(t, X_{t-}, y) p_i(y) \nu(dy) \quad \text{for } i \geq 2, \\ Z_t^{(1)} &= \int_{\mathbb{R}} \theta^{(1)}(t, X_{t-}, y) p_1(y) \nu(dy) + \frac{\partial \theta}{\partial x}(t, X_{t-}) (\int_{\mathbb{R}} y^2 \nu(dy))^{1/2}. \end{aligned}$$

Proof: Indeed by Itô's lemma applied to $\theta(s, X_s)$ from $s = t$ to $s = T$ we have

$$\theta(T, X_T) - \theta(t, X_t) = \int_t^T \frac{\partial\theta}{\partial t}(s, X_{s-})ds + \int_t^T \frac{\partial\theta}{\partial x}(s, X_{s-})dX_s \qquad (10)$$
$$+ \sum_{t<s\leq T} \left[\theta(s, X_s) - \theta(s, X_{s-}) - \frac{\partial\theta}{\partial x}(s, X_{s-})\Delta X_s\right].$$

If we apply Lemma 3 to $h(s, y) = \theta(s, X_{s-} + y) - \theta(s, X_{s-}) - \frac{\partial\theta}{\partial x}(s, X_{s-})y$, we obtain

$$\sum_{t<s\leq T} \left[\theta(s, X_s) - \theta(s, X_{s-}) - \frac{\partial\theta}{\partial x}(s, X_{s-})\Delta X_s\right]$$
$$= \sum_{i=1}^{\infty} \int_t^T \left(\int_{\mathbb{R}} \theta^{(1)}(s, X_{s-}, y)p_i(y)\nu(dy)\right) dH_s^{(i)}$$
$$+ \int_t^T \int_{\mathbb{R}} \theta^{(1)}(s, X_{s-}, y)\ \nu(dy)ds. \qquad (11)$$

Hence, substituting (11) into (10) yields

$$g(X_T) - \theta(t, X_t)$$
$$= \int_t^T \frac{\partial\theta}{\partial t}(s, X_{s-})ds + \int_t^T \frac{\partial\theta}{\partial x}(s, X_{s-})dX_s$$
$$+ \sum_{i=1}^{\infty} \int_t^T \left(\int_{\mathbb{R}} \theta^{(1)}(s, X_{s-}, y)p_i(y)\nu(dy)\right) dH_s^{(i)}$$
$$+ \int_t^T \int_{\mathbb{R}} \theta^{(1)}(s, X_{s-}, y)\ \nu(dy)ds. \qquad (12)$$

Notice that

$$X_t = Y_t^{(1)} + tE(X_1) = (\int_{\mathbb{R}} y^2\nu(dy))^{1/2} H_t^{(1)} + tE(X_1),$$

and

$$E(X_1) = a + \int_{\{|x|\geq 1\}} y^2\nu(dy).$$

We also have $Y_0 = E[Y_0] = E[g(\mathrm{X}_T)]$ so we can rewrite (12) as

$$\begin{aligned} g(X_T) &= E[g(X_T)] + \int_t^T \frac{\partial\theta}{\partial x}(s, X_{s-})(\int_{\mathbb{R}} y^2\nu(dy))^{-1/2} dH_s^{(1)} \\ &\quad + \sum_{i=1}^{\infty}\int_t^T \left(\int_{\mathbb{R}} \theta^{(1)}(s, X_{s-}, y) p_i(y)\nu(dy)\right) dH_s^{(i)}, \end{aligned}$$

which completes the proof of the Proposition. $\diamondsuit$

If $\int_{\mathbb{R}} |y|\nu(dy) < \infty$, and we take $a = \int_{\{|y|<1\}} y\nu(dy)$, then the equation (8) reduces to

$$\frac{\partial\theta}{\partial t}(t,x) + \int_{\mathbb{R}} (\theta(t, x+y) - \theta(t,x))\nu(dy) = 0,$$
$$\theta(T,x) = g(x),$$

and taking into account that $p_1(y) = y(\int_{\mathbb{R}} y^2\nu(dy))^{-1/2}$ in Proposition 4 we have

$$Z_t^{(i)} = \int_{\mathbb{R}} [\theta(t, X_{t-} + y) - \theta(t, X_{t-})\ ]\, p_i(y)\nu(dy).$$

Now by taking expectations we derive that the solution $\theta(t,x)$ to our PDIE equation has the stochastic representation

$$\theta(t,x) = E[g(X_T)|X_t = x].$$

This is an extension of the classical Feynman-Kac Formula.

Remark:

Consider the very special case where we have a compensated Poisson process $X_t = N_t - \lambda t$. Then

$$H_t^{(1)} = \frac{1}{\sqrt{\lambda}}(N_t - \lambda t) = \frac{X_t}{\sqrt{\lambda}} \text{ and } H_t^{(i)} = 0 \text{ for } i = 2, 3, ...$$

Note that $p_1(x) = \frac{x}{\sqrt{\lambda}}$ and $p_i(x) = 0$, $i = 2, 3, ...$ Moreover the PDIE (8) reduces to

$$(\theta(s, x+1) - \theta(s,x)) - \lambda\frac{\partial\theta}{\partial x}(s,x) + \frac{\partial\theta}{\partial t}(t,x) = 0,$$
$$\theta(T,x) = g(x).$$

The Clark-Ocone Formula is now given by

$$g(X_T) = E[g(X_T)] + \int_t^T \theta(s, X_{s-} + 1) - \theta(s, X_{s-}) dX_s.$$

4.2 Nonlinear Clark-Haussman-Ocone Formula and Feynman-Kac Formula

Let us consider the BSDE

$$-dY_t = f(t, Y_t, Z_t)dt - \sum_{i=1}^{\infty} Z_t^{(i)} dH_t^{(i)}; \qquad\qquad Y_T = g(X_T) \qquad (13)$$

or equivalently

$$Y_t = g(X_T) + \int_t^T f(s, Y_{s-}, Z_s)ds - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} dH_s^{(i)}.$$

Suppose that $\theta = \theta(t,x)$ satisfies the following PDIE:

$$\begin{gathered}\frac{\partial \theta}{\partial t}(t,x) + \int_{\mathbb{R}} \theta^{(1)}(t,x,y)\nu(dy) + a' \frac{\partial \theta}{\partial x}(t,x) \\ + f\left(t, \theta(t,x)\ , \left\{\theta^{(i)}(t,x)\right\}_{i=1}^{\infty}\right) = 0, \\ \theta(T,x) = g(x).\end{gathered} \qquad (14)$$

where as in the previous section, we define $\theta^{(1)}(t,x,y)$ by (9),

$$\theta^{(1)}(t,x) = \int_{\mathbb{R}} \theta^{(1)}(t,x,y)p_1(y)\nu(dy) + \frac{\partial \theta}{\partial x}(t,x)(\int_{\mathbb{R}} y^2 \nu(dy))^{1/2},$$

and for $i \geq 2$

$$\theta^{(i)}(t,x) = \int_{\mathbb{R}} \theta^{(1)}(t,x,y)p_i(y)\nu(dy).$$

Assume that θ is a $C^{1,2}$ function such that $\frac{\partial \theta}{\partial x}$ and $\frac{\partial^2 \theta}{\partial x^2}$ is bounded by a polynomial function of x. Then the (unique) adapted solution of (13) is given by

$$\begin{aligned} Y_t &= \theta(t, X_t) \\ Z_t^{(i)} &= \int_{\mathbb{R}} \theta^{(1)}(t, X_{t-}, y)p_i(y)\nu(dy) \quad \text{for } i \geq 2, \\ Z_t^{(1)} &= \int_{\mathbb{R}} \theta^{(1)}(t, X_{t-}, y)p_1(y)\nu(dy) + \frac{\partial \theta}{\partial x}(t, X_{t-})(\int_{\mathbb{R}} y^2 \nu(dy))^{1/2}. \end{aligned}$$

Indeed, applying Itô's lemma to $\theta(s, X_s)$ from $s = t$ to $s = T$ and using Lemma 3 we obtain the equality (12). Now, using (14) we get

$$g(X_T) - \theta(t, X_t) = -\int_t^T f\left(s, \theta(s, X_{s-})\ , \left\{\ \theta^{(k)}(s, X_{s-})\right\}_{k=1}^{\infty}\right) ds$$

$$+\int_t^T \left[\int_{\mathbb{R}} \theta^{(1)}(s, X_{s-}, y) p_1(y) \nu(dy) + \frac{\partial \theta}{\partial x}(s, X_{s-}) (\int_{\mathbb{R}} y^2 \nu(dy))^{1/2}\right] dH_s^{(2)}$$

$$+\sum_{i=2}^{\infty} \int_t^T \left[\int_{\mathbb{R}} \theta^{(1)}(s, X_{s-}, y) p_i(y) \nu(dy)\right] dH_s^{(i)}.$$

which completes the proof of the proposition.$\diamondsuit$

Notice that taking expectations we get

$$\theta(t,x) = E[g(X_T)|X_t = x] +$$
$$E\left[\int_t^T f\left(s, \theta(s, X_{s-})\ , \left\{\ \theta^{(i)}(s, X_{s-})\right\}_{i=1}^{\infty}\right) ds | X_t = x\right].$$

Remark:

Consider again the very special case where we have a Poisson process N_t with $E[N_t] = \lambda t$. Set $X_t = N_t - \lambda t$. Then the PDIE (14) reduces to

$$\begin{aligned} (\theta(t, x+1) - \theta(t,x)) - \lambda \frac{\partial \theta}{\partial x}(t,x) + \frac{\partial \theta}{\partial t}(t,x) + \\ f\left(t, \theta(t,x)\ , \theta(t, x+1) - \theta(t,x)\right) = 0, \\ \theta(T, x) = g(x). \end{aligned} \tag{15}$$

And we derive the nonlinear Feynman-Kac Formula:

$$\begin{aligned} \theta(t,x) &= E[g(X_T)|X_t = x] \\ &+ E\left[\int_t^T f\left(s, \theta(s, X_{s-}), \theta(s, X_{s-}+1) - \theta(s, X_{s-})\right) ds | X_t = x\right]. \end{aligned}$$

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