Tail probabilities for a risk process with subexponential jumps in a regenerative and diffusion environment

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Abstract

In this paper we find a nonexponential Lundberg approximation of the ruin probability in a Cox model, in which a governing process has a regenerative structure and the claims are light-tailed or have an intermediate regularly varying distribution. Examples include an intensity process being reflected Brownian motion, square functions of the Ornstein-Uhlenbeck process and splitting reflected Brownian bridges. In particular, we consider a non-Markovian intensity process.

Keywords: ruin probability, Cox process, diffusion process, exponential change of measure.

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1 Introduction

This paper is concerned with a risk theory subject to a combination of two features which repeatedly have been argued to be relevant in practical applications, namely stochastic modulation and regular varying claim size distributions. We consider a canonical *surplus process* $\{S(t), t \geq 0\}$ given by:

$$S(t) = \sum_{i=1}^{N(t)} U_i - t \ ,$$

where $\{N(t), t \geq 0\}$ is a Cox process with an underlying cádlág process $\{X(t), t \geq 0\}$. That is, if a realization of the process $\{X(t), t \geq 0\}$ is $x(t) \in \mathcal{D}[0, +\infty)$, then for a non-negative function $\lambda : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ the process $\{N(t), t \geq 0\}$ has the same law as a non-homogeneous Poisson process $\left\{N^{(x)}(t), t \geq 0\right\}$ with an intensity function $\overline{\lambda}(t) = \lambda(x(t))$. The process $\{\lambda(X(t)), t \geq 0\}$ is called an intensity process. Thus stochastic modulation means that the surplus process is not time-homogeneous, but evolves in some random environment. A detailed discussion of Cox processes and their impact on risk theory is to be found in Grandell (1991) and Rolski *et al.* (1999). The *claim sizes* $U_1, U_2, \ldots$ are i.i.d. r.v.'s independent of the process $\{N(t), t \geq 0\}$ with a common distribution function $F_U(x)$. Let u be an initial reserve and assume that $S(t) \to -\infty$ *a.e.* as $t \to +\infty$. An *infinite horizon ruin probability* is then

$$\psi(u) = \mathbb{P}(\sup_{t \geq 0} S(t) > u) \ . \tag{1.1}$$

The model, in which $\{N(t), t \geq 0\}$ is Coxian, is called the Bjőrk-Grandell model which goes to the pioneering paper Bjőrk and Grandell (1988). In that paper one derives by a martingale approach an exponential upper bound of $\psi(u)$ when an intensity process has piecewise realizations and claim sizes are light-tailed. Further generalizations can be found in Embrechts (1993) (finite time non-Markovian intensities) and Grigolionis (1992). Grandell and Schmidli (2000) and Palmowski (2000) find a Lundberg upper bound and a Lundberg approximation of $\psi(u)$ when the intensity process is governed by a diffusion process and claim sizes are light-tailed. These papers fail to capture the second main feature considered in this paper, namely, that of regularly varying tails. Relevance of heavy-tail conditions can be found e.g. in Embrechts and Veraverbeke (1982) and Klűppelberg (1989). Asmussen et al. (1994) find the nonexponential asymptotics in the Bjőrk-Grandell model when the governing process is a finite-state Markov process and the claim size has a heavy tailed distribution. Asmussen *et al.* (1999) generalize it to the case when $\{S(t), t \geq 0\}$ has a regenerative structure. In this paper we apply this result to get the asymptotics of $\psi(u)$ when a rate of income of the claim at time t is a function $\lambda(x)$ of the regenerative process $\{X(t), t \geq 0\}$, in particular when $\{X(t), t \geq 0\}$ is a reccurent diffusion process.

Denote by $0 = T_0 \leq T_1 \leq T_2 \leq T_3 \leq \ldots$ the regenerative epochs of the regenerative process $\{X(t), t \geq 0\}$. Let T be the generic time $T_{n+1} - T_n$. Define the r.v. $Z = \int_{T_n}^{T_{n+1}} \lambda(X(t))\ \mathrm{d}t$. Denote by $F(x)$ the "heavier" distribution from distributions of variables Z and U. Let $F(x)$ have a regularly varying distribution. We will write $f(x) \sim g(x)$ as $x \to +\infty$ if $\lim_{x \to +\infty} f(x)/g(x) = 1$. Similarly, $f(x) \sim g(x)$ as $x \to 0$ means that $\lim_{x \to 0} f(x)/g(x) = 1$. If $\mathbb{E}T < +\infty$, then under some mild assumptions

$$\psi(u) \sim CF^s(u) \qquad \text{as } u \to +\infty,$$

where F^s is a residual distribution of F and constant C is given explicit. Thus even in the case of light-tailed claims one can get the nonexponential asymptotics. The asymptotics of $\psi(u)$ in this case depends on the distribution of the interarrival time T only via its mean $\mathbb{E}T < +\infty$. If $\mathbb{E}T = +\infty$, then its tail also has an impact on the rate of the asymptotics of the ruin probability (see Theorem 3.2 (iv) and Section 5.2). The method of the proof of the main Theorem 3.2 is based on the Karamata Tauberian Theorem, the Kingman-Taylor expansion of the Laplace transform (see Stam (1973), Cohen (1973) and Asmussen *et al.* (1999), Cor. 3.1-3.2).

To apply this result for the specific governing process $\{X(t), t \geq 0\}$ one has to determine the asymptotic tails of r.v.'s $Z = \int_{T_n}^{T_{n+1}} \lambda(X(t))\, dt$ and T and its means. We refer to Asmussen *et al.* (1999) for similar functionals. In the second part of this paper we calculate some examples presenting there main techniques useful in solving this problem. The random variable Z is light-tailed if there exists $\delta > 0$ such that $\mathbb{E}e^{\delta Z} < +\infty$. To prove it we generalize Wentzell (1975), p. 265, in the following way. Consider a family of diffusion processes $\{X_w(t), t \geq 0\}$ parametrized by $w \geq 0$ starting at $X_w(0) = x_w$. Let τ_w be an exiting time from a compact set D. If there exists $w_0 > 0$ such that $\mathbb{E}_{x_w}^{X_w} \tau_w$ is uniformly bounded for all $0 < w \leq w_0$, then $\mathbb{E}_{x_w}^{X_w} e^{w\tau_w}$ is also uniformly bounded. The asymptotics of the tail of the distribution of Z and its mean we calculate using the Laplace transform. In most cases we take the square function $\lambda(x)$. Then the method of computing m.g.f. and Laplace transform consists in changing probability so that the quadratic functional disappears and the remaining problem is to calculate m.g.f. and Laplace transform of some hitting or exiting time. In other words, we linearize the original problem by transferring the computional problem for a variable belonging to a second Wiener chaos to computations for a variable in a first chaos. One can calculate the Laplace transform of hitting and exiting time using the Feynman-Kac formula (see Ito and McKean (1974), Wentzell (1975) and Borodin and Salminen (1996)).

The rest of the paper is organized as follows. In Section 2 we recall the Karamata-Tauberian theorem. The main Theorem 3.2 is stated in Section 3. We consider the following examples of the governing process $\{X(t), t \geq 0\}$ and the function $\lambda(x)$: semi-Markov process and $\lambda(x) = x$ (Section 4), reflected Brownian motion at 0 and 1 and $\lambda(x) = x$ (Section 5.1), Brownian motion and $\lambda(x) = e^{-\gamma|x|}$ (Section 5.2), Ornstein-Uhlenbeck process and $\lambda(x) = x^2 + k$ (Section 5.3) and $\lambda(x) = (x + p)^2$ (Section 5.4), finally splitting Brownian bridges and $\lambda(x) = |x|$ (Section 5.5).

2 Preliminaries

The main technique useful in finding the asymptotics of $\psi(u)$ is the Karamata Tauberian Theorem, which we recall now. The critical index is defined in extended real numbers by

$$\alpha_K = \inf\{v : \mathbb{E}|K|^v = +\infty\} .$$

That is, if there exists $\delta > 0$ such that $\mathbb{E}e^{\delta K} < +\infty$, then $\alpha_K = +\infty$. Moreover, we say that r.v. K has a regularly varying distribution if $\mathbb{P}(K > x) \sim x^{-\alpha_K} l_K(x)$ as $x \to +\infty$ for a slowly varying function $l_K(x)$. Denote

$$m_{i,K} = \mathbb{E}|K|^i .$$

The Karamata Tauberian Theorem relates the tail behaviour of a distribution function to the asymptotic behaviour of its Laplace transform at the origin (see Goldie *et al.* (1987), p. 333). For variable K let $\alpha_K < +\infty$ and define $n = [\alpha_K]$. Then by Kingman and Taylor (1966) the Laplace transform $\tilde{F}^K(s)$ of r.v. K may be expanded in a Taylor series as far as the s^n term:

$$\tilde{F}^K(s) = \sum_{k=0}^{n} m_{k,K}(-s)^k/k! + o(s^n), \qquad \text{as } s \to 0 \ .$$

Let

$$f_n^K(s) = (-1)^n \left(\tilde{F}^K(s) - \sum_{k=0}^{n} m_{k,K}(-s)^k/k! \right) \ .$$

Theorem 2.1. *Let $l_K(x)$ be a slowly varying function. The following are equivalent:*

$$f_n^K(s) \sim s^{\alpha_K} l_K(1/s), \qquad \text{as } s \to 0 \tag{2.2}$$

$$\mathbb{P}(K > x) \sim \frac{(-1)^n}{\Gamma(1-\alpha_K)} x^{-\alpha_K} l_K(x), \qquad \text{as } x \to +\infty. \tag{2.3}$$

From Feller (1971), Th. 2, p. 445 we have the following theorem.

Theorem 2.2. *Consider some function $L(x)$ and $\alpha > 0$. For the slowly varying function $l_L(x)$ the following are equivalent:*

$$\tilde{F}^L(s) \sim s^{-\alpha} l_L(\frac{1}{s}), \qquad \text{as } s \to 0 \tag{2.4}$$

$$L(x) \sim \frac{1}{\Gamma(1+\alpha)} x^{\alpha} l_L(x), \qquad \text{as } x \to +\infty. \tag{2.5}$$

3 Main Theorem

Assume that $\{\lambda(X(t)), t \geq 0\}$ is the intensity process for the regenerative process $\{X(t), t \geq 0\}$ and nonnegative function $\lambda(x)$. Then the surplus process $\{S(t), t \geq 0\}$ also has a regenerative structure. We let S be the increment of $\{S(t), t \geq 0\}$ during the generic cycle $T_{n+1} - T_n$, that is

$$S = \sum_{i=N(T_n)+1}^{N(T_{n+1})} U_i - (T_{n+1} - T_n) \ .$$

Let

$$S^+ = \sum_{i=N(T_n)+1}^{N(T_{n+1})} U_i$$

and

$$Z = \int_0^{T_1} \lambda(X(s)) \, \mathrm{d}s \ .$$

Condition A.
We assume that

$$\mathbb{P}(S^+ > x) \sim \mathbb{P}(S > x) \sim x^{-\alpha_S} l_S(x) \ , \tag{A}$$

where $\alpha_S > 0$ and $l_S(x)$ is a slowly varying function.

By Asmussen *et al.* (2000), Lem. 5.1 we have the following lemma.

Lemma 3.1. *If*
Condition B.

$$\exists \delta > 0 : \qquad \mathbb{E}e^{\delta Z} < +\infty \ , \tag{B}$$

then condition (A) holds.

Note that:

$$S(T_n) + \sum_{i=N(T_n)+1}^{N(T_{n+1})} U_i - (T_{n+1} - T_n) \le \sup_{T_n \le t < T_{n+1}} S(t) \le S(T_n) + \sum_{i=N(T_n)+1}^{N(T_{n+1})} U_i \ .$$

Thus under (A) following Asmussen and Klűppelberg (1996) and Asmussen *et al.* (1999), Th. 3.3 we have:

$$\psi(u) \sim \mathbb{P}(\max_{n \ge 1}(Y_1 + Y_2 + \ldots + Y_n) > u) \ ,$$

where Y_n are i.i.d. random variables such that $Y_n \stackrel{D}{=} S$.

Theorem 3.1. *Assume that (A) holds.*
(i) If $\nu = |\mathbb{E}S| < +\infty$ *and* $\mathbb{E}S < 0$, *then*

$$\psi(u) \sim \frac{1}{1-\alpha_S} \frac{1}{\nu} l_S(u) u^{-\alpha_S + 1} \ . \tag{3.1}$$

(ii) If $\nu = +\infty$, $l_S(x) = c_1$ *and*

$$\mathbb{P}(T > x) \sim c_2 x^{-\beta} \tag{3.2}$$

for $0 < \beta < 1$ *and* $\beta < \alpha_S$, *then*

$$\psi(u) \sim \frac{\sin(\beta\pi)}{\beta\pi} \frac{c_1}{c_2} \quad u^{\beta - \alpha_S} \quad \int_0^{+\infty} y^{\beta-1}(1+y)^{-\alpha_S} \, \mathrm{d}y \quad . \tag{3.3}$$

Proof. Part (i) is the corollary from Asmussen *et al.* (1999), Cor. 3.1. We now prove (ii). Denote by $G^+(x)$ and $G^-(x)$ the ascending and descending weak ladder height distributions of random walk $Y_1 + Y_2 + \ldots + Y_n$. Thus $G^+(x)$ and $G^-(x)$ are concentrated on $[0, +\infty)$ and $(-\infty, 0]$ respectively. From the Wiener-Hopf factorization (see Borovkov (1976), (33), p. 165) we have

$$G^-(-x) \sim \frac{c_2}{1-p} x^{-\beta} \qquad \text{as } x \to +\infty, \tag{3.4}$$

where $p = \psi(0)$. Let

$$H^-(t) = \sum_{k=0}^{+\infty} (G^-)^{*k}(-t), \qquad t \ge 0$$

and $\tilde{F}^{G^-}(s) = \int_0^{+\infty} e^{-sx} \, \mathrm{d}G^-(-x)$. Then the Laplace transform of H^- is equal

$$\tilde{F}^{H^-}(s) = \frac{1}{1 - \tilde{F}^{G^-}(s)} \ .$$

From the Karamata Tauberian Theorem 2.1 the following holds:

$$\lim_{s \to 0} \frac{\tilde{F}^{H^-}(s)}{s^{-\beta}} = \lim_{s \to 0} \frac{s^\beta}{1 - \tilde{F}^{G^-}(s)} = \frac{1-p}{\Gamma(1-\beta) c_2} \ .$$

Thus by the Karamata Tauberian Theorem 2.2

$$H^-(t) \sim \frac{1-p}{\Gamma(1-\beta)\Gamma(1+\beta)c_2} t^\beta = \frac{(1-p)\sin(\beta\pi)}{\beta\pi c_2} t^\beta ,$$

which completes the proof of (ii) in view of Borovkov (1976), p. 180 and Lemma 2, p. 173.

□

Note that the Laplace transform of S^+ is equal

$$\tilde{F}^{S^+}(s) = \mathbb{E}e^{-sS^+} = \mathbb{E}^X(\mathbb{E}^U e^{-sU})^{N(T_{n+1})-N(T_n)} = \mathbb{E}^X e^{-\log((\mathbb{E}^U e^{-sU})^{-1})\int_{T_0}^{T_1}\lambda(X(t))\,dt} ,$$

where $\mathbb{E}^X$ and $\mathbb{E}^U$ are expectations with respect to the law of process $\{X(t), t \geq 0\}$ and r.v. U. That is,

$$\tilde{F}^{S^+}(s) = \tilde{F}^Z(\log \tilde{F}^U(s)^{-1}) . \tag{3.5}$$

Moreover, if $\nu < +\infty$, then $\nu = \mathbb{E}T - \mathbb{E}Z\mathbb{E}U = m_{1,T} - m_{1,Z}m_{1,U}$. Thus if
Condition S.

$$\nu < +\infty \qquad \text{and} \qquad m_{1,T} > m_{1,U}m_{1,Z} , \tag{S}$$

then a stability condition $S(t) \to -\infty$ a.e. as $t \to +\infty$ holds. We assume that at least one of the variables U and Z has a regularly varying distribution. Thus

$$\alpha_{Z,U} = \min\{\alpha_U, \alpha_Z\} < +\infty .$$

In particular, if U or Z has a regularly varying distribution, then
Condition U.

$$\mathbb{P}(U > x) \sim l_U(x)x^{-\alpha_U} \tag{U}$$

or
Condition Z.

$$\mathbb{P}(Z > x) \sim l_Z(x)x^{-\alpha_Z} \tag{Z}$$

for slowly varying functions $l_U(x)$ and $l_Z(x)$ respectively. Define

$$l_{Z,U}(x) = \begin{cases} l_Z(x) & \text{if } Z \text{ is reg. var. and } \alpha_Z < \alpha_U \\ l_U(x) & \text{if } U \text{ is reg. var. and } \alpha_U \leq \alpha_Z. \end{cases} \tag{3.6}$$

Using the Karamata Tauberian Theorem we can prove the following theorem (see also Asmussen *et al.* (1999), Schmidli (1999), Stam (1973) and Grandell (1997)).

Theorem 3.2. *Assume that condition (A) and (Z) or (U) hold.*
(i) If $1 < \alpha_Z < \alpha_U$ *and (S) holds, then*

$$\psi(u) \sim C_1 l_Z(u) u^{-\alpha_Z+1} , \tag{3.7}$$

where

$$C_1 = \frac{1}{\alpha_Z - 1} m_{1,U}^{\alpha_Z} \frac{1}{m_{1,T} - m_{1,U}m_{1,Z}} . \tag{3.8}$$

(ii) If $1 < \alpha_U < \alpha_Z$ *and (S) holds, then*

$$\psi(u) \sim C_2 l_U(u) u^{-\alpha_U+1} \ , \tag{3.9}$$

where

$$C_2 = \frac{1}{\alpha_U - 1} m_{1,Z} \frac{1}{m_{1,T} - m_{1,U} m_{1,Z}} \ . \tag{3.10}$$

(iii) If $\alpha_Z = \alpha_U$ *and (S) holds, then*

$$\psi(u) \sim C_3 l_U(u) u^{-\alpha_U+1} \ , \tag{3.11}$$

where $C_3 = C_1 + C_2$.
(iv) Assume that T *fulfils (3.2) for* $0 < \beta < 1$. *Then*

$$\psi(u) \sim C_4 u^{\beta - \alpha_{Z,U}} \ , \tag{3.12}$$

where

$$C_4 = \frac{\sin(\beta\pi)}{\beta\pi} \frac{m_{1,U}^{\alpha_Z} c_1}{c_2} \int_0^{+\infty} y^{\beta-1}(1+y)^{-\alpha_Z} \, \mathrm{d}y \ , \tag{3.13}$$

when $l_Z(x) = c_1$, $\beta < \alpha_Z < \alpha_U$; *and*

$$C_4 = \frac{\sin(\beta\pi)}{\beta\pi} \frac{m_{1,Z} c_1}{c_2} \int_0^{+\infty} y^{\beta-1}(1+y)^{-\alpha_U} \, \mathrm{d}y \ , \tag{3.14}$$

when $l_U(x) = c_1$, $\beta < \alpha_U < \alpha_Z$.

Proof. We prove (i). The points (ii)-(iv) can be proved in a very similar way. To prove (i) by Theorem 3.1 it suffices to prove that

$$\mathbb{P}(S^+ > x) \sim m_{1,U}^{\alpha_Z} l_Z(x) x^{-\alpha_Z} \ . \tag{3.15}$$

Let $k = [\alpha_Z]$ and $l = [\alpha_U]$. We will write $g(s) = O_1(f(s))$ if $\lim_{s\to 0} \frac{g(s)}{f(s)} = 1$. By the Karamata Tauberian Theorem 2.1:

$$\tilde{F}^Z(s) = 1 - m_{1,Z} s + \frac{1}{2} m_{2,Z} s^2 - \ldots + \frac{(-1)^k}{k!} m_{k,Z} s^k + O_1((-1)^k \Gamma(1-\alpha_Z) s_Z^{\alpha} l_Z(1/s)) \tag{3.16}$$

and

$$\tilde{F}^U(s) = 1 - m_{1,U} s + \frac{1}{2} m_{2,U} s^2 - \ldots + \frac{(-1)^l}{l!} m_{l,U} s^l + O_1((-1)^l \Gamma(1-\alpha_U) s_U^{\alpha} l_U(1/s)) \ . \tag{3.17}$$

Hence by (3.5) we have:

$$\tilde{F}^{S^+}(s) = 1 - m_{1,Z} \log(\tilde{F}^U(s)^{-1}) + \frac{1}{2} m_{2,Z} \left(\log(\tilde{F}^U(s)^{-1})\right)^2 - \ldots + \frac{(-1)^k}{k!} \left(\log(\tilde{F}^U(s)^{-1})\right)^k +$$

$$+ O_1\left[(-1)^k \Gamma(1-\alpha_Z) \left(\log(\tilde{F}^U(s)^{-1})\right)^{\alpha_Z} l_Z\left(1/\log(\tilde{F}^U(s)^{-1})\right)\right] \ . \tag{3.18}$$

Note that for $x > 0$ such that $|x - 1| < 1$

$$\log(1/x) = \sum_{k=1}^{+\infty} \frac{(-1)^k}{k}(x-1)^k \ .$$

Hence

$$\log(\tilde{F}^U(s)^{-1}) = \sum_{k=1}^{+\infty} \frac{(-1)^k}{k}(\tilde{F}^U(s)-1)^k \ .$$

From above, (3.16) - (3.18) under assumption $\alpha_Z < \alpha_U$ we have $k \le l$ and

$$\begin{aligned}
\tilde{F}^{S^+}(s) &= 1 - m_{1,Z}m_{1,U}s + \frac{1}{2}(m_{2,Z}m_{1,U}^2 - m_{1,Z}m_{2,U})s^2 - \ldots + m_{k,S^+}s^k + \\
&+ O_1[(-1)^k\Gamma(1-\alpha_Z)(\tilde{F}^U(s)-1)^{\alpha_Z} l_Z(1/(\tilde{F}^U(s)-1))] \\
&= 1 - m_{1,Z}m_{1,U}s + \ldots + m_{k,S^+}s^k + O_1[(-1)^k\Gamma(1-\alpha_Z)m_{1,U}^{\alpha_Z}s^{\alpha_Z}l_Z(1/m_{1,U}s)] \\
&= 1 - m_{1,Z}m_{1,U}s + \ldots + m_{k,S^+}s^k + O_1[(-1)^k\Gamma(1-\alpha_Z)m_{1,U}^{\alpha_Z}s^{\alpha_Z}l_Z(1/s)] \ .
\end{aligned}$$

Thus

$$f_k^{S^+}(s) \sim (-1)^k\Gamma(1-\alpha_Z)m_{1,U}^{\alpha_Z}s^{\alpha_Z}l_Z(1/s)$$

as $s \to 0$, which completes the proof in view of (3.15) and the Karamata Tauberian Theorem 2.1.

□

Remark 3.1. Similar results can be also obtained in a so-called delayed case, when $T_0 > 0$. Denote $S_0^+ = \sum_{i=1}^{N(T_0)} U_i$ and $Z_0 = \int_0^{T_0} \lambda(X(t))\mathrm{d}t$. If $\mathbb{P}(S_0^+ > x) = o(l_S(x)x^{-\alpha_S+1})$, then the ruin probability $\psi_0(u)$ in the delayed case is asymtotically equivalent to the ruin probability $\psi(u)$ in the so-called zero-delayed case (when $T_0 = 0$). That is,

$$\psi_0(u) \sim \psi(u), \qquad \text{as } u \to +\infty.$$

This is the case when claim size U has the regularly varying distribution given in condition (U) and there exists a $\delta > 0$ such that $\mathbb{E}e^{\delta Z_0} < +\infty$. See Asmussen *et al.* (2000), Cor. 3.2, for other relations between $\psi(u)$ and $\psi_0(u)$.

Corollary 3.1. *If conditions (B),(U) and (S) are fulfilled, then*

$$\psi(u) \sim C_2 l_U(u)u^{-\alpha_U+1} \ , \tag{3.19}$$

where C_2 is given in (3.10).

4 Semi-Markov model

Let $\{T_n\}_{n=1}^{+\infty}$ be the zero-delayed renewal process. That is, $T_{n+1} - T_n$ are i.i.d. r.v.'s. On time interval $[T_n, T_{n+1})$ the process $\{X(t), t \ge 0\}$ is equal to a positive r.v. Δ_n. Moreover, let $\lambda(x) = x$. Thus $Z = T\Delta$. We can change the distributions of T and Δ in such a way that we can get all possible cases (i) -(iv) in the main Theorem 3.2 (see Grandell (1997), Schmidli (1999)). In particular, we can consider the Ammeter (1948) model when $T = 1$. Then obviously condition (A) holds. From Theorem 3.2 (i) we obtain the following theorem.

Theorem 4.1. *Assume that there exists a* $\delta > 0$ *such that* $\mathbb{E}\exp\{\delta U\} < +\infty$ *and*

$$\mathbb{P}(\Delta > x) \sim x^{-\alpha_\Delta} l_\Delta(x), \qquad as\ x \to +\infty$$

for the slowly varying function $l_\Delta(x)$ *and* $\alpha_\Delta > 1$. *If* $m_{1,U} m_{1,\Delta} < 1$, *then we have the following asymptotics:*

$$\psi(u) \sim \frac{1}{\alpha_\Delta - 1} m_{1,U}^{\alpha_\Delta} \frac{1}{1 - m_{1,U} m_{1,\Delta}} \quad l_\Delta(u) u^{-\alpha_\Delta + 1} .$$

Hence you can get the regularly varying asymptotics of the ruin probability $\psi(u)$ even when the claim sizes U are light-tailed.

5 Diffusion processes

On a probability space $(\mathcal{C}[0, +\infty), \mathcal{F}, \{\mathcal{F}_t^X\}_{t \geq 0}, \mathbb{P}^X)$ consider a canonical diffusion process $\{X(t), t \geq 0\}$, where $\{\mathcal{F}_t^X\}_{t \geq 0}$ is a natural filtration and $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t^X$. Let the process $\{X(t), t \geq 0\}$ have the following infinitesimal generator

$$(\boldsymbol{A} f)(x) = \frac{1}{2} a(x) \frac{d^2}{dx^2} f(x) + b(x) \frac{d}{dx} f(x)$$

for $f \in \mathcal{C}^2(\mathbb{R})$. Assume that there exists a constant L so that

$$a^2(x) + |b(x)| \leq L(1 + |x|) \tag{5.20}$$

and that there exists, for each constant $C > 0$, a constant L_C so that

$$|a^2(x) - a^2(y)| + |b(x) - b(y)| \leq L_C |x - y| \qquad \text{for } |x| \leq C \text{ and } |y| \leq C. \tag{5.21}$$

Assume also that the diffusion process $\{X(t), t \geq 0\}$ is recurrent in a sense that any possible state is reached from any other state with probability 1. Let $X(0) = 0$ and $T_0 = 0$. In this paper we consider two kinds of regeneration moments: $T_n = n$ ($n \in N$) and

$$T_{n+1} = \inf\{t \geq S_n : X(t) = 0\} , \tag{5.22}$$

where

$$S_n = \inf\{t \geq T_n : |X(t)| = 1\} \qquad n = 0, 1, 2, \dots .$$

In this case $Z = \int_0^{T_1} \lambda(X(s))\, \mathrm{d}s$.

5.1 Reflected Brownian motion and $\lambda(x) = x$

Assume that the claim size U has the regularly varying distribution given in condition (U). Let $\{B(t), t \geq 0\}$ be a Brownian motion starting at $B(0) = 0$. Set $s(y) = (-1)^{[y]}$ and

$$S(x) = \int_0^x s(y)\, \mathrm{d}y .$$

Thus $S(x)$ is a "saw-tooth" function with $S(x) = |x|$ for $-1 \leq x \leq 1$ and with a period 2. Assume that $\lambda(x) = x$. Then the intensity process $\{X(t) = S(B(t)), t \geq 0\}$ is a reflected Brownian motion with boundaries 0 and 1. The regeneration moments are defined by (5.22). Note that

$$\mathbb{E}_0^X e^{\delta T} = \mathbb{E}_1^B e^{\delta T'} \mathbb{E}_0^B e^{\delta S_0} = \left(\mathbb{E}_0^B e^{\delta S_0}\right)^2 , \tag{5.23}$$

where

$$T^{'} = \inf\{t \geq 0 : |B(t) - 1| = 1\}$$

and $\mathbb{E}_x^B$ is the expectation with respect to $\mathbb{P}^B$ when the Brownian motion $\{B(t), t \geq 0\}$ starts at x. By Wentzell (1975), p. 259, we have

$$\mathbb{E}_0^B S_0 = 1 \ . \tag{5.24}$$

Thus

$$m_{1,T} = 2\mathbb{E}_0^B S_0 = 2 \ . \tag{5.25}$$

Moreover, by Wentzell (1975), p. 265, we have the following lemma.

Lemma 5.1. *If τ is an exiting time by a diffusion process from a compact set D and $\mathbb{E}\tau \leq M < +\infty$, then $\mathbb{E}e^{\delta\tau} \leq 1 + \frac{\delta}{1-\delta M}\mathbb{E}\tau \leq 1 + \frac{\delta M}{1-\delta M}$ for $0 \leq \delta < M^{-1}$.*

Thus from Lemma 5.1, (5.23)-(5.24) there exists a $\delta > 0$ such that

$$\mathbb{E}_0^X e^{\delta T} < +\infty \ . \tag{5.26}$$

Note also that $0 \leq X(t) \leq 1$, hence condition (B) is fulfilled:

$$\mathbb{E}_0^X e^{\delta Z} = \mathbb{E}_0^X e^{\delta \int_0^{T_1} X(t)\ \mathrm{d}t} \leq \mathbb{E}_0^X e^{\delta T} < +\infty \ . \tag{5.27}$$

If (S), (U) hold, then from Corollary 3.1 we have that

$$\psi(u) \sim C_2 l_U(u) u^{-\alpha_U + 1} \ ,$$

where C_2 is given in (3.10). To calculate C_2 explicit we find by the Markov property and the symmetry of the Brownian motion that

$$\begin{aligned} m_{1,Z} = \mathbb{E}_0^X \int_0^{T_1} X(s)\ \mathrm{d}s &= \mathbb{E}_0^B \int_0^{S_0} |B(t)|\ \mathrm{d}t + \mathbb{E}_1^B \int_0^{T^{'}} (1 - |1 - B(t)|)\ \mathrm{d}t = \\ &= \mathbb{E}_0^B \int_0^{S_0} |B(t)|\ \mathrm{d}t + \mathbb{E}_1^B T^{'} - \mathbb{E}_0^B \int_0^{S_0} |B(t)|\ \mathrm{d}t = 1 \ . \end{aligned}$$

Summarizing we have the following theorem.

Theorem 5.1. *Assume that the claim size U has the regularly varying distribution (U) with $\alpha_U > 1$ and $\mathbb{E}U < 2$. Moreover, let the intensity process $\{X(t), t \geq 0\}$ be the reflecting Brownian motion reflecting at barriers 0 and 1. Then*

$$\psi(u) \sim \frac{1}{\alpha_U - 1} \frac{1}{2 - m_{1,U}} \quad l_U(u) u^{-\alpha_U + 1} \ .$$

5.2 Brownian Motion and $\lambda(x) = e^{-\gamma|x|}$

Assume that the claim size has the regularly varying distribution (U) with the index $\alpha_U > \frac{1}{2}$ and $l_U(x) = c_1$ for some constant c_1. Let the governing process $\{X(t) = B(t), t \geq 0\}$ be the Brownian motion starting at $B(0) = 0$ and $\lambda(x) = e^{-\gamma|x|}$. That is, $\{\exp(-\gamma|B(t)|), t \geq 0\}$ is the intensity Markov process. The regeneration moments are defined by (5.22). Then by the symmetry and the Markov property of the Brownian motion we have:

$$T \overset{D}{=} S_0 + \hat{T} \ ,$$

where

$$\hat{T} = \inf\{t \geq 0 : B(t) = 0, B(0) = 1\} .$$

Note that $\mathbb{E}_0^B S_0 = 1$ and $\mathbb{E}_1^B \hat{T} = +\infty$. Hence

$$\mathbb{E}_0^B T = +\infty . \tag{5.28}$$

Moreover, by Karatzas and Shreve (1988), p. 96,

$$\mathbb{P}(T > t) \sim \frac{1}{\sqrt{\pi}} t^{-\frac{1}{2}} . \tag{5.29}$$

Thus $\alpha_T = \frac{1}{2}$. Note that

$$\mathbb{E}_0^B e^{\delta Z} = \mathbb{E}_0^B e^{\delta Z_1} \mathbb{E}_1^B e^{\delta Z_2} ,$$

where $Z_1 = \int_0^{\hat{T}} e^{-\gamma|B(t)|}\, \mathrm{d}t$ and $Z_2 = \int_0^{S_0} e^{-\gamma|B(t)|}\, \mathrm{d}t$. Moreover, by (5.24) and Lemma 5.1:

$$\mathbb{E}_0^B e^{\delta Z_2} \leq \mathbb{E}_0^B e^{\delta S_0} < +\infty \tag{5.30}$$

for some $\delta > 0$. Let $T(R) = \inf\{t \geq 0 : B(t) = R, B(0) = 1\}$. Then by the Monotone Convergence Theorem:

$$\mathbb{E}_1^B e^{\delta Z_1} = \lim_{R \to +\infty} \mathbb{E}_1^B \exp\{\delta \int_0^{\hat{T} \wedge T(R)} e^{-\gamma|B(t)|}\, \mathrm{d}t\} .$$

Thus from the Feynman-Kac formula (see also Chung and Zhao (1995), Th. 9.22) we get that for sufficently small $\delta > 0$ the following holds:

$$\mathbb{E}_1^B e^{\delta Z_1} = \frac{J_0(2\frac{\sqrt{2\delta}}{\gamma\sqrt{\exp\{\gamma\}}})}{J_0(2\frac{\sqrt{2\delta}}{\gamma})} < +\infty , \tag{5.31}$$

where $J_\nu(x)$ is the Bessel function of the first kind. Then condition (B) follows from (5.30) - (5.31). From Theorem 3.2 (iv) and Lemma 3.1 we have the following theorem.

Theorem 5.2. *Assume that the claim size has the regularly varying distribution (U) with index $\alpha_U > \frac{1}{2}$ and $l_U(x) = c_1$ for some constant c_1. Then*

$$\psi(u) \sim \frac{4c_1}{\pi^{3/2}\gamma} \quad u^{\frac{1}{2}-\alpha_U} \quad \int_0^{+\infty} y^{-1/2}(1+y)^{-\alpha_U}\, \mathrm{d}y \quad .$$

5.3 Ornstein-Uhlenbeck process and $\lambda(x) = x^2 + k$

Let $\{X(t), t \geq 0\}$ be a one-parameter Ornstein-Uhlenbeck process with a parameter b such that $X(0) = 0$. That is, $\{X(t), t \geq 0\}$ is the diffusion process with the infinitesimal generator

$$(\boldsymbol{A}f)(x) = \frac{1}{2}\frac{d^2}{dx^2}f(x) - bx\frac{d}{dx}f(x) , \tag{5.32}$$

where $f \in \mathcal{C}^2(\mathbb{R})$. Let the regeneration moments be defined via (5.22). We take $\lambda(x) = x^2 + k$, for $k \geq 0$. Hence the intensity process $\{X^2(t) + k, t \geq 0\}$ is still the Markov process (see the discussion in Lawrance (1972), p. 225 - 228). We prove that condition (B) holds, that is that

$$\mathbb{E}_0^X e^{\delta Z} = \mathbb{E}_0^X \exp\{\delta \int_0^{T_1} (X^2(t) + k)\ \mathrm{d}t\} < +\infty \tag{5.33}$$

for some $\delta > 0$. Then under (U) and (S) by Corollary 3.1 we have:

$$\psi(u) \sim C_2 l_U(u) u^{-\alpha_U + 1}\ , \tag{5.34}$$

where C_2 is given in (3.10).

The method of calculating functional (5.33) consists in changing probability so that the quadratic functional disappears and the remaining problem is to compute m.g.f.'s of some hitting and exiting time. We introduce the following exponential change of measure

$$\frac{dQ_{|\mathcal{F}_t^X}}{d\mathbb{P}^X_{|\mathcal{F}_t^X}} = M(t)\ , \tag{5.35}$$

where

$$M(t) = \exp\{-\frac{\kappa^2 - b^2}{2} \int_0^t X^2(s)\ \mathrm{d}s - (\kappa - b) \int_0^t X(s)\ \mathrm{d}X(s)\} = \tag{5.36}$$

$$= \exp\{-\frac{\kappa^2 - b^2}{2} \int_0^t X^2(s)\ \mathrm{d}s - \frac{\kappa - b}{2}(X^2(t) - X^2(0) - t)\} \tag{5.37}$$

is an exponential martingale (see Stroock (1987), Th. 4.6 and Rogers and Williams (1987), Th. 27.1). The second equality follows by integration-by-parts for semimartingales. By Stroock (1987), Th. 4.4 and Parthasarathy (1967), Th. 4.2, there exists a unique probability measure Q on $(\mathcal{C}[0, +\infty), \mathcal{F}, \{\mathcal{F}_t^X\}_{t \geq 0})$ fulfilling (5.35). Moreover, by Yor (1992), Leblanc *et al.* (2000) and Palmowski and Rolski (2000) on a new probability space process $\{X(t), t \geq 0\}$ is the Ornstein-Uhlenbeck process with parameter κ. Denote by $\mathbb{E}_0^Q$ the expectation with respect to the measure Q. Let $\kappa = \sqrt{b^2 - 2\delta}$ for $\delta < b^2/2$. Then by the Optional Sampling Theorem we have

$$\mathbb{E}_0^X e^{\delta Z} = \mathbb{E}_0^X e^{\delta \int_0^{T_1} X^2(t)\ \mathrm{d}t + \delta k T_1} = \mathbb{E}_0^Q e^{\delta \int_0^{T_1} X^2(t)\ \mathrm{d}t + \delta k T_1} M(T_1)^{-1} =$$

$$= \mathbb{E}_0^Q \exp\{\frac{\kappa - b}{2}(X^2(T_1) - X^2(0) - T_1) + \delta k T_1\} = \mathbb{E}_0^Q e^{\frac{b - \kappa + 2\delta k}{2} T_1}\ . \tag{5.38}$$

Let $\hat{\delta} = \frac{b - \kappa + 2\delta k}{2}$. Note that the following monotone convergence holds:

$$\hat{\delta} \to 0^+ \qquad \text{as } \delta \to 0^+. \tag{5.39}$$

Thus it suffices to find $\hat{\delta} > 0$ such that

$$\mathbb{E}_0^Q e^{\hat{\delta} T_1} < +\infty\ .$$

From the Markov property and the symmetry of the Ornstein-Uhlenbeck process we have

$$\mathbb{E}_0^Q e^{\hat{\delta} T_1} = \mathbb{E}_0^Q e^{\hat{\delta} S_0} \mathbb{E}_1^Q e^{\hat{\delta} \hat{T}}\ , \tag{5.40}$$

where $\hat{T} = \inf\{t \geq 0 : X(t) = 0 \quad \text{and} \quad X(0) = 1\}$. Note that now, the parameters of the process $\{X(t), t \geq 0\}$ under the new probability measure Q depend on δ and hence also on $\hat{\delta}$. For this case we prepare some lemmas. Firstly, we generalize Lemma 5.1 in the following way.

Lemma 5.2. *Consider a family of diffusion processes* $\{X_w(t), t \geq 0\}$ *parametrized by* $w \geq 0$ *starting at* $X(0) = x_w$. *If* τ_w *is an exiting time by a diffusion* $\{X_w(t), t \geq 0\}$ *from a compact set* D *and* $\mathbb{E}_{x_w}^{X_w} \tau_w \leq M$ *for all* $0 < w \leq w_0 < M^{-1}$, *then* $\mathbb{E}_{x_w}^{X_w} e^{w\tau_w} \leq 1 + \frac{w_0 M}{1 - w_0 M}$ *for* $0 < w \leq w_0$.

Remark 5.1. Assume that $\{X_w(t), t \geq 0\}$ has the following infinitesimal generator

$$(\boldsymbol{A}_w f)(x) = \frac{1}{2} a_w(x) \frac{d^2}{dx^2} f(x) + b_w(x) \frac{d}{dx} f(x)$$

for $f \in \mathcal{C}^2(\mathbb{R})$, where functions $a_w(x)$ and $b_w(x)$ fulfil (5.20) and (5.21). If there exists $w_0 > 0$ such that

$$\inf_{w \leq w_0} \inf_{x \in D} a_w(x) > 0 \quad \text{and} \quad \sup_{w \leq w_0} \sup_{x \in D} |b(x)| \leq B < +\infty$$

for some constant B, then by Lemma 5.2 and Wentzell (1975), p. 258, $\mathbb{E}_{x_w}^{X_w} e^{w\tau_w}$ is uniformly bounded for $0 < w \leq w_0$.

Lemma 5.3. *Let* $\{X_w(t), t \geq 0\}$ *be the family of diffusion processes parametrized by* w *starting at* $X(0) = x_w$ *and*

$$H_z^w = \inf\{t \geq 0 : X_w(t) = z\}$$

be a hitting time. If there exists $w_0 > 0$ *such that* $\mathbb{E}_{x_w}^{X_w} H_z^w \leq M$ *for all* $0 < w \leq w_0 < M^{-1}$ *and some* M, *then* $\mathbb{E}_{x_w}^{X_w} e^{wH_z^w}$ *is also uniformly bounded for* $0 < w \leq w_0$.

Proof. Without loss of generality we can assume that $x_w > z$ for $0 < w \leq w_0$. By the Monotone Convergence Theorem

$$\mathbb{E}_{x_w}^{X_w} e^{wH_z^w} = \lim_{R \to +\infty} \mathbb{E}_{x_w}^{X_w} e^{wH_z^w \wedge T^w(R)} \ ,$$

where

$$T^w(R) = \inf\{t \geq 0 : X_w(t) = R\} \ .$$

Note that $\mathbb{E}_{x_w}^{X_w} H_z^w \wedge T^w(R) \leq \mathbb{E}_{x_w}^{X_w} H_z^w \leq M$ for all $0 < w \leq w_0$. Thus by Lemma 5.2

$$\mathbb{E}_{x_w}^{X_w} e^{wH_z^w} = \lim_{R \to +\infty} \mathbb{E}_{x_w}^{X_w} e^{wH_z^w \wedge T^w(R)} \leq 1 + \frac{Mw_0}{1 - w_0 M} < +\infty \ .$$

□

By Remark 5.1 and (5.39) there exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$

$$\mathbb{E}_0^Q e^{\hat{\delta} S_0} < +\infty \ . \tag{5.41}$$

Moreover, if

$$\mathbb{E}_1^Q \hat{T} < M$$

for given M and $0 < \delta \leq \delta_0$, then by Lemma 5.3 and (5.39) for $0 < \delta \leq \delta_0$ we have:

$$\mathbb{E}_1^Q e^{\hat{\delta} \hat{T}} < +\infty \ . \tag{5.42}$$

We calculate $\mathbb{E}_1^Q \hat{T}$ using the Laplace transform method. Denote by $D_{-\mu}(x)$ a parabolic cylinder function given by

$$D_{-\mu}(x) = e^{-x^2/4} 2^{-\mu/2} \sqrt{\pi} \left\{ \frac{1}{\Gamma((\mu+1)/2)} \left(1 + \sum_{k=1}^{+\infty} \frac{\mu(\mu+2)\dots(\mu+2k-2)}{3\cdot 5 \cdots (2k-1)k!} \left(\frac{x^2}{2}\right)^k \right) - \frac{x\sqrt{2}}{\Gamma(\mu/2)} \left(1 + \sum_{k=1}^{+\infty} \frac{(\mu+1)(\mu+3)\dots(\mu+2k-1)}{3\cdot 5 \cdots (2k+1)k!} \left(\frac{x^2}{2}\right)^k \right) \right\} . \tag{5.43}$$

Moreover, let

$$s_1(x) = \sum_{k=2}^{+\infty} \frac{2\dots(2k-2)}{3\cdot 5\cdots(2k-1)k!} \left(\frac{x^2}{2}\right)^k + \frac{x^2}{2} \tag{5.44}$$

and

$$s_2(x) = \sum_{k=1}^{+\infty} \frac{1}{(2k+1)k!} \left(\frac{x^2}{2}\right)^k . \tag{5.45}$$

From Borodin and Salminen (1996), p. 429, we have the following lemma.

Lemma 5.4. *Let $\{X(t), t \geq 0\}$ be the Ornstein-Uhlenbeck process with infinitesimal generator (5.32) and*

$$H_z = \inf\{t \geq 0 : X(t) = z\} .$$

Then

$$L^{H_z}(s) = \mathbb{E}_x^X e^{-sH_z} = \begin{cases} \frac{e^{(x^2 b)/2} D_{-s/b}(-\sqrt{2b}x)}{e^{(z^2 b)/2} D_{-s/b}(-\sqrt{2b}z)}, & \text{for } x \leq z \\ \\ \frac{e^{(x^2 b)/2} D_{-s/b}(\sqrt{2b}x)}{e^{(z^2 b)/2} D_{-s/b}(\sqrt{2b}z)}, & \text{for } z \leq x \end{cases}$$

and

$$\mathbb{E}_x^X H_z = \begin{cases} \frac{1}{b}\left[(s_1(z\sqrt{2b}) - s_1(x\sqrt{2b})) + \sqrt{b\pi}(z-x) + \sqrt{b\pi}(zs_2(z\sqrt{2b}) - xs_2(x\sqrt{2b}))\right], & \text{for } x \leq z \\ \\ \frac{1}{b}\left[\sqrt{b\pi}(xs_2(x\sqrt{2b}) - zs_2(z\sqrt{2b})) - (s_1(x\sqrt{2b}) - s_1(z\sqrt{2b})) + \sqrt{b\pi}(x-z)\right], & \text{for } x \geq z. \end{cases}$$

From Lemma 5.4

$$\mathbb{E}_1^Q \hat{T} = -\frac{d}{ds} L^{\hat{T}}(s)_{|s=0+} = \frac{1}{\kappa}\left[\sqrt{\kappa\pi} s_2(\sqrt{2\kappa}) - s_1(\sqrt{2\kappa}) + \sqrt{\kappa\pi}\right] \leq \frac{2}{\sqrt{b}}(s_2(\sqrt{2b}) + 1) \tag{5.46}$$

for $\delta \leq \frac{3b^2}{8}$ (then $\kappa \leq b/2$). Thus by (5.38), (5.40), (5.41) - (5.42) the condition (5.33) is fulfilled and hence the asymptotics (5.34) holds.

To calculate constant C_2 in (5.34) explicit we have to compute $m_{1,T}$ and $m_{1,Z}$. Note that

$$m_{1,T} = \mathbb{E}_0^X T = \mathbb{E}_0^X S_0 + \mathbb{E}_1^X \hat{T} . \tag{5.47}$$

By Lemma 5.4:

$$\mathbb{E}_1^X \hat{T} = \frac{1}{b}\left[\sqrt{b\pi} s_2(\sqrt{2b}) - s_1(\sqrt{2b}) + \sqrt{b\pi}\right] . \tag{5.48}$$

We calculate $\mathbb{E}_0^X S_0$ using the Laplace transform method. Denote:

$$S(\mu,x,y) = \frac{\Gamma(\mu)}{\pi} e^{(x^2+y^2)/4} \left(D_{-\mu}(-x)D_{-\mu}(y) - D_{-\mu}(x)D_{-\mu}(-y)\right) .$$

By Borodin and Salminen (1996), p. 434, we have the following lemma.

Lemma 5.5. *Let $\{X(t), t \geq 0\}$ be the Ornstein-Uhlenbeck process with the infinitesimal generator (5.32) and*

$$H_{a,z} = \inf\{t \geq 0 : X(t) \notin (a,z)\} .$$

Then for $a \leq x \leq z$

$$L^{H_{a,z}}(s) = \mathbb{E}_x^X e^{-sH_{a,z}} = \frac{S(\frac{s}{b}, z\sqrt{2b}, x\sqrt{2b}) + S(\frac{s}{b}, x\sqrt{2b}, a\sqrt{2b})}{S(\frac{s}{b}, z\sqrt{2b}, a\sqrt{2b})}$$

and

$$\mathbb{E}_x^X H_{a,z} = \frac{A(x,a,z)}{b(z(1+s_2(z\sqrt{2b})) - a(1+s_2(a\sqrt{2b})))} ,$$

where

$$\begin{aligned} A(x,a,z) = {} & z(1+s_2(z\sqrt{2b}))(s_1(a\sqrt{2b}) - s_1(x\sqrt{2b})) + \\ & + a(1+s_2(a\sqrt{2b}))(s_1(x\sqrt{2b}) - s_1(z\sqrt{2b})) + \\ & + x(1+s_2(x\sqrt{2b}))(s_1(z\sqrt{2b}) - s_1(a\sqrt{2b})) . \end{aligned}$$

Lemma 5.5 gives:

$$m_{1,S_0} = \mathbb{E}_0^X S_0 = \frac{1}{b} s_1(\sqrt{2b}) . \tag{5.49}$$

By (5.47) - (5.49) we have:

$$m_{1,T} = \frac{\sqrt{\pi}}{\sqrt{b}} \left[s_2(\sqrt{2b}) + 1 \right] . \tag{5.50}$$

We change the measure by (5.35) using the martingale

$$M(t) = \exp\{-\frac{\tilde{\kappa}^2 - b^2}{2} \int_0^t X^2(s)\, \mathrm{d}s - \frac{\tilde{\kappa} - b}{2}(X^2(t) - X^2(0) - t)\} \tag{5.51}$$

for $\tilde{\kappa} = \sqrt{b^2 + 2s}$. Then we get

$$L^Z(s) = \mathbb{E}_0^X e^{-s\int_0^T (X^2(t)+k)\,\mathrm{d}t} = \mathbb{E}_0^Q e^{-\hat{s}T} = \mathbb{E}_0^Q e^{-\hat{s}S_0} \mathbb{E}_1^Q e^{-\hat{s}\hat{T}} ,$$

where $\hat{s} = \frac{\kappa - b + 2sk}{2}$ and under the probability measure Q process $\{X(t), t \geq 0\}$ is the Ornstein-Uhlenbeck process with the parameter $\tilde{\kappa}$. From Lemmas 5.5 and 5.4:

$$\mathbb{E}_0^Q e^{-\hat{s}S_0} = \frac{S(\hat{s}/\tilde{\kappa}, \sqrt{2\tilde{\kappa}}, 0) + S(\hat{s}/\tilde{\kappa}, 0, -\sqrt{2\tilde{\kappa}})}{S(\hat{s}/\tilde{\kappa}, \sqrt{2\tilde{\kappa}}, -\sqrt{2\tilde{\kappa}})} ,$$

and

$$\mathbb{E}_1^Q e^{-\hat{s}\hat{T}} = \frac{e^{\tilde{\kappa}/2} D_{-\frac{\hat{s}}{\tilde{\kappa}}}(\sqrt{2\tilde{\kappa}})}{D_{-\frac{\hat{s}}{\tilde{\kappa}}}(0)} .$$

Thus

$$m_{1,Z} = -\frac{d}{ds} L^Z(s)_{|s=0^+} = \frac{\sqrt{\pi}}{\sqrt{b}} (s_2(\sqrt{2b}) + 1) \left(\frac{1}{2b} + k \right) . \tag{5.52}$$

Summarizing, from (5.34) we have the following theorem.

Theorem 5.3. *Assume that $\{X^2(t)+k, t\geq 0\}$ is the intensity process for $k\geq 0$ and for the Ornstein-Uhlenbeck process $\{X(t), t\geq 0\}$ with the parameter b starting at $X(0)=0$. If the claim size has the regularly varying distribution (U) and $m_{1,U}<\frac{2b}{1+2bk}$, then*

$$\psi(u)\sim\frac{1}{\alpha_U-1}\frac{(1/2b+k)}{1-m_{1,U}(1/2b+k)}\quad l_U(u)u^{-\alpha_U+1}\ .$$

5.4 Ornstein-Uhlenbeck process and $\lambda(x)=(x+p)^2$

Let $\{X(t), t\geq 0\}$ be the Ornstein-Uhlenbeck process with the parameter b starting at $X(0)=0$. Let the regeneration moments be defined by (5.22). We take $\lambda(x)=(x+p)^2$. Hence the intensity process $\{(X(t)+p)^2, t\geq 0\}$ is non-Markovian. We prove condition (B) similarly like in the previous section. Then by Corollary 3.1 under conditions (U) and (S) the asymptotics (3.19) holds. We introduce the exponential change of measure (5.35), where

$$M(t)=\exp\{\frac{b^2}{2}\int_0^t X^2(w)\ \mathrm{d}w+\frac{b}{2}(X^2(t)-X^2(0)-t)\}\ . \tag{5.53}$$

By the Cameron-Martin-Girsanov Theorem the process $\{X(t), t\geq 0\}$ is the Brownian motion. Hence

$$\begin{aligned}\mathbb{E}_0^X e^{\delta Z}&=\mathbb{E}_0^Q M^{-1}(T)e^{\delta\int_0^T(X(t)+p)^2\ \mathrm{d}t}\\&=\mathbb{E}_0^Q e^{\frac{-\kappa^2}{2}\int_0^T(X(t)-\tilde{p})^2\ \mathrm{d}t+(\frac{\kappa^2}{2}\tilde{p}^2+\delta p^2+\frac{b}{2})T}\ ,\end{aligned}$$

where $\kappa=\sqrt{b^2-2\delta}$ and $\tilde{p}=\frac{2p\delta}{\kappa^2}$. Let

$$\tilde{S}_0=\inf\{t\geq 0:|X(t)+\tilde{p}|=1\},\qquad \tilde{T}=\inf\{t\geq\tilde{S}_0:X(t)=-\tilde{p}\}\ .$$

Then

$$\mathbb{E}_0^X e^{\delta Z}=\mathbb{E}_{-\tilde{p}}^Q e^{\frac{\kappa^2}{2}\int_0^{\tilde{T}}X^2(t)\ \mathrm{d}t+(\frac{\kappa^2}{2}\tilde{p}^2+\delta p^2+\frac{b}{2})\tilde{T}}\ .$$

We change again the measure in the following way:

$$\frac{d\tilde{Q}_{|\mathcal{F}_t^X}}{dQ^X_{|\mathcal{F}_t^X}}=\tilde{M}(t)\ ,$$

where

$$\tilde{M}(t)=\exp\{-\frac{\kappa^2}{2}\int_0^t X^2(w)\ \mathrm{d}w-\frac{\kappa}{2}(X^2(t)-X^2(0)-t)\}\ .$$

Then under the probability measure $\tilde{Q}$ the process $\{X(t), t\geq 0\}$ is the Ornstein-Uhlenbeck process with the parameter κ. We have

$$\mathbb{E}_0^X e^{\delta Z}=\mathbb{E}_{-\tilde{p}}^{\tilde{Q}}\exp\{(-\frac{\kappa}{2}+\frac{\kappa^2}{2}\tilde{p}^2+\delta p^2+\frac{b}{2})\tilde{T}\}=\mathbb{E}_{-\tilde{p}}^{\tilde{Q}}e^{\tilde{\delta}\tilde{T}}\ , \tag{5.54}$$

where

$$\tilde{\delta}=\frac{b}{2}-\frac{\kappa}{2}+p\delta\tilde{p}+\delta p^2>0\ .$$

Note that monotonically

$$\tilde{\delta}\to 0^+\qquad\text{as }\delta\to 0^+. \tag{5.55}$$

Denote

$$\tilde{T}_0 = \inf\{t \geq 0 : X(t) = -\tilde{p}\} \ .$$

By the Markov property

$$\mathbb{E}^{\tilde{Q}}_{-\tilde{p}} e^{\tilde{\delta}\tilde{T}} = \mathbb{E}^{\tilde{Q}}_{-\tilde{p}} e^{\tilde{\delta}\tilde{S}_0} \Big[\mathbb{E}^{\tilde{Q}}_{-\tilde{p}-1} e^{\tilde{\delta}\tilde{T}_0} \cdot \tilde{Q}(X(\tilde{S}_0) = -\tilde{p}-1) + \\ + \mathbb{E}^{\tilde{Q}}_{-\tilde{p}+1} e^{\tilde{\delta}\tilde{T}_0} \cdot \tilde{Q}(X(\tilde{S}_0) = -\tilde{p}+1) \Big] \leq \mathbb{E}^{\tilde{Q}}_{-\tilde{p}} e^{\tilde{\delta}\tilde{S}_0} \left(\mathbb{E}^{\tilde{Q}}_{-\tilde{p}-1} e^{\tilde{\delta}\tilde{T}_0} + \mathbb{E}^{\tilde{Q}}_{-\tilde{p}+1} e^{\tilde{\delta}\tilde{T}_0} \right) \ .$$

From Remark 5.1 and (5.55) there exists $\delta_0 > 0$ such that $\mathbb{E}^{\tilde{Q}}_{-\tilde{p}} e^{\tilde{\delta}\tilde{S}_0}$ is uniformly bounded for all $0 < \delta \leq \delta_0$. Thus to prove (B) it suffices by Lemma 5.3 and (5.55) to find $\delta_0 > 0$ such that $\mathbb{E}^{\tilde{Q}}_{-\tilde{p}-1}\tilde{T}_0$ and $\mathbb{E}^{\tilde{Q}}_{-\tilde{p}+1}\tilde{T}_0$ are uniformly bounded for all $0 < \delta \leq \delta_0$. Lemma 5.4 gives:

$$\mathbb{E}^{\tilde{Q}}_{-\tilde{p}-1}\tilde{T}_0 = \frac{1}{\kappa}[s_1(\tilde{p}\sqrt{2\kappa}) + \sqrt{\kappa\pi} + \sqrt{\kappa\pi}(\tilde{p}+1)s_2(\sqrt{\kappa\pi}(\tilde{p}+1))] \leq \\ \leq \frac{2}{b}\left[s_1(\sqrt{2b}) + \sqrt{b\pi} + 2\sqrt{b\pi}s_2(2\sqrt{2b}) \right] \ , \\ \mathbb{E}^{\tilde{Q}}_{-\tilde{p}+1}\tilde{T}_0 \leq \frac{2}{b}\left[s_1(\sqrt{2b}) + \sqrt{b\pi} + 2\sqrt{b\pi}s_2(\sqrt{2b}) \right]$$

for all $\delta < \frac{3b^2}{8} \wedge \frac{b^2}{8|p|}$ (then $|\tilde{p}| \leq 1$ and $b \geq \kappa \geq \frac{b}{2}$). We now calculate $m_{1,Z}$ needed for obtaining constant C_2 in (3.19) explicit. Constant $m_{1,T}$ is given in (5.50). Note that the Laplace transform of Z equals:

$$L^Z(s) = \mathbb{E}^X_0 e^{-sZ} = \mathbb{E}^{\tilde{Q}}_{-\tilde{p}_0} e^{-\tilde{s}\tilde{T}} = \\ = \mathbb{E}^{\tilde{Q}}_{-\tilde{p}_0} e^{-\tilde{s}\tilde{S}_0} \left(\mathbb{E}^{\tilde{Q}}_{-\tilde{p}_0-1} e^{-\tilde{s}\tilde{T}_0} \cdot \tilde{Q}(X(\tilde{S}_0) = -\tilde{p}_0 - 1) + \mathbb{E}^{\tilde{Q}}_{-\tilde{p}_0+1} e^{-\tilde{s}\tilde{T}_0} \cdot \tilde{Q}(X(\tilde{S}_0) = -\tilde{p}_0 + 1) \right) \ ,$$

where under the measure $\tilde{Q}$ the process $\{X(t), t \geq 0\}$ is the Ornstein-Uhlenbeck process with the parameter $\tilde{\kappa} = \sqrt{b^2 + 2s}$. Moreover, $\tilde{p}_0 = -\frac{2ps}{\tilde{\kappa}^2}$ and

$$\tilde{s} = \frac{\tilde{\kappa}}{2} - \frac{b}{2} - ps\tilde{p}_0 + sp^2 > 0$$

for sufficiently small s. From Lemmas 5.5 and 5.4 we have

$$\mathbb{E}^{\tilde{Q}}_{-\tilde{p}_0-1} e^{-\tilde{s}\tilde{T}_0} = \frac{e^{\tilde{p}_0^2\tilde{\kappa}/2} D_{-\frac{\tilde{s}}{\tilde{\kappa}}}(-\tilde{p}_0\sqrt{2\tilde{\kappa}})}{e^{(\tilde{p}_0+1)^2\tilde{\kappa}/2} D_{-\frac{\tilde{s}}{\tilde{\kappa}}}(-(\tilde{p}_0+1)\sqrt{2\tilde{\kappa}})} \ ,$$

$$\mathbb{E}^{\tilde{Q}}_{-\tilde{p}_0+1} e^{-\tilde{s}\tilde{T}_0} = \frac{e^{\tilde{p}_0^2\tilde{\kappa}/2} D_{-\frac{\tilde{s}}{\tilde{\kappa}}}(\tilde{p}_0\sqrt{2\tilde{\kappa}})}{e^{(1-\tilde{p}_0)^2\tilde{\kappa}/2} D_{-\frac{\tilde{s}}{\tilde{\kappa}}}(-(1-\tilde{p}_0)\sqrt{2\tilde{\kappa}})} \ ,$$

$$\mathbb{E}^{\tilde{Q}}_{-\tilde{p}_0} e^{-\tilde{s}\tilde{S}_0} = \frac{S(\frac{\tilde{s}}{\tilde{\kappa}}, (1-\tilde{p}_0)\sqrt{2\tilde{\kappa}}, -\tilde{p}_0\sqrt{2\tilde{\kappa}}) + S(\frac{\tilde{s}}{\tilde{\kappa}}, -\tilde{p}_0\sqrt{2\tilde{\kappa}}, -(\tilde{p}_0+1)\sqrt{2\tilde{\kappa}})}{S(\frac{\tilde{s}}{\tilde{\kappa}}, (1-\tilde{p}_0)\sqrt{2\tilde{\kappa}}, -(\tilde{p}_0+1)\sqrt{2\tilde{\kappa}})} \ .$$

Denote:

$$\mathrm{Erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{v^2}\, \mathrm{d}v, \qquad \mathrm{Erfid}(x,y) = \mathrm{Erfi}(\frac{x}{\sqrt{2}}) - \mathrm{Erfi}(\frac{y}{\sqrt{2}})$$

and

$$C(\nu, x, y) = \frac{\Gamma(\nu+1)}{\pi} e^{(x^2+y^2)/4} (D_{-\nu-1}(-x)D_{-\nu}(y) + D_{-\nu-1}(x)D_{-\nu}(-y)) \ .$$

By Borodin and Salminen (1996) the following holds:

$$\tilde{Q}(X(\tilde{S}_0) = -\tilde{p}_0 - 1) = 1 - \tilde{Q}(X(\tilde{S}_0) = -\tilde{p}_0 + 1) = \frac{\mathrm{Erfid}((1-\tilde{p}_0)\sqrt{2\tilde{\kappa}}, -\tilde{p}_0\sqrt{2\tilde{\kappa}})}{\mathrm{Erfid}((1-\tilde{p}_0)\sqrt{2\tilde{\kappa}}, -(\tilde{p}_0+1)\sqrt{2\tilde{\kappa}})} \ .$$

Note that:

$$\frac{d}{dx}D_{-\nu}(x) = -\frac{x}{2}D_{-\nu}(x) - \nu D_{-\nu-1}(x), \qquad \frac{d}{ds}D_{-s}(x) = e^{-x^2/4}\left[s_2(x) - \frac{x}{\sqrt{2\pi}}(1+s_1(x))\right]$$

and

$$\frac{\partial}{\partial x}S(\nu,x,y) = C(\nu,x,y), \qquad \frac{\partial}{\partial y}S(\nu,x,y) = -C(\nu,x,y) \ .$$

Hence

$$\begin{aligned} m_{1,Z} &= -\frac{d}{ds}(\mathbb{E}^X_0 e^{-sZ})_{|s=0^+} = -\frac{d}{ds}(\mathbb{E}^{\tilde{Q}}_{-\tilde{p}_0} e^{-\tilde{s}\tilde{S}_0})_{|s=0^+} - \\ &-\frac{1}{2}\left(\frac{d}{ds}(\mathbb{E}^{\tilde{Q}}_{-\tilde{p}_0-1}e^{-\tilde{s}\tilde{T}_0})_{|s=0^+} + \frac{d}{ds}(\mathbb{E}^{\tilde{Q}}_{-\tilde{p}_0+1}e^{-\tilde{s}\tilde{T}_0})_{|s=0^+}\right) = \\ &= \frac{\sqrt{2}}{4\sqrt{\pi}\mathrm{Erfi}(\sqrt{b})^2}\left\{(e^b+1)\left[2s_1(\sqrt{2b})\mathrm{Erfi}(\sqrt{b})(\frac{1}{2b^2}+\frac{p^2}{b}) + \frac{2}{\sqrt{\pi}}\frac{1}{b^{3/2}}e^b\right] + \right. \\ &\left. -\frac{2\sqrt{2}}{b^{3/2}}e^b\mathrm{Erfi}(\sqrt{b})\right\} + \frac{1}{b} + (\frac{1}{2b^2}+\frac{p^2}{b})\left[s_2(\sqrt{2b}) + \frac{\sqrt{b}}{\sqrt{\pi}}(1+s_1(\sqrt{2b}))\right] \ . \end{aligned} \tag{5.56}$$

Summarizing we have the following theorem.

Theorem 5.4. *Assume that $\{(X(t)+p)^2, t \geq 0\}$ is the intensity process for $p \in \mathbb{R}$ and for the Ornstein-Uhlenbeck process $\{X(t), t \geq 0\}$ with the parameter b starting at $X(0) = 0$. If the claim size has the regularly varying distribution (U) and (S) holds, then*

$$\psi(u) \sim \frac{1}{\alpha_U - 1} m_{1,Z} \frac{1}{m_{1,T} - m_{1,U}m_{1,Z}} \quad l_U(u)u^{-\alpha_U+1} \ ,$$

where $m_{1,T}$ and $m_{1,Z}$ are given in (5.50) and (5.56) respectively.

5.5 Splitting Brownian bridges and $\lambda(x) = |x|$

We construct the governing process $\{X(t), t \geq 0\}$ by splitting independent Brownian bridges defined on interval $[n, n+1]$ ($n \in \mathbb{N}$). That is, $X(t) = Z_n(t)$ if $t \in [n, n+1]$, where $\{Z_n(t), t \in [n, n+1]\}$ is a Brownian bridge (see Karatzas and Shreve (1988), p. 358, for construction of the Brownian bridge). We assume that processes $\{Z_n(t), t \geq 0\}$ are independent. Hence $T_n = n$ are moments of regeneration and $T = 1$. Let $\lambda(x) = |x|$. Thus on each interval $[n, n+1]$ the intensity process is the reflecting Brownian bridge. By Karatzas and Shreve (1988), p. 360,

$$Z_1(t) \stackrel{D}{=} B(t) - tB(1) \ . \tag{5.57}$$

Thus

$$Z = \int_0^1 |X(t)| \ \mathrm{d}t = \int_0^1 |Z_1(t)| \ \mathrm{d}t \stackrel{D}{=} \int_0^1 |B(t) - tB(1)| \ \mathrm{d}t \ . \tag{5.58}$$

Note that

$$Z \le \int_0^1 (\sup_{0\le t\le 1} |B(t)| + t|B(1)|)\, \mathrm{d}t \le 2 \sup_{0\le t\le 1} |B(t)| \ . \tag{5.59}$$

Hence

$$\mathbb{P}(Z > x) \le \mathbb{P}(\sup_{0\le t\le 1} |B(t)| > \frac{x}{2}) \le 2\mathbb{P}(\sup_{t\in[0,1]} B(t) > \frac{x}{2}) = 4\frac{1}{\sqrt{2\pi}} \int_{x/2}^{+\infty} e^{-\frac{y^2}{2}}\, \mathrm{d}y$$

by Adler (1999) and Karatzas and Shreve (1988), p. 96. Thus condition (B) is fulfilled. Moreover, by the Fubbini Theorem:

$$\begin{aligned} m_{1,Z} &= \int_0^1 \left(\mathbb{E}_0^B |B(t) - tB(1)|\right)\, \mathrm{d}t = \\ &= \int_0^1 \frac{1}{2\pi\sqrt{1-t}} \int_{-\infty}^{+\infty} e^{-x^2/2} \int_{-\infty}^{+\infty} |x - ty| e^{-\frac{(x-y)^2}{2(1-t)}}\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}t \\ &= \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^1 \frac{1}{\sqrt{t^2 - t + 1}} (t^2 + t\sqrt{1-t})\, \mathrm{d}t = \\ &= \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{1}{2} - \frac{1}{8}\log(3) + \int_0^1 \frac{t\sqrt{1-t}}{\sqrt{t^2-t+1}}\, \mathrm{d}t\right] \ . \end{aligned}$$

Taking substitution $y^2 := 1 - t$ we get:

$$\begin{aligned} m_{1,Z} &= \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{1}{2} - \frac{1}{8}\log(3) - 2\int_0^1 \sqrt{y^4 - y^2 + 1}\, \mathrm{d}y + 2\int_0^1 \frac{\mathrm{d}y}{\sqrt{y^4 - y^2 + 1}}\right] = \\ &= \frac{\sqrt{2}}{\sqrt{\pi}} \left[\mathrm{EllipticK}(\frac{\sqrt{3}}{2}) - \frac{2\sqrt{2}}{3}\mathrm{EllipticF}(\frac{2}{3}\sqrt{2}, \frac{1}{4}\sqrt{10}) - \frac{1}{6} - \frac{1}{8}\log(3)\right] \simeq 0.79788 \ , \end{aligned} \tag{5.60}$$

where

$$\mathrm{EllipticF}(z, k) = \int_0^z \frac{1}{\sqrt{1-y^2}\sqrt{1-k^2y^2}}\, \mathrm{d}y$$

is the incomplete integral of the first kind and $\mathrm{EllipticK}(k) = \mathrm{EllipticF}(1, k)$ is the complete elliptic integral of the first kind (see Abramowitz and Stegun (1972), Chapter 17). By Corollary 3.1 we have the following theorem.

Theorem 5.5. *Assume that the intensity process is constructed by splitting reflecting Brownian bridges. If the claim size has the regularly varying distribution (U) with* $\alpha_U > 1$ *and* $m_{1,Z} m_{1,U} < 1$*, then*

$$\psi(u) \sim \frac{1}{\alpha_U - 1} m_{1,Z} \frac{1}{1 - m_{1,U} m_{1,Z}} \quad l_U(u) u^{-\alpha_U + 1} \ ,$$

where $m_{1,Z}$ *is given in (5.60).*

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