Recurrence and Transience Criteria for Directed-edge-reinforced Random Walk and Random Walk in Random Environment on Some Tube-like Graphs

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Abstract

We introduce directed-edge-reinforced random walk and prove that the process is equivalent to a random walk in random environment. Using Oseledec's multiplicative ergodic theorem, we obtain recurrence and transience criteria for random walk in random environment on graphs with a certain linear structure and apply them to directed-edge-reinforced random walk.

1 Introduction

Let G be a finite or infinite connected locally finite graph with all edges directed. We introduce *directed-edge-reinforced random walk* (DRRW) on G as follows. Each edge is given a strictly positive real number as initial weight. In each step the random walker traverses a directed edge pointing from her current location to an adjacent vertex with probability proportional to the weight of the edge chosen. Each time an edge is traversed, its weight is increased by 1.

We prove that DRRW is equivalent to a random walk in random environment (RWRE) with independent environment (Theorem 4). If all initial values are equal to 1, then the transition probabilities at vertex v are distributed according to a uniform distribution on the d-dimensional simplex with d equal to the out degree of v; transition probabilities at different vertices are independent. In case of general initial values the transition probabilities have a Dirichlet distribution.

We are interested in the question on which graphs DRRW is recurrent. It follows immediately from Solomon's criterion for RWRE [Sol75] that DRRW on

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$\mathbb{Z}$ with all initial values equal to 1 is recurrent. For the two-dimensional integer lattice the recurrence question seems to be open.

In this article we study the recurrence problem for DRRW and RWRE on $\mathbb{Z} \times G$ with G a finite connected graph. We assume that the transition probabilities of the RWRE at the different levels $\{i\} \times G$, $i \in \mathbb{Z}$, are i.i.d.. Using Oseledec's multiplicative ergodic theorem, we find necessary and sufficient conditions for recurrence and transience of RWRE in terms of the Lyapunov exponents of certain random matrices. This approach is similar to the one in [Key84] where RWRE on $\mathbb{Z}$ with jumps of bounded size is studied. We prove that RWRE is recurrent if the transition probabilities have a certain symmetry property. In particular, we obtain recurrence of DRRW with all initial values equal.

After finishing this paper, we learned that Bolthausen and Goldscheid [BG00] have a characterization of recurrence and transience of RWRE on $\mathbb{Z} \times G$ in terms of the top Lyapunov exponent of certain non-negative random matrices. It seems that Corollaries 1 to 4 can be proved using their results. However they do not discuss reinforced random walks, and the random matrices we consider are more intuitively defined.

Every graph can be turned into a directed graph by replacing each edge by two directed edges with opposite directions. DRRW on this graph differs from so called edge-reinforced random walk which has been introduced by Coppersmith and Diaconis in 1987. Edge-reinforced random walk (ERRW) is a nearest neighbour random walk on a non-directed graph. Each edge has a strictly positive number as a weight. Each time an edge is traversed, its weight is increased by 1, independent of the direction in which the edge is traversed. The random walker moves in each step to an adjacent vertex with a probability proportional to the weight of the traversed edge.

The first time a vertex v is visited, the probabilities to leave vertex v depend in case of ERRW on the edge that has been traversed to reach v, whereas they do not depend on this edge for DRRW. The small difference in the definition of ERRW and DRRW results in a significant difference of the processes: On finite graphs for example, ERRW is equivalent to a reversible RWRE with dependent environment [KR00], whereas DRRW is equivalent to a non-reversible RWRE with independent environment.

For ERRW not much is known about recurrence. It is easy to show that ERRW on $\mathbb{Z}$ is recurrent, but even for $\mathbb{Z} \times \{1, 2\}$ we do not know of any recurrence proof.

The exposition is organized as follows: In Section 2, we define DRRW and RWRE on a general graph, and state our results. In Section 3, we study the potential equations on $\mathbb{Z} \times G$. They can be written in terms of products of random matrices. In Section 4, we use Oseledec's multiplicative ergodic theorem to obtain an abstract characterization of recurrence and transience for RWRE. We obtain also a sufficient criterion for recurrence which is easy to verify. In Section 5, we prove that DRRW is equivalent to a RWRE, and apply the criteria from section 4 to DRRW.

2 Definitions and Results

Let $G = (V, E)$ be a connected graph with vertex set V and edge set E. We assume that each vertex has only finitely many neighbours and all edges are directed. Between two vertices u and v with $u \neq v$ there may be two parallel edges, one from u to v and one from v to u. For simplicity of notation, we do not allow parallel edges with the same direction. All proofs remain valid without this assumption. For a directed edge $e = (u, v)$ from vertex u to vertex v, we call $\hat{e} := v$ the *head* of e and $\check{e} := u$ the *tail* of e. We do not require that head and tail of an edge are different.

Each edge is given a strictly positive weight. At time 0 the weights are non-random; edge e has weight $a(e)$. We denote by $w_n(e)$ the weight of edge e at time n (just after the n^{th} step) and by $w_n(v)$ the sum of the weights of the edges incident to vertex v.

Let $v_0 \in V$. We define *directed-edge-reinforced random walk* with starting point v_0 to be a sequence $X = (X_0, X_1, X_2, \dots)$ with X_i taking values in V, $P(X_0 = v_0) = 1$ and

$$P(X_{n+1} = v | X_0, X_1, \dots, X_n) = \begin{cases} \dfrac{w_n(X_n, v)}{w_n(X_n)} & \text{if } (X_n, v) \in E \\ 0 & \text{otherwise,} \end{cases}$$

and the weights satisfy $w_0(e) = a(e)$,

$$w_{n+1}(e) = \begin{cases} w_n(e) + 1 & \text{if } (X_n, X_{n+1}) = e \\ w_n(e) & \text{otherwise.} \end{cases}$$

Next we define *random walk in random environment* (RWRE) on G. An *environment* is a function $\omega : E \to [0, 1]$ with the property that $\omega(e) > 0$ for all $e \in E$ and

$$\sum_{\{v \in V : (u,v) \in E\}} \omega(u, v) = 1$$

for all $u \in V$. We denote the set of all environments by Ω. Let $P_{v_0,\omega}$ denote the distribution of the Markov chain on G induced by the environment ω with starting point v_0: $P_{v_0,\omega}(Y_0 = v_0) = 1$ and

$$P_{v_0,\omega}(Y_{n+1} = v | Y_0, Y_1, \dots, Y_n) = \begin{cases} \omega(Y_n, v) & \text{if } (Y_n, v) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{P}$ be a probability measure on Ω. The measure

$$P_{v_0}(\cdot) := \int_\Omega P_{v_0,\omega}(\cdot) \mathbb{P}(d\omega)$$

is the distribution of the *random walk in random environment* with environment distributed according to $\mathbb{P}$ and starting point v_0. We call the environment *independent* if under $\mathbb{P}$ the transition probabilities $\{\omega(v, \cdot); v \in V\}$ are independent.

Definition 1 *We call a sequence* $(u_0, u_1, u_2, \dots)$ *with* $n \geq 1$, $u_i \in V$ *and* $(u_{i-1}, u_i) \in E$ *for all* $i \geq 1$ *an* infinite path *with* starting point u_0. *We call an infinite path* recurrent *if it contains each vertex infinitely often and* transient *if it contains each vertex at most finitely often. We call DRRW or RWRE on a graph* G recurrent *(*transient*) if with probability one the paths are recurrent (transient).*

Proposition 1 *RWRE with independent environment is either recurrent or transient.*

Proof. This follows from the same arguments as Lemma 1 in [Kal81], page 761. □

Let $G = (V, E)$ be a finite connected graph with all edges directed. We study recurrence of DRRW and RWRE on $G' = \mathbb{Z} \times G$. More precisely, $G' = (V', E')$ with $V' = \mathbb{Z} \times V$ and

$$E' = \{(u_i, v_i) : (u, v) \in E\} \cup \{(v_i, v_{i-1}), (v_i, v_{i+1}) : v \in V, i \in \mathbb{Z}\}$$

with $v_i = (i, v)$ for $v \in V$. We set $V_i = \{i\} \times V$ and define the length of v_i by $|v_i| = i$. We call G' a *tube* and V_i *level* i of the tube.

Let ω be an environment on $\mathbb{Z} \times G$, and let $\omega_i = (\omega(e) : \check{e} \in V_i, e \in E')$ denote the transition probabilities to leave level i. For the rest of this article we make the following **assumption on the environment:** ω_i, $i \in \mathbb{Z}$, are independent and identically distributed with

$$\int_\Omega \log \omega(e) \mathbb{P}(d\omega) > -\infty \quad \text{for all } e \in E'. \tag{1}$$

We apply Oseledec's multiplicative ergodic theorem to products of random matrices A_i describing the potential equations for the environment. This yields to the following characterization of recurrence and transience in terms of Lyapunov exponents:

Theorem 1 *Let* $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2d}$ *be the Lyapunov exponents of the sequence* A_i, $i \geq 0$, *defined in Section 3 by (4). The RWRE on* $\mathbb{Z} \times G$

1. *is recurrent iff* $\lambda_d = \lambda_{d+1} = 0$,
2. *satisfies* $\lim_{n\to\infty} |Y_n| = -\infty$ *iff* $\lambda_d = 0$ *and* $\lambda_{d+1} > 0$,
3. *satisfies* $\lim_{n\to\infty} |Y_n| = +\infty$ *iff* $\lambda_d < 0$ *and* $\lambda_{d+1} = 0$.

In general it seems impossible to calculate the sign of $\lambda_d + \lambda_{d+1}$. If G has only one vertex, all the Lyapunov exponents can be calculated, and we obtain Solomon's criterion [Sol75]:

Corollary 1 *Let* $q(\omega) = \frac{\omega(v_1, v_0)}{\omega(v_1, v_2)}$. *The RWRE on* $\mathbb{Z}$

1. *is recurrent if* $\mathbb{E} \log q = 0$,

2. *satisfies* $\lim_{n\to\infty}|Y_n| = -\infty$ *if* $\mathbb{E}\log q > 0$,

3. *satisfies* $\lim_{n\to\infty}|Y_n| = +\infty$ *if* $\mathbb{E}\log q < 0$.

We define the *reflected environment* $\widetilde{\omega}$ by

$$\begin{aligned}\widetilde{\omega}(e) &= \omega(e), \text{ for } e \in E' \text{ with } \check{e}, \hat{e} \in V_i \text{ for some } i,\\ \widetilde{\omega}(v_i, v_{i-1}) &= \omega(v_i, v_{i+1}),\\ \widetilde{\omega}(v_i, v_{i+1}) &= \omega(v_i, v_{i-1}), \text{ for } v \in V, i \in \mathbb{Z}.\end{aligned}$$

$\widetilde{\omega}$ is obtained from ω by interchanging for all i, the probability to jump from v_i to v_{i-1} and the probability to jump from v_i to v_{i+1}. We use Theorem 1 to derive the following sufficient criteron for recurrence:

Corollary 2 *If* $\mathbb{P}$ *is invariant under the reflection* $\widetilde{\ }$, *then RWRE on* $\mathbb{Z} \times G$ *is recurrent.*

Corollary 1 and Corollary 2 can be applied to DRRW. Recall that $a(e)$ denotes the initial weight of edge e.

Corollary 3 *Let* $a, b > 0$, *and let* $a(i, i-1) = a$, $a(i, i+1) = b$ *for all* $i \in \mathbb{Z}$. *DRRW on* $\mathbb{Z}$

1. *is recurrent if* $a = b$,

2. *satisfies* $\lim_{n\to\infty}|X_n| = -\infty$ *if* $a > b$,

3. *satisfies* $\lim_{n\to\infty}|X_n| = +\infty$ *if* $a < b$.

Corollary 4 *Let* $a_v > 0$, $v \in V$, *and let* $a(v_i, v_{i-1}) = a(v_i, v_{i+1}) = a_v$ *for all* $v \in V, i \in \mathbb{Z}$. *Then DRRW on* $\mathbb{Z} \times G$ *is recurrent.*

3 Potential Equations

In this section, we study the potential equations on $\mathbb{Z} \times G$. Let ω be an environment on $\mathbb{Z} \times G$. The potential equations for ω are given by:

$$\begin{aligned}x_{v_i}(\omega) &= \omega(v_i, v_{i-1}) x_{v_{i-1}}(\omega) + \omega(v_i, v_{i+1}) x_{v_{i+1}}(\omega)\\ &\quad + \sum_{\{u \in V : (v,u) \in E\}} \omega(v_i, u_i) x_{u_i}(\omega). \qquad (2)\end{aligned}$$

for $i \in \mathbb{Z}$, $v \in V$. These equations are for example satisfied for $i \geq 1$ and $x_{v_i}(\omega)$ equal to the probability of never reaching level 0 under the law $P_{v_i,\omega}$.

Let $d = |V|$ be the cardinality of the vertex set of G. We denote the $d \times d$ identity matrix by I_d and the $d \times d$ zero matrix by 0_d. We define the $d \times d$ matrix $B_i(\omega) = (B_i(\omega)(u, v); u, v \in V)$ by

$$\begin{aligned}B_i(\omega)(v, v) &= \frac{1}{\omega(v_i, v_{i+1})}, \; v \in V,\\ B_i(\omega)(v, u) &= -\frac{\omega(v_i, u_i)}{\omega(v_i, v_{i+1})}, \text{ if } (u, v) \in E,\end{aligned}$$

and $B_i(\omega)(u,v)=0$ for all other choices of u,v. We denote by $C_i(\omega)$ the $d\times d$ diagonal matrix with

$$C_i(\omega)(v,v) = -\frac{\omega(v_i,v_{i-1})}{\omega(v_i,v_{i+1})},\ v\in V,$$

and $C_i(\omega)(u,v)=0$ if $u\neq v$. The potential equations (2) can be rewritten as

$$\mathbf{x}_{i+1}(\omega) = A_i(\omega)\mathbf{x}_i(\omega) \tag{3}$$

with

$$A_i(\omega) = \begin{pmatrix} B_i(\omega) & C_i(\omega) \\ I_d & 0_d \end{pmatrix}, \tag{4}$$

$$\mathbf{x}_i(\omega) = \left((x_{v_i}(\omega); v\in V), (x_{v_{i-1}}(\omega); v\in V)\right)^t, \tag{5}$$

where we denote the transpose of a matrix M by M^t. Iterating equation (3) gives for $n\geq 1$

$$\mathbf{x}_{n+1}(\omega) = S_n(\omega)\mathbf{x}_1(\omega) \quad \text{with } S_n(\omega) = A_n(\omega)A_{n-1}(\omega)\cdots A_1(\omega). \tag{6}$$

Lemma 1 *For any $\omega\in\Omega$, the matrix $A_i(\omega)$ is invertible. Its inverse is given by*

$$A_i^{-1}(\omega) = U\widetilde{A}_i(\omega)U$$

where

$$U = \begin{pmatrix} 0_d & I_d \\ I_d & 0_d \end{pmatrix}, U^2 = I_{2d},$$

and $\widetilde{A}_i(\omega) = A_i(\widetilde{\omega})$.

Proof. This is an easy calculation. □

4 Characterizations of Recurrence and Transience

For our further analysis we need the multiplicative ergodic theorem of Oseledec. For $b\in\mathbb{R}^r$ we denote by $||b||$ the Euclidean norm of b. For an $(r\times r)$-matrix M we use the norm

$$||M|| = \sup\{||Mb|| : b\in\mathbb{R}^r, ||b||\leq 1\}.$$

Theorem 2 (Oseledec's multiplicative ergodic theorem[Ose68]) *Let $M_i, i\geq 0$, be independent and identically distributed real-valued $(r\times r)$-matrices on some probability space $(\Omega_o,\mathcal{F},\mu)$, and suppose $\log^+||M_1||$ is integrable, where $\log^+ x = \max\{0,\log x\}$. Let $T_n = M_nM_{n-1}\cdots M_1$. Then there exist constants*

$$-\infty\leq\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_r<\infty$$

and a strictly increasing non-random sequence of integers $1=i_1<i_2<\cdots<i_s<i_{s+1}=r+1$ satisfying $\lambda_{i_1}<\lambda_{i_2}<\cdots<\lambda_{i_s}$ and $\lambda_{i_j}=\lambda_k$ for $i_j\leq k<i_{j+1}$, $1\leq j\leq s$ such that for μ-almost all $\omega\in\Omega_o$ the following is true:

1. *For every* $b \in \mathbb{R}^r$, $\lim_{n\to\infty} n^{-1} \log ||T_n(\omega)b||$ *exists or is* $-\infty$.

2. *For every* $j \le s$,

$$W(j,\omega) := \left\{ b \in \mathbb{R}^r : \lim_{n\to\infty} n^{-1} \log ||T_n(\omega)b|| \le \lambda_{i_j} \right\}$$

   is a random linear subspace of $\mathbb{R}^r$ *with dimension* $i_{j+1} - 1$.

3. *If* $W(0,\omega) = \{\mathbf{0}\}$, *then* $b \in W(j,\omega) \setminus W(j-1,\omega)$ *implies*

$$\lim_{n\to\infty} n^{-1} \log ||T_n(\omega)b|| = \lambda_{i_j}.$$

4. $\lim_{n\to\infty} (T_n^t(\omega)T_n(\omega))^{1/2n} =: \Lambda(\omega)$ *exists and all entries of* $\Lambda(\omega)$ *are finite. The eigenvalues of* $\Lambda(\omega)$ *are* $\exp(\lambda_i)$, $1 \le i \le r$. *For every* $j \le s$, *the orthogonal complement of* $W(j-1,\omega)$ *in* $W(j,\omega)$ *is the eigenspace of* $\Lambda(\omega)$ *corresponding to the eigenvalue* $\exp(\lambda_{i_j})$.

The λ_{i_j} *are called* Lyapunov exponents.

The following identity will be useful to calculate the Lyapunov exponents in a special case.

Lemma 2 *Suppose* $|\det M_1| > 0$ μ*-almost surely and the expected value* $c := \int_{\Omega_0} \log|\det M_1(\omega)|\mu(d\omega)$ *is finite. Then* $\sum_{i=1}^r \lambda_i = c$.

Proof. Let $\Lambda_n(\omega) = (T_n^t(\omega)T_n(\omega))^{1/2n}$. Then $\det \Lambda_n = \left(\prod_{i=1}^n |\det M_i|\right)^{1/n}$ and consequently,

$$\log(\det \Lambda_n) = n^{-1} \sum_{i=1}^n \log|\det M_i|.$$

By Oseledec's theorem the left-hand-side converges to $\log(\det \Lambda) = \sum_{i=1}^r \lambda_i$. The right-hand-side converges to c by Birkhoff's ergodic theorem. □

For a proof of the following lemma see for example [CMP98], Lemma 4.1.

Lemma 3 *Let* M_i, $i \ge 0$, *be independent and identically distributed real-valued* $(r \times r)$*-matrices. Suppose* $M_1(\omega)$ *is invertible for almost all* ω, *and* $log^+||M_1||$ *and* $log^+ \|M_1^{-1}\|$ *are integrable. Let* $\lambda_1, \lambda_2, \dots, \lambda_{2d}$ *be the Lyapunov exponents of the sequence* $M_i, i \ge 0$. *Then the Lyapunov exponents of the sequence* M_i^{-1}, $i \ge 0$, *are given by* $-\lambda_1, -\lambda_2, \dots, -\lambda_{2d}$.

We want to apply Oseledec's Theorem and Lemma 3 to the matrices A_i and $\widetilde{A}_i$ defined in Section 3.

Lemma 4 *The following expected values are finite:* $\mathbb{E}\log^+ ||A_1||$, $\mathbb{E}\log^+ \|A_1^{-1}\|$, $\mathbb{E}\log^+ \|\widetilde{A}_1\|$, $\mathbb{E}\log^+ \|(\widetilde{A}_1)^{-1}\|$.

Proof. Since $||A_1|| \geq 1$, $\log^+ ||A_1|| = \log ||A_1||$. For simplicity, we write A instead of A_1 for the rest of this proof, and we denote the entries of A by A_{ij}, $1 \leq i,j \leq 2d$. Observe that for any matrix $||A|| \leq (2d)^2 \max_{1\leq i,j\leq 2d} |A_{ij}|$ and therefore

$$\log ||A|| \leq 2\log(2d) + \sum_{\{1\leq i,j\leq 2d: A_{ij}\neq 0\}} \log |A_{ij}|.$$

Using (1), it follows that $\mathbb{E}\log |A_{ij}|$ is finite for all non-zero entries of A. Hence $\mathbb{E}\log^+ ||A_1|| < \infty$, and the same argument applies to $\widetilde{A}_1$. Together with Lemma 1, the assertions about the inverses follow. □

Next we determine the sign structure of the Lyapunov exponents for the sequence A_i.

Lemma 5 *Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2d}$ be the Lyapunov exponents of the sequence A_i, $i \geq 0$, and let $\widetilde{\lambda}_1 \leq \widetilde{\lambda}_2 \leq \ldots \leq \widetilde{\lambda}_{2d}$ be the Lyapunov exponents of the sequence $\widetilde{A}_i$, $i \geq 0$. Then for $1 \leq i \leq 2d$*

$$\widetilde{\lambda}_i = -\lambda_{2d-i+1}.$$

Proof. By Lemma 1,

$$T_n := A_n^{-1}A_{n-1}^{-1}\cdots A_1^{-1} = U\widetilde{A}_nUU\widetilde{A}_{n-1}U\cdots U\widetilde{A}_1U = U\widetilde{S}_nU$$

with $\widetilde{S}_n = \widetilde{A}_n\widetilde{A}_{n-1}\cdots\widetilde{A}_1$. Therefore for $b \in \mathbb{R}^r$, $||T_nb|| = ||\widetilde{S}_nUb||$ and $\lim_{n\to\infty} n^{-1}\log ||T_nb|| = \lim_{n\to\infty} n^{-1}\log ||\widetilde{S}_nUb||$. By Oseledec's Theorem, the Lyapunov exponents of the sequences A_i^{-1}, $i \geq 0$ and $\widetilde{A}_i$, $i \geq 0$ agree. The claim follows from Lemma 3. (The integrability assumptions from Oseledec's Theorem and Lemma 3 are satisfied by Lemma 4.) □

Theorem 3 *The Lyapunov exponents $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2d}$ of the sequence A_i, $i \geq 0$, satisfy one of the following conditions:*

1. $\lambda_d < 0$ *and* $\lambda_{d+1} = 0$
2. $\lambda_d = 0$ *and* $\lambda_{d+1} = 0$
3. $\lambda_d = 0$ *and* $\lambda_{d+1} > 0$

Proof. We denote by τ_0 the hitting time of level 0:

$$\tau_0 = \min\{n \geq 0 : X_n \in V_0\}.$$

For $u, v \in V$ and $i \in \mathbb{Z}$ we set

$$x^u_{v_i}(\omega) = P_{v_i,\omega}(\tau_0 < \infty, X_{\tau_0} = u_0).$$

By the Markov property, $x^u_{v_i}(\omega)$ satisfies the potential equations (2) for $i \geq 1$. From (6) we know that $\mathbf{x}^u_{n+1}(\omega) = S_n(\omega)\mathbf{x}^u_1(\omega)$ with $\mathbf{x}^u_n(\omega)$ defined by

(5). Since all components of $\mathbf{x}_{n+1}^u(\omega)$ have values in $[0,1]$, it follows that $\lim_{n\to\infty} n^{-1} \log \left\| S_n(\omega)\mathbf{x}_1^u(\omega)\right\| \leq 0$. Observe that $x_{u_0}^u(\omega) = 1$ and $x_{v_0}^u(\omega) = 0$ for all $v \in V \setminus \{u\}$. Hence the vectors $\mathbf{x}_1^u(\omega)$, $u \in V$, are linearly independent, and the linear space

$$W(\omega) := \left\{ b \in \mathbb{R}^{2d} : \lim_{n\to\infty} n^{-1} \log ||S_n(\omega) b|| \leq 0 \right\}$$

has dimension $\dim(W(\omega)) \geq d$. Thus $\lambda_d \leq 0$.

Recall that the matrix $\widetilde{A}_i$ is obtained from A_i by reflection of the environment. Therefore the previous argument applies to $\widetilde{A}_i$, $i \geq 0$, and the d^{th} Lyapunov exponent satisfies $\widetilde{\lambda}_d \leq 0$. By Lemma 5, $\lambda_{d+1} = -\widetilde{\lambda}_d \geq 0$.

Let $\mathbf{1}$ denote the vector in $\mathbb{R}^{2d}$ with all components equal to 1. Since $A_i \mathbf{1} = \mathbf{1}$ for all i, $S_n \mathbf{1} = \mathbf{1}$ for all n and $\lim_{n\to\infty} n^{-1} \log ||S_n \mathbf{1}|| = 0$. Thus either $\lambda_d = 0$ or $\lambda_{d+1} = 0$. $\square$

The sign structure of the Lyapunov exponents can be used to characterize recurrence and transience of the RWRE. We will make use of the following lemma:

Lemma 6 *RWRE on $\mathbb{N}_0 \times G$ is transient iff there exists $c > 0$ such that*

$$\lim_{n\to\infty} n^{-1} \log \left[\max_{v\in V} P_{v_n,\omega}(\tau_0 < \infty, X_{\tau_0} = u_0) \right] \leq -c$$

for all $u \in V$, $\mathbb{P}$-almost all ω.

Proof. We denote by τ_i the hitting time of level i:

$$\tau_i = \min\{n \geq 0 : X_n \in V_i\}.$$

First we consider a fixed environment ω. If level 0 is reached from a starting point at level n, then level $n-1$ must be reached before level 0. Hence

$$P_{v_n,\omega}(\tau_0 < \infty, X_{\tau_0} = u_0) = P_{v_n,\omega}(\tau_{n-1} < \infty, \tau_0 < \infty, X_{\tau_0} = u_0).$$

An application of the Markov property at time τ_{n-1} shows that the last probability equals

$$\begin{aligned} & E_{v_n,\omega}(P_{X_{\tau_{n-1}},\omega}(\tau_0 < \infty, X_{\tau_0} = u_0); \tau_{n-1} < \infty) \\ \leq \quad & P_{v_n,\omega}(\tau_{n-1} < \infty) \max_{t\in V} P_{t_{n-1},\omega}(\tau_0 < \infty, X_{\tau_0} = u_0). \end{aligned}$$

Repeating this argument yields

$$\max_{v\in V} P_{v_n,\omega}(\tau_0 < \infty, X_{\tau_0} = u_0) \leq \prod_{i=0}^{n-1} \max_{v\in V} P_{v_{i+1},\omega}(\tau_i < \infty).$$

We conclude

$$n^{-1} \log \left[\max_{v\in V} P_{v_n,\omega}(\tau_0 < \infty, X_{\tau_0} = u_0) \right] \leq n^{-1} \sum_{i=0}^{n-1} \log(Z_i(\omega)) \tag{7}$$

with $Z_i(\omega) = \max_{v\in V} P_{v_{i+1},\omega}(\tau_i < \infty)$. Observe that Z_i, $i \geq 0$, is a stationary sequence under $\mathbb{P}$ with $\mathbb{E}\log(Z_1) \leq 0$. By Birkhoff's ergodic theorem, $n^{-1}\sum_{i=0}^{n-1}\log(Z_i)$ converges almost surely as $n\to\infty$ to a limit U with $\mathbb{E}(U) = \mathbb{E}[\log(Z_1)]$.

Suppose $\mathbb{E}\log(Z_1) = 0$. Then $Z_1 = 1$ $\mathbb{P}$-almost surely. If for some v, $P_{v_1,\omega}(\tau_0 < \infty) < 1$, then $P_{v_1,\omega}(\tau_0 < \infty) < 1$ for all v. Consequently, $P_{v_1,\omega}(\tau_0 < \infty) = 1$ for all v $\mathbb{P}$-almost surely, and similarly $P_{v_{i+1},\omega}(\tau_i < \infty) = 1$ for all v and $i \geq 0$ $\mathbb{P}$-almost surely. We conclude that in this case, the RWRE on $\mathbb{N}_0 \times G$ is recurrent.

Suppose $\mathbb{E}\log(Z_1) < 0$. Then there exists $c > 0$ such that on a set of positive $\mathbb{P}$-measure $\lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1}\log(Z_i) \leq -c$. Since $x^u_{v_i} = P_{v_i,\omega}(\tau_0 < \infty, X_{\tau_0} = u_0)$ satisfies the potential equations, we conclude from Oseledec's theorem that the limit of the left-hand-side of (7) is $\leq -c$ $\mathbb{P}$-almost surely. In particular, $\lim_{n\to\infty}\max_{v\in V} P_{v_n,\omega}(\tau_0 < \infty, X_{\tau_0} = u_0) = 0$ and the RWRE is transient. □

Lemma 7 *RWRE on $\mathbb{N}_0 \times G$ is transient iff $\lambda_d < 0$.*

Proof. Suppose RWRE on $\mathbb{N}_0 \times G$ is transient. Then by Lemma 6,

$$W(\omega) := \left\{ b \in \mathbb{R}^{2d} : \lim_{n\to\infty} n^{-1}\log ||S_n(\omega)b|| < 0 \right\}$$

has dimension $\dim(W(\omega)) \geq d$. By Oseledec's Theorem, $\lambda_d < 0$.

Suppose $\lambda_d < 0$. Then by Theorem 3, $\dim(W(\omega)) = d$. For the rest of this proof we fix a typical environment ω and suppress the dependence on ω in the notation. Let $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_d$ be a basis of W. For $1 \leq j \leq d$, we let $\mathbf{c}_i$ denote the vector in $\mathbb{R}^d$ consisting of the last d components of $\mathbf{b}_i$. We claim that $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_d$ are linearly independent. Suppose not. Then there exists a non-zero vector $\mathbf{f}_1 \in W$ with the last d components equal to 0. We set $\mathbf{f}_{n+1} = S_n\mathbf{f}_1$ with $\mathbf{f}_n = ((f_{v_n}, v \in V), (f_{v_{n-1}}, v \in V))^t$, and we define the function $f : \mathbb{N}_0 \times G \to \mathbb{R}, v_i \mapsto f_{v_i}$. We say that the function f is *induced* by $\mathbf{f}_1$. By definition, f_{v_i} satisfies the potential equations (2) for $i \geq 1$ and $f_{v_0} = 0$ for $v \in V$. Since $\mathbf{f}_1 \in W$, $\lim_{n\to\infty} f_{v_n} = 0$ for $v \in V$. In particular, v is bounded and attains its maximum in $\mathbb{N} \times G$. By a standard argument, we conclude that f is identically zero. This is a contradiction to $\mathbf{f}_1 \neq \mathbf{0}$, and we conclude that $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_d$ are linearly independent.

Hence there exists a vector $\mathbf{g}_1 \in W$ with the last d components equal to 1. We denote by g the function on $\mathbb{N}_0 \times G$ induced by $\mathbf{g}_1$. We set

$$f(v) = P_{v,\omega}(\tau_0 < \infty)$$

for $v \in \mathbb{N}_0 \times G$. Then $f(v_i)$ and $g(v_i)$ satisfy the potential equations (2) for $i \geq 1$.

Fix $v_i \in \mathbb{N} \times G$. Let $N_n = f(Y_{n\wedge\tau_0})$, $n \geq 0$, with $n \wedge \tau_0 = \min\{n, \tau_0\}$. Then $N_n, n \geq 0$, is a bounded martingale under $P_{v_i,\omega}$. Hence it converges in L^1 to a limit N_∞, and

$$\begin{aligned} f(v_i) &= E_{v_i,\omega}N_0 = E_{v_i,\omega}N_\infty \\ &= E_{v_i,\omega}\left(\lim_{n\to\infty} f(Y_{n\wedge\tau_0}); \tau_0 < \infty\right) + E_{v_i,\omega}(N_\infty; \tau_0 = \infty). \end{aligned}$$

Using $f(Y_{\tau_0}) = 1$, we obtain

$$f(v_i) = f(v_i) + E_{v_i,\omega}(N_\infty; \tau_0 = \infty).$$

Thus $E_{v_i,\omega}(N_\infty; \tau_0 = \infty) = 0$. We set $N'_n = f(Y_{n\wedge\tau_0}) - g(Y_{n\wedge\tau_0})$, $n \geq 0$. Then $N'_n, n \geq 0$, is a bounded martingale under $P_{v_i,\omega}$. Hence it converges in L^1 to a limit N'_∞, and

$$\begin{aligned} f(v_i) - g(v_i) &= E_{v,\omega} N'_0 = E_{v_i,\omega} N'_\infty = E_{v_i,\omega}\left(f(Y_{\tau_0}) - g(Y_{\tau_0}); \tau_0 < \infty\right) \\ &\quad + E_{v_i,\omega}(N_\infty; \tau_0 = \infty) - E_{v_i,\omega}\left(\lim_{n\to\infty} g(Y_{n\wedge\tau_0}); \tau_0 = \infty\right) \\ &= -E_{v_i,\omega}\left(\lim_{n\to\infty} g(Y_{n\wedge\tau_0}); \tau_0 = \infty\right), \end{aligned}$$

because $f(Y_{\tau_0}) = g(Y_{\tau_0}) = 1$. If $P_{v_i,\omega}(\tau_0 = \infty) > 0$, the Markov chain is transient and $\lim_{n\to\infty} |Y_n| = \infty$ $P_{v_i,\omega}$-almost surely. Since $\mathbf{g} \in W$, we have $\lim_{n\to\infty} g(u_n) = 0$ for $u \in V$, and we conclude $g(v_i) = f(v_i)$. Since $v_i \in \mathbb{N} \times G$ is arbitrary, we have shown $f = g$. Using $\mathbf{g}_1 \in W$ we conclude from Lemma 6 that the RWRE is transient. □

Proof of Theorem 1. Suppose $\lambda_d < 0$ and $\lambda_{d+1} = 0$. By Lemma 7, the RWRE restricted to $\mathbb{N}_0 \times G$ is transient, thus $\lim_{n\to\infty} |Y_n| = +\infty$. To apply Lemma 7 to the RWRE restricted to $-\mathbb{N}_0 \times G$, we consider the reflected environment on $\mathbb{N}_0 \times G$. For the potential equations, this means that A_i is replaced by $\widetilde{A}_i$. By Lemma 5, $\widetilde{\lambda}_d = -\lambda_{d+1} = 0$, hence by Lemma 7 the RWRE restricted to $-\mathbb{N}_0 \times G$ is recurrent. This implies that the RWRE on $\mathbb{Z} \times G$ satisfies $\lim_{n\to\infty} |Y_n| = +\infty$.

Suppose $\lambda_d = \lambda_{d+1} = 0$. Using the same argument as above, we see that RWRE restricted to $\mathbb{N}_0 \times G$ and $-\mathbb{N}_0 \times G$ are recurrent. Hence the RWRE on $\mathbb{Z} \times G$ is recurrent.

The case $\lambda_d = 0$ and $\lambda_{d+1} > 0$ is treated similarly. □

Proof of Corollary 1. We apply Theorem 1 with G equal to the graph with precisely one vertex. Then

$$A_1(\omega) = \begin{pmatrix} \frac{1}{\omega(v_1,v_2)} & -\frac{\omega(v_1,v_0)}{\omega(v_1,v_2)} \\ 1 & 0 \end{pmatrix}.$$

Using Lemma 2, we obtain

$$\lambda_1 + \lambda_2 = \mathbb{E} \log |\det A_1| = \int_\Omega \log \left[\frac{\omega(v_1,v_0)}{\omega(v_1,v_2)}\right] \mathbb{P}(d\omega).$$

The statement follows from Theorem 1. □

Proof of Corollary 2. Suppose $\mathbb{P}$ is invariant under reflection. Then $\widetilde{A}_i$ has the same distribution as A_i and consequently both sequences have the same Lyapunov exponents. Using Lemma 5 and Theorem 3, we conclude that $\lambda_d = -\lambda_{d+1} = 0$. Theorem 1 implies recurrence of the RWRE. □

5 Application to DRRW

Recall the definition of directed-edge-reinforced random walk (DRRW) from Section 2. First we show that DRRW on a general graph G' is equivalent to a RWRE, then we apply Corollary 1 and Corollary 2.

We recall the definition of a *generalized Polya urn* with parameters $a_1, \dots, a_k > 0$. The urn contains "balls" of k different colours, a_i balls of colour i at time 0. At each time unit, a ball is drawn from the urn and returned with an additional ball of the same colour. The sequence of colours of the balls drawn from the urn is called a generalized Polya urn process with parameters $a_1, \dots, a_k$. Clearly, this process is well-defined for any strictly positive paramaters a_i although the analogy with balls makes only sense for integer-valued parameters.

For $a > 0$ we denote the value of the gamma function at a by $\Gamma(a)$. We recall the density of the *Dirichlet distribution* with parameters $a_1, \dots, a_k$:

$$D(a_1, \dots, a_k)(x_1, \dots, x_k) = \frac{\Gamma(a_1 + \dots + a_k)}{\prod_{i=1}^k \Gamma(a_i)} \prod_{i=1}^k x_i^{a_i - 1}, x_i > 0, \sum_{i=1}^k x_i = 1.$$

We attach to each vertex of the graph G' a Polya urn, urns at different vertices being independent. For a vertex v we denote by E'_v the set of edges in G' with tail v: $E'_v = \{e \in E' : \check{e} = v\}$. We assume that the parameters of the urn at vertex v are $(a(e), e \in E'_v)$, so the urn at vertex v contains balls of as many different types as there are edges with tail v. We define a nearest neighbour random walk on G' starting at v_0 as follows: If the random walker is at vertex v her next step is decided with the Polya urn at vertex v; the ball drawn from the urn determines which edge she traverses next. Writing down the finite-dimensional distributions for the location of the random walker, it is easy to see that they agree with the finite-dimensional distributions of the DRRW. Using de Finetti's theorem, each Polya urn can be replaced by a Dirichlet distribution with the same parameters ([MW92], Section 2).

We have proved the following theorem:

Theorem 4 *DRRW is equivalent to a RWRE with an independent environment. The transition probabilities at vertex v are distributed according to a Dirichlet distribution with parameters* $(a(e), e \in E'_v)$.

Proof of Corollary 3. By Theorem 4, DRRW on $\mathbb{Z}$ is equivalent to a RWRE with $\omega_i, i \in \mathbb{Z}$, independent and identically distributed. The transition probabilities at vertex v have a beta distribution with parameters a and b. To verify the integrability condition (1), let $\epsilon > 0$ such that $a - \epsilon > 0$. There exists $c > 0$ such that $\log(1/p) \leq cp^{-\epsilon}$ for all $p \in]0, 1]$. Hence

$$\begin{aligned} 0 &\leq \int_\Omega -\log \omega(e) \mathbb{P}(d\omega) = \int_0^1 (-\log p) D(a,b)(p, 1-p) dp \\ &\leq c \int_0^1 p^{-\epsilon} D(a,b)(p, 1-p) dp. \end{aligned}$$

The last integral is finite because the integrand is up to a constant the beta density $D(a-\epsilon, b)$. It remains to compute the sign of

$$I := \int_\Omega \log\left[\frac{\omega(v_1, v_0)}{\omega(v_1, v_2)}\right] \mathbb{P}(d\omega) = c\int_0^1 \log\left(\frac{p}{1-p}\right) p^{a-1}(1-p)^{b-1} dp$$

with c a normalizing constant. We write the last integral as a sum of two integrals, the first over $[0, 1/2]$, the second over $[1/2, 1]$. Substituting $q = 1-p$ in the first integral yields

$$I = c\int_{1/2}^1 \log\left(\frac{p}{1-p}\right) p^{a-1}(1-p)^{b-1}\left[1 - \left(\frac{p}{1-p}\right)^{b-a}\right] dp.$$

Since $p/(1-p) \geq 1$ for $p \in [1/2, 1]$, the statement follows from Corollary 1. □

Proof of Corollary 4. We apply Corollary 2. Using the assumption on the initial values and the symmetry property of the Dirichlet distribution, we see that the distribution of the environment is invariant under reflection. The other assumptions are verified similarly to the proof of Corollary 3. □

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