Asymptotics for Random Walks with Dependent Heavy-Tailed Increments

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Abstract

In this paper we consider a random walk $\{S_n\}$ with dependent heavy-tailed increments and negative drift. We study the asymptotics for the tail probability $\mathbb{P}(\sup_n S_n > x)$ as $x \to \infty$. If the increments of $\{S_n\}$ are independent, then the exact asymptotic behaviour of $\mathbb{P}(\sup_n S_n > x)$ is well-known. We investigate the case that the increments are given as a one-sided asymptotically stationary linear process. It turns out that the tail behaviour of $\sup_n S_n$ heavily depends on the coefficients of this linear process.

Keywords: random walk, dependent increments, heavy tails, subexponential distribution, tail asymptotics
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1 Introduction

In a sequence of recent papers, see e.g. [2, 3, 4, 5, 8, 10], the tail behaviour of the supremum of negatively drifted random walks with dependent heavy-tailed increments is studied. In the present paper, we continue these studies, where we consider a stochastic model which can be justified as follows. Suppose that the nominal return of some manufacturing or financial system per unit time is equal to some constant $a > 0$. However, in a variety of practical situations, this nominal return is not exactly achieved by the actual returns

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in the individual unit time period. We therefore assume that the actual return in the nth period is subject to some random perturbations $\eta_1, \ldots, \eta_n$, which arise in the first n periods, due to unexpected claim costs or additional income. For example, the perturbation η_n incurred in the nth period may not be fully reported during that period and may also affect the actual returns of later periods. More precisely, the fraction $c_0\eta_n$ of η_n is reported in the nth period, the fraction $c_1\eta_n$ in the period $n+1$, the fraction $c_2\eta_n$ in the period $n+2$, and so on, where $c_0, c_1, \ldots \in [0,1]$ with $\sum_{i=0}^{\infty} c_i = 1$. Thus, supposing that the system begins to work at time zero, the actual return in the kth period is given by the expression $a - \sum_{j=1}^{k} c_{k-j}\eta_j$. Furthermore, the sum $S_n = \xi_1 + \cdots \xi_n$, where $\xi_k = \sum_{j=1}^{k} c_{k-j}\eta_j - a$, can be seen as total (cumulative) claim surplus in the nth period.

However, the results proved in the present paper hold under more general conditions on the coefficients $c_0, c_1, \ldots$. Namely, they can be arbitrary fixed real numbers such that $\sum_{i=0}^{\infty} |c_i| < \infty$. Coefficients greater than one and negative coefficients could, for example, be interpreted as declaration of too high costs and reimbursement in later periods, respectively.

The question whether the claim surplus process $\{S_n,\ n \geq 1\}$ is "well-behaved" or dangerous, is often answered by studying the asymptotics for the tail probability $\mathbb{P}(\sup_n S_n > x)$ as $x \to \infty$. In the present paper, we derive conditions under which the exact asymptotic behaviour of $\mathbb{P}(\sup_n S_n > x)$ can be determined. It turns out that this asymptotic tail behaviour heavily depends on the choice of the coefficients $c_0, c_1, \ldots$

1.1 The Model

Let $\{\eta_n, n = 1, 2, \ldots\}$ be a sequence of independent and identically distributed random variables with $\mathbb{E}\eta_n = 0$. The distribution of η_n will be denoted by F, i.e. $F(x) = \mathbb{P}(\eta_n \leq x)$ for $x \in \mathbb{R}$. The right tail of F is denoted by $\overline{F}(x) = 1 - F(x)$. We will also use the notations, $x \geq 0$,

$$G_+(x) = \int_x^{\infty} \overline{F}(y)\, \mathrm{d}y \quad \text{and} \quad G_-(x) = \int_x^{\infty} F(-y)\, \mathrm{d}y,$$

where the integrals are finite for each x, due to the existence of $\mathbb{E}\eta_n$. For any real numbers $B \geq 0$ and $b \geq 0$, which are not both equal to 0, let $G_{B,b}$ be the following function (here $x/0 = \infty$ for $x \geq 0$):

$$G_{B,b}(x) = BG_+(x/B) + bG_-(x/b)\,, \quad x \geq 0\,.$$

By definition, $G_+ = G_{1,0}$, $G_- = G_{0,1}$ and

$$G_{B,b}(x) = \int_x^\infty \Big(\overline{F}(y/B) + F(-y/b)\Big)\mathrm{d}y.$$

Let $a > 0$ and let $c_k \in \mathbb{R}$, $k \in \mathbb{N}$, be some constants, not all of which are equal to 0; $\mathbb{N} = \{0, 1, \ldots\}$. Let the random variable ξ_k be given by

$$\xi_k = \sum_{j=1}^{k} c_{k-j}\eta_j - a.$$

Consider the partial sums

$$S_0 = 0, \quad S_n = \xi_1 + \cdots + \xi_n, \; n \geq 1.$$

Then, the sequence $\{S_n,\, n \in \mathbb{N}\}$ is called a *random walk* with asymptotically stationary dependent increments and with negative drift. The following representation of the partial sums S_n is useful. With the notation

$$\overline{c}_k = \sum_{i=0}^{k} c_i\,, \quad k \in \mathbb{N}, \tag{1}$$

we have the representation in terms of *weighted sums*:

$$S_n = \sum_{j=1}^{n} \overline{c}_{n-j}\eta_j - na. \tag{2}$$

We assume everywhere that

$$\sum_{k=0}^{\infty} |c_k| < \infty. \tag{3}$$

In Lemma 1 below, we use this condition in order to show that $\{S_n\}$ satisfies the strong law of large numbers, i.e. with probability 1, $S_n/n \to -a < 0$ as $n \to \infty$. Hence, the supremum $\sup_{n\in\mathbb{N}} S_n$ of the random walk $\{S_n\}$ is a well-defined random variable, which is finite with probability 1.

1.2 Main Results

The purpose of this paper is to derive conditions, under which the asymptotic behaviour of the tail $\mathbb{P}(\sup_{n\in\mathbb{N}} S_n > x)$ can be easily related to the asymptotic behaviour of the functions $G_+(x)$ and $G_-(x)$ as $x \to \infty$.

In Section 2, we derive an asymptotic lower bound for the probability $\mathbb{P}(\sup_n S_n > x)$. We prove in Theorem 2 that, if we take any different natural numbers $m_1,\ m_2 \in \mathbb{N}$ and put $C = \max\{0, \overline{c}_{m_1}\} \geq 0$ and $c = \min\{0, \overline{c}_{m_2}\} \leq 0$, then

$$\liminf_{x\to\infty} \frac{\mathbb{P}\Big(\sup_n S_n > x\Big)}{G_{C,|c|}(x)} \geq \frac{1}{a}, \tag{4}$$

provided that $C+|c| > 0$ and $G_{C,|c|}$ is a *long-tailed function* (for its definition see Section 2).

In Section 3, we get the asymptotic upper bound (Theorem 3). Let $\overline{C} = \sup\{0, \overline{c}_k, k \in \mathbb{N}\} \geq 0$ and $\overline{c} = \inf\{0, \overline{c}_k, k \in \mathbb{N}\} \leq 0$, where $\overline{C} + |\overline{c}| > 0$ since not all c_k are equal to 0. If $G_{\overline{C},|\overline{c}|} \in \mathcal{S}$ (i.e. $G_{\overline{C},|\overline{c}|}$ is *subexponential*; for its definition see Section 3), then

$$\limsup_{x\to\infty} \frac{\mathbb{P}\Big(\sup_n S_n > x\Big)}{G_{\overline{C},|\overline{c}|}(x)} \leq \frac{1}{a}\ . \tag{5}$$

Combining (4) and (5) we immediately obtain the following asymptotic tail behaviour of $\sup_n S_n$, where $f(x) \sim g(x)$ means $\lim_{x\to\infty} f(x)/g(x) = 1$.

Theorem 1. *Let one of the following conditions hold:*

(i) $\overline{C} = \overline{c}_{m_1} > 0$ *for some* m_1 *and* $\overline{c} = \overline{c}_{m_2} < 0$ *for some* m_2*;*

(ii) $\overline{C} = \overline{c}_{m_1} > 0$ *for some* m_1*, and* $\overline{c} = 0$*;*

(iii) $\overline{C} = 0$ *and* $\overline{c} = \overline{c}_{m_2} < 0$ *for some* m_2*.*

If $G_{\overline{C},|\overline{c}|} \in \mathcal{S}$*, then*

$$\mathbb{P}\Big(\sup_n S_n > x\Big) \sim a^{-1} G_{\overline{C},|\overline{c}|}(x) \qquad \text{as } x \to \infty\ .$$

In Section 4 we consider the only possible remaining cases which are not covered by Theorem 1, namely $\overline{C} > 0$ and $\overline{C} > \overline{c}_m$ for any m or $\overline{c} < 0$ and $\overline{c} < \overline{c}_m$ for any m. We show that then the assertion of Theorem 1 holds under the additional condition of (intermediate) regularly varying at infinity tails.

Notice that our results generalize a well-known theorem on the asymptotic tail behaviour of the supremum of negatively drifted random walks with independent subexponential increments, which concerns the case $c_0 = 1$, $c_1 = c_2 = \ldots = 0$; see [7] and also [1, 6, 11]. Recently, related extensions of this theorem to the case of random walks with dependent increments have been proved in [2, 3, 4, 5, 8]. An extension, which is similar to our results, has been derived in [10], where F is assumed to have regularly varying left and right tails. This assumption made in [10] is essential for the application of Karamata-type arguments. Our proving technique used in Sections 2 and 3 is different; it allows us to omit the assumption that F is regularly varying in Theorem 1.

1.3 Strong Law of Large Numbers

Lemma 1. *With probability* 1, $S_n/n \to -a$ *as* $n \to \infty$.

Proof. Condition (3) implies that the sequence $\overline{c}_n$ has a limit as $n \to \infty$, say $c \in \mathbb{R}$. Then, for any $n, N \in \mathbb{N}$ with $n \geq N$, we have

$$\frac{S_n + na}{n} = \frac{c}{n}\sum_{j=1}^{n} \eta_j + \frac{1}{n}\sum_{j=1}^{n-N} (\overline{c}_{n-j} - c)\eta_j + \sum_{j=0}^{N-1} (\overline{c}_j - c)\frac{\eta_{n-j}}{n}.$$

By the strong law of large numbers for i.i.d. random variables,

$$\frac{c}{n}\sum_{j=1}^{n} \eta_j \to 0 \quad \text{as} \quad n \to \infty$$

with probability 1. Furthermore, since $\mathbb{E}|\eta_1|$ is finite, we have for any fixed $j \geq 0$:

$$|\eta_{n-j}|/n \to 0 \quad \text{as} \quad n \to \infty$$

with probability 1. Hence, for any fixed $N \in \mathbb{N}$,

$$\limsup_{n\to\infty} \Big| \frac{S_n + na}{n} \Big| \leq \limsup_{n\to\infty} \Big| \frac{1}{n} \sum_{j=1}^{n-N} (\overline{c}_{n-j} - c)\eta_j \Big|.$$

Fix $\varepsilon > 0$. Then, there exists N such that $|\overline{c}_n - c| \leq \varepsilon$ for any $n \geq N$. Thus,

$$\limsup_{n\to\infty} \Big| \frac{1}{n} \sum_{j=1}^{n-N} (\overline{c}_{n-j} - c)\eta_j \Big| \leq \varepsilon \limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} |\eta_j| = \varepsilon \mathbb{E}|\eta_j|.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, the lemma is proved. □

2 Lower Bounds

We first state some asymptotic properties of long-tailed distributions. They will be used in Section 2.2 in order to derive an asymptotic lower bound for the tail of supremum of sums.

2.1 Properties of Long-Tailed Distributions

Let $\mathcal{L}$ be the collection of all non-increasing functions $f : \mathbb{R} \to (0, \infty)$ such that, for each $y \in \mathbb{R}$,

$$\lim_{x\to\infty} f(x+y)/f(x) = 1.$$

The distribution F is called *right long-tailed* if $\overline{F}(x) \in \mathcal{L}$. For simplicity of notation, we will write $F \in \mathcal{L}$ if the distribution F is right long-tailed. Notice that $G_+ \in \mathcal{L}$ if $F \in \mathcal{L}$.

The distribution F is called *left long-tailed* if $F(-x) \in \mathcal{L}$. By $\mathcal{L}^-$ we denote the family of all distributions on $\mathbb{R}$ with this property. Notice that the distribution F of a random variable η belongs to $\mathcal{L}^-$ if and only if the distribution of $-\eta$ belongs to $\mathcal{L}$.

Lemma 2. *Let $f \in \mathcal{L}$. Then there exists an increasing function $g : \mathbb{R} \to \mathbb{R}_+ = [0, \infty)$ with $g(x) \to \infty$ as $x \to \infty$ such that*

$$\lim_{x\to\infty} f(x+g(x))/f(x) = 1.$$

Proof. From the definition of class $\mathcal{L}$ we get that there exists an increasing sequence of real numbers $\{x_n,\ n \geq 1\}$ such that $x_n \geq n$ and

$$f(x+n)/f(x) \geq 1 - 1/n \quad \text{for each } x \geq x_n.$$

Define

$$g(x) = \begin{cases} 0 \ \text{ if } x < x_1, \\ n \ \text{ if } x_n \leq x < x_{n+1}. \end{cases}$$

Since $x_n \to \infty$, we have $g(x) \to \infty$ as $x \to \infty$ and, for $x_n \leq x < x_{n+1}$,

$$f(x+g(x))/f(x) \geq 1 - 1/n\ ,$$

which implies

$$\liminf_{x\to\infty} f(x+g(x))/f(x) \geq 1\ .$$

On the other hand, for any nonnegative function g, $f(x+g(x)) \leq f(x)$. This completes the proof. □

Corollary 1. *Assume that* $f \in \mathcal{L}$. *Then,*

$$\lim_{x\to\infty} \inf_{y\ge x} f(y+g(x))/f(y) = 1 \ .$$

Proof. Fix $\varepsilon > 0$. Then, by the result of Lemma 2, there exists an x_0 such that $f(x+g(x))/f(x) \ge 1-\varepsilon$ for each $x \ge x_0$. Thus, due to the monotonicity of g, for each $y \ge x \ge x_0$,

$$f(y+g(x))/f(y) \ge f(y+g(y))/f(y) \ge 1-\varepsilon. \qquad \square$$

Lemma 3. *Let the sequence* $T_1, T_2, \ldots$ *of random variables be such that* $T_n/n \to 0$ *as* $n \to \infty$ *with probability* 1. *Then there exists a non-decreasing function* $h : \mathbb{N} \to \mathbb{R}_+$ *with* $h(n) = o(n)$ *as* $n \to \infty$ *such that*

$$\lim_{z\to\infty} \mathbb{P}\Big(\bigcap_{n\ge1} \Big\{|T_n| \le z + h(n)\Big\}\Big) = 1.$$

Proof. Since $T_n/n \to 0$ with probability 1, there exists a sequence of integers $\{N_k,\ k \ge 1\}$ such that $N_k \to \infty$ and

$$\mathbb{P}\Big(\bigcup_{n\ge N_k} \Big\{|T_n| > n/k\Big\}\Big) \le 2^{-k} \tag{6}$$

for all $k = 1, 2, \ldots$, where without loss of generality we can assume that $N_{k+1} \ge N_k + 1$. Define

$$h^*(n) = \begin{cases} n & \text{if } n < N_1, \\ n/k & \text{if } N_k \le n < N_{k+1}. \end{cases} \tag{7}$$

Since $N_k \to \infty$, $h^*(n) = o(n)$. We have, for any fixed $M \in \mathbb{N}$,

$$\mathbb{P}\Big(\bigcup_{n\ge1}\Big\{|T_n| > z + h^*(n)\Big\}\Big) \le \sum_{n=1}^{N_M-1} \mathbb{P}\Big(|T_n| > z\Big) + \mathbb{P}\Big(\bigcup_{n\ge N_M} \Big\{|T_n| > h^*(n)\Big\}\Big).$$

Therefore,

$$\limsup_{z\to\infty} \mathbb{P}\Big(\bigcup_{n\ge1}\Big\{|T_n| > z + h^*(n)\Big\}\Big) \le \mathbb{P}\Big(\bigcup_{n\ge N_M} \Big\{|T_n| > h^*(n)\Big\}\Big).$$

Using (6) and (7), we get the following estimates:

$$
\begin{aligned}
\mathbb{P}\Big(\bigcup_{n\ge N_M}\Big\{|T_n|>h^*(n)\Big\}\Big) &\le \sum_{k=M}^{\infty}\mathbb{P}\Big(\bigcup_{N_k\le n<N_{k+1}}\Big\{|T_n|>h^*(n)\Big\}\Big)\\
&\le \sum_{k=M}^{\infty}\mathbb{P}\Big(\bigcup_{n\ge N_k}\Big\{|T_n|>n/k\Big\}\Big)\\
&\le \sum_{k=M}^{\infty}2^{-k}=2^{-M+1}\ .
\end{aligned}
$$

Since M is arbitrary, letting $M\to\infty$ yields

$$
\limsup_{z\to\infty}\mathbb{P}\Big(\bigcup_{n\ge 1}\Big\{|T_n|>z+h^*(n)\Big\}\Big)=0\ ,
$$

which is equivalent to

$$
\lim_{z\to\infty}\mathbb{P}\Big(\bigcap_{n\ge 1}\Big\{|T_n|\le z+h^*(n)\Big\}\Big)=1\ .
$$

Putting now $h(n)\equiv\max\{h^*(k),\ k\le n\}$, we obtain a non-decreasing function $h(n)=o(n)$ which satisfies the assertion of the lemma. □

Lemma 4. *Let $a>0$ and $n_1\ge 1$. Let $h:\mathbb{N}\to\mathbb{R}_+$ be a function such that $h(n)=o(n)$ as $n\to\infty$. If $G_+\in\mathcal{L}$, then, as $z\to\infty$:*

$$
\sum_{n=n_1}^{\infty}\overline{F}(z+na)\sim a^{-1}G_+(z);\qquad \sum_{n=n_1}^{\infty}\overline{F}(z+na+h(n))\sim a^{-1}G_+(z).
$$

Proof. For any distribution F we have

$$
\begin{aligned}
\sum_{n=n_1}^{\infty}\overline{F}(z+na) &\le \sum_{n=1}^{\infty}\int_{n-1}^{n}\overline{F}(z+ay)\,\mathrm{d}y\\
&= \int_0^{\infty}\overline{F}(z+ay)\,\mathrm{d}y=a^{-1}G_+(z)\ .
\end{aligned}
\tag{8}
$$

On the other hand,

$$
\begin{aligned}
\sum_{n=n_1}^{\infty}\overline{F}(z+na) &\ge \sum_{n=n_1}^{\infty}\int_{n}^{n+1}\overline{F}(z+ay)\,\mathrm{d}y\\
&= \int_{n_1}^{\infty}\overline{F}(z+ay)\,\mathrm{d}y=a^{-1}G_+(z+an_1)\sim a^{-1}G_+(z)
\end{aligned}
$$

as $z \to \infty$, since $G_+ \in \mathcal{L}$. Thus, the first equivalence of the lemma is proved. To prove the second equivalence, fix $\varepsilon > 0$. First,

$$\sum_{n=n_1}^{\infty} \overline{F}(z + na + h(n)) \le \sum_{n=n_1}^{\infty} \overline{F}(z + na) \le a^{-1} G_+(z).$$

On the other hand, since $h(n) = o(n)$, there exists $N \ge n_1$ such that $h(n) \le \varepsilon n$ for any $n \ge N$. Therefore,

$$\sum_{n=n_1}^{\infty} \overline{F}(z + na + h(n)) \ge \sum_{n=N}^{\infty} \overline{F}(z + n(a+\varepsilon)) \sim (a+\varepsilon)^{-1} G_+(z),$$

as $z \to \infty$, in view of the first equivalence of the lemma. Due to the arbitrary choice of $\varepsilon > 0$, it implies the second equivalence of the lemma. $\square$

Let $b_k \in \mathbb{R}$, $k \in \mathbb{N}$, be a bounded convergent sequence. Thus, the supremum $b = \sup_k |b_k|$ is finite. Put

$$T_n = \sum_{k=1}^{n} b_{n-k} \eta_k$$

and, for any natural numbers $n \ge 1$, $m \ge 0$, $n > m$,

$$T_n^{(m)} = \sum_{k=1}^{n-m-1} b_{n-k} \eta_k.$$

By definition, for $n > m$,

$$T_n = T_n^{(m)} + \sum_{k=n-m}^{n} b_{n-k} \eta_k.$$

The sequences $\{T_n\}$ and $\{T_n^{(m)}\}$ fulfill the condition of Lemma 3. Indeed, since $\mathbb{E}\eta_i = 0$, we have $\lim_{n\to\infty} T_n/n = 0$ and $\lim_{n\to\infty} T_n^{(m_1)}/n = 0$ with probability 1 by the strong law of large numbers (see Lemma 1). Hence, for any function $g(x)$ with $g(x) \to \infty$, there exists a function $h(n)$ with $h(n) = o(n)$ such that

$$\lim_{x\to\infty} \mathbb{P}\Big(\bigcap_{n\ge 1} \Big\{ |T_n| \le g(x) + h(n) \Big\} \Big) = 1 \tag{9}$$

and

$$\lim_{x\to\infty} \mathbb{P}\Big(\bigcap_{n\ge m_1} \Big\{ |T_n^{(m)}| \le g(x) + h(n) \Big\} \Big) = 1. \tag{10}$$

Furthermore, for $n > m_1 \in \mathbb{N}$, let B_n denote the event

$$B_n = \bigcap_{j=1}^{n-m_1-1} \Big\{ |T_j| \le g(x)+h(j) \Big\} \cap \Big\{ |T_n^{(m_1)}| \le g(x)+h(n) \Big\}$$
$$\cap \Big\{ b_{m_1}\eta_{n-m_1} > x+(2+m_1 b)g(x)+na+2h(n) \Big\} \cap \bigcap_{j=n-m_1+1}^{n} \Big\{ |\eta_j| \le g(x) \Big\}$$

and, for $n > m_2 \in \mathbb{N}$,

$$B_n^- = \bigcap_{j=1}^{n-m_2-2} \Big\{ |T_j| \le g(x)+h(j) \Big\} \cap \Big\{ |T_n^{(m_2)}| \le g(x)+h(n) \Big\}$$
$$\cap \Big\{ b_{m_2}\eta_{n-m_2} > x+(2+m_2 b)g(x)+na+2h(n) \Big\} \cap \bigcap_{j=n-m_2+1}^{n} \Big\{ |\eta_j| \le g(x) \Big\}.$$

Lemma 5. *Let m_1, $m_2 \in \mathbb{N}$ be any natural numbers such that $b_{m_1} \ge 0$ and $b_{m_2} \le 0$. Then the events B_n, $n > m_1$, and B_n^-, $n > m_2$, are pairwise disjoint.*

Proof. Let us consider, for example, any two of B_n, say B_k and B_n, $m_1 < k < n$. If $n \le k + m_1$, then for $\omega \in B_n$ we have

$$b_{m_1}\eta_{n-m_1}(\omega) > x + (2+m_1 b)g(x) + na + 2h(n) \ge b_{m_1} g(x),$$

whereas for $\omega \in B_k$,

$$b_{m_1}\eta_{n-m_1}(\omega) \le b_{m_1}|\eta_{n-m_1}(\omega)| \le b_{m_1} g(x).$$

If $n > k + m_1$, then for $\omega \in B_k$ we have

$$\begin{aligned} T_k(\omega) &= T_k^{(m_1)}(\omega) + b_{m_1}\eta_{k-m_1}(\omega) + \sum_{j=k-m_1+1}^{k} b_{k-j}\eta_j(\omega) \\ &> -g(x) - h(k) + x + (2+m_1 b)g(x) + ka + 2h(k) - m_1 b g(x) \\ &\ge g(x) + h(k), \end{aligned}$$

whereas for $\omega \in B_n$ it holds that $|T_k(\omega)| \le g(x) + h(k)$. The rest of the proof follows by similar arguments. □

Lemma 6. *Let $b_{m_1} > 0$ and $G_+ \in \mathcal{L}$. Let $g(x) \to \infty$ be a function such that $G_+((x+g(x))/b_{m_1}) \sim G_+(x/b_{m_1})$ as $x \to \infty$. Then*

$$\lim_{x\to\infty} \frac{\mathbb{P}\Big(\bigcup_{n>m_1} B_n\Big)}{b_{m_1} G_+(x/b_{m_1})} = \frac{1}{a}.$$

Proof. The function $g(x)$ exists due to Lemma 2. By Lemma 5,

$$\begin{aligned}
&\mathbb{P}\Big(\bigcup_{n>m_1} B_n\Big) = \sum_{n>m_1} \mathbb{P}(B_n) \\
&= \sum_{n>m_1} \mathbb{P}\Big(\bigcap_{j=1}^{n-m_1-1} \Big\{|T_j| \le g(x)+h(j)\Big\} \cap \Big\{|T_n^{(m_1)}| \le g(x)+h(n)\Big\}\Big) \\
&\quad \times \mathbb{P}^{m_1}\Big(|\eta_1| \le g(x)\Big)\mathbb{P}\Big(b_{m_1}\eta_{n-m_1} > x+(2+m_1 b)g(x)+na+2h(n)\Big).
\end{aligned}$$

This gives the following upper bound

$$\mathbb{P}\Big(\bigcup_{n>m_1} B_n\Big) \le \sum_{n>m_1} \mathbb{P}\Big(b_{m_1}\eta_1 > x+na\Big) \tag{11}$$

and the lower bound

$$\begin{aligned}
&\mathbb{P}\Big(\bigcup_{n>m_1} B_n\Big) \\
&\ge \mathbb{P}\Big(\bigcap_{n\ge 1}\Big\{|T_n| \le g(x)+h(n)\Big\} \cap \bigcap_{n\ge m_1} \Big\{|T_n^{(m_1)}| \le g(x)+h(n)\Big\}\Big) \\
&\quad \times \mathbb{P}^{m_1}\Big(|\eta_1| \le g(x)\Big) \sum_{n>m_1} \mathbb{P}\Big(b_{m_1}\eta_1 > x+(2+m_1 b)g(x)+na+2h(n)\Big).
\end{aligned} \tag{12}$$

We have the convergence $\mathbb{P}\Big(|\eta_1| \le g(x)\Big) \to 1$ as $x \to \infty$, as well as (9) and (10). Hence, inequalities (11) and (12), and Lemma 4 with $z = (x + (2+m_1 b)g(x))/b_{m_1}$ lead to the assertion of the lemma. □

Lemma 7. *Let $b_{m_1} \ge 0$, $b_{m_2} \le 0$, and $b_{m_1} + |b_{m_2}| > 0$. Let $G_{b_{m_1},|b_{m_2}|} \in \mathcal{L}$ and let $g(x) \to \infty$ be a function such that, with $m = \max\{m_1, m_2\}$, $G_{b_{m_1},|b_{m_2}|}(x + (2+mb)g(x)) \sim G_{b_{m_1},|b_{m_2}|}(x)$ as $x \to \infty$. Then,*

$$\liminf_{x\to\infty} \frac{\mathbb{P}\Big(\bigcup_{n>m}(B_n \cup B_n^-)\Big)}{G_{b_{m_1},|b_{m_2}|}(x)} \ge \frac{1}{a}.$$

Remark. Notice that $G_{B,b} \in \mathcal{L}$ if both $G_+ \in \mathcal{L}$ and $G_- \in \mathcal{L}^-$. Another sufficient condition for $G_{B,b} \in \mathcal{L}$ is $G_+ \in \mathcal{L}$ and $G_-(x/b) = o(G_+(x/B))$ as $x \to \infty$. Notice that the function $g(x)$ in Lemma 7 exists, since $(2+mb)g(x)$ can be taken as the function $g(x)$ in Lemma 2.

Proof of Lemma 7. By Lemma 5,

$$\mathbb{P}\Big(\bigcup_{n>m} (B_n \cup B_n^-)\Big) = \mathbb{P}\Big(\bigcup_{n>m} B_n\Big) + \mathbb{P}\Big(\bigcup_{n>m} B_n^-\Big).$$

Following now the guidelines of the proof of Lemma 6, we deduce that for any $\varepsilon > 0$ there exists x_0 such that, for $x \geq x_0$,

$$\mathbb{P}\Big(\bigcup_{n>m_1} B_n\Big) \geq (1-\varepsilon) \sum_{n>m_1} \mathbb{P}\Big(b_{m_1}\eta_1 > x+(2+mb)g(x)+na+2h(n)\Big);$$
$$\mathbb{P}\Big(\bigcup_{n>m_2} B_n^-\Big) \geq (1-\varepsilon) \sum_{n>m_2} \mathbb{P}\Big(|b_{m_2}|\eta_1 < -x-(2+mb)g(x)-na-2h(n)\Big).$$

Therefore, according to Lemma 4,

$$\begin{aligned}\mathbb{P}\Big(\bigcup_{n>m} (B_n \cup B_n^-)\Big) &\geq (1-\varepsilon) \sum_{n>m} \Big(\overline{F}\Big(\frac{x+(2+mb)g(x)+na+2h(n)}{b_{m_1}}\Big) \\ &\qquad + F\Big(-\frac{x+(2+mb)g(x)+na+2h(n)}{|b_{m_2}|}\Big)\Big) \\ &\sim (1-\varepsilon)a^{-1}G_{b_{m_1},|b_{m_2}|}(x + (2+mb)g(x)) \\ &\sim (1-\varepsilon)a^{-1}G_{b_{m_1},|b_{m_2}|}(x),\end{aligned}$$

for $x \to \infty$. □

2.2 Asymptotic Lower Bounds for the Tail of the Supremum

We are now in a position to derive an asymptotic lower bound for the tail $\mathbb{P}(\sup_n S_n > x)$ as $x \to \infty$.

Theorem 2. *Let m_1, $m_2 \in \mathbb{N}$ be any different natural numbers. Put $C = \max\{0, \overline{c}_{m_1}\} \geq 0$ and $c = \min\{0, \overline{c}_{m_2}\} \leq 0$. If $C + |c| > 0$ and $G_{C,|c|} \in \mathcal{L}$, then*

$$\liminf_{x\to\infty} \frac{\mathbb{P}\Big(\sup_n S_n > x\Big)}{G_{C,|c|}(x)} \geq \frac{1}{a}\,. \tag{13}$$

Proof. Put $b_k = \overline{c}_k$ and $T_n = S_n + na$ in Lemma 7. Let $m = \max\{m_1, m_2\}$ and let $g(x) \to \infty$ be a function such that $G_{C,|c|}(x+(2+mb)g(x)) \sim G_{C,|c|}(x)$ as $x \to \infty$, which exists due to Lemma 2; see the remark after Lemma 7. For $n > m$, consider the events

$$\widetilde{B}_n = \Big\{|T_n^{(m_1)}| \leq g(x) + h(n)\Big\} \cap \Big\{C\eta_{n-m_1} > x + (2+m_1 b)g(x) + na + 2h(n)\Big\} \cap \bigcap_{j=n-m_1+1}^{n} \Big\{|\eta_j| \leq g(x)\Big\}$$

and

$$\widetilde{B}_n^- = \Big\{|T_n^{(m_2)}| \leq g(x) + h(n)\Big\} \cap \Big\{|c|\eta_{n-m_2} > x + (2+m_2 b)g(x) + na + 2h(n)\Big\} \cap \bigcap_{j=n-m_2+1}^{n} \Big\{|\eta_j| \leq g(x)\Big\}$$

where $h(n)$ is the function considered in (9) and (10). By definition, $B_n \subseteq \widetilde{B}_n \subseteq \{S_n > x\}$ and $B_n^- \subseteq \widetilde{B}_n^- \subseteq \{S_n > x\}$. Thus,

$$\mathbb{P}\Big(\sup_n S_n > x\Big) \geq \mathbb{P}\Big(\bigcup_n (B_n \cup B_n^-)\Big).$$

Now the assertion follows from Lemma 7. □

The following statements are immediate consequences of the proof of Theorem 2.

Corollary 2. *Let $\overline{c}_m > 0$ for some $m \geq 0$ and $G_+ \in \mathcal{L}$. Then,*

$$\liminf_{x\to\infty} \frac{\mathbb{P}\big(\sup_n S_n > x\big)}{\overline{c}_m G_+(x/\overline{c}_m)} \geq \frac{1}{a}.$$

Corollary 3. *Let $\overline{c}_m < 0$ for some $m \geq 0$ and $G_- \in \mathcal{L}^-$. Then,*

$$\liminf_{x\to\infty} \frac{\mathbb{P}\big(\sup_n S_n > x\big)}{|\overline{c}_m| G_-(x/|\overline{c}_m|)} \geq \frac{1}{a}.$$

3 Upper Bound

We first introduce the class of subexponential distributions. They will be used in this section in order to derive asymptotic upper bounds for the tail $\mathbb{P}(\sup_n S_n > x)$ as $x \to \infty$.

The distribution G on $\mathbb{R}_+$ is called *subexponential* if $G(x) < 1$ for all $x \geq 0$ and

$$\lim_{x\to\infty} \frac{\overline{G*G}(x)}{\overline{G}(x)} = 2\,, \tag{14}$$

where $\overline{G*G}(x)$ denotes the tail of the convolution $G*G(x) = \int_0^x G(x-y)\,G(\mathrm{d}y)$. By $\mathcal{S}$ we denote the family of all subexponential distributions. It is well known that $\mathcal{S} \subset \mathcal{L}$.

For simplicity of notation, we will write $G_{B,b} \in \mathcal{S}$ if $G_{B,b}(x)/G_{B,b}(0)$, $x \geq 0$, is the tail of a subexponential distribution. In particular, $G_+ \in \mathcal{S}$ if the *integrated tail* $G_+(x)/G_+(0)$, $x \geq 0$, of the distribution function $F(x)$ is the tail of a subexponential distribution.

It is well known that the tail behaviour of the supremum of partial sums of i.i.d. random variables is given by

$$\mathbb{P}\Big(\sup_{n\geq 1}\Big\{\sum_{k=1}^{n} \eta_k - na\Big\} > x\Big) \sim a^{-1} G_+(x) \quad \text{as } x\to\infty, \tag{15}$$

provided that $G_+ \in \mathcal{S}$; see [7] and also [1, 6, 11]. It turns out that $G_+ \in \mathcal{S}$ is not only sufficient, but also necessary for (15); see [9].

Lemma 8. *Let $b_k \in \mathbb{R}$, $k \in \mathbb{N}$, $B \geq \sup\{0, b_k, k \in \mathbb{N}\}$, and $b \leq \inf\{0, b_k, k \in \mathbb{N}\}$. Let there exist a limit*

$$\lim_{k\to\infty} b_k = \widetilde{b}. \tag{16}$$

If $B + |b| > 0$ and $G_{B,|b|} \in \mathcal{S}$, then

$$\limsup_{x\to\infty} \frac{\mathbb{P}\Big(\sup_n \Big\{\sum_{k=1}^n b_{n-k}\eta_k - na\Big\} > x\Big)}{G_{B,|b|}(x)} \leq \frac{1}{a}.$$

Remark. The condition $G_{B,|b|} \in \mathcal{S}$ is fulfilled, e.g., if $G_+ \in \mathcal{S}$ and $G_-(x/b) = (\gamma + o(1))G^+(x/B)$ as $x \to \infty$, for some $\gamma \geq 0$.

Proof of Lemma 8. Our proving argument is based on a truncation technique. For any real $z > 0$ and for any random variable η, put

$$\eta^{[z]}(\omega) \equiv \begin{cases} B\eta(\omega) & \text{if } \eta(\omega) > z, \\ \widetilde{b}\eta(\omega) & \text{if } -z \leq \eta(\omega) \leq z, \\ b\eta(\omega) & \text{if } \eta(\omega) < -z. \end{cases}$$

For $x > \max\{B, -b, |\widetilde{b}|\}z$,

$$\mathbb{P}(\eta^{[z]} > x) = \mathbb{P}(B\eta > x) + \mathbb{P}(b\eta > x) = \overline{F}(x/B) + F(-x/|b|). \quad (17)$$

Since $G_{B,|b|} \in \mathcal{S}$, the integrated tail distribution of $\eta_1^{[z]}$ is subexponential. Furthermore, for any $\omega \in \Omega$, and for any $b' \in [b, B]$, we have

$$\begin{aligned} b'\eta(\omega) &\le \begin{cases} B\eta(\omega) & \text{if } \eta(\omega) > z, \\ b'\eta(\omega) & \text{if } -z < \eta(\omega) \le z, \\ b\eta(\omega) & \text{if } \eta(\omega) < -z \end{cases} \\ &= \begin{cases} \eta^{[z]}(\omega) & \text{if } \eta(\omega) > z, \\ \eta^{[z]}(\omega) + (b' - \tilde{b})\eta(\omega) & \text{if } -z < \eta(\omega) \le z, \\ \eta^{[z]}(\omega) & \text{if } \eta(\omega) < -z \end{cases} \\ &\le \eta^{[z]}(\omega) + |\widetilde{b} - b'|z\,. \end{aligned}$$

Therefore,

$$\sum_{k=1}^{n} b_{n-k}\eta_k \le \sum_{k=1}^{n} \eta_k^{[z]} + z\sum_{k=0}^{n-1} |\widetilde{b} - b_k|.$$

Fix $\varepsilon \in (0, a/2)$. Since $b_k \to \widetilde{b}$, there exists K such that $|b_k - \widetilde{b}| \le \varepsilon$ for any $k \ge K$. Hence,

$$\begin{aligned} \sum_{k=1}^{n} b_{n-k}\eta_k &\le \sum_{k=1}^{n} \eta_k^{[z]} + z\sum_{k=0}^{K} |\widetilde{b} - b_k| + n\varepsilon \\ &\equiv \sum_{k=1}^{n} \eta_k^{[z]} + \widehat{b}z + n\varepsilon\,, \end{aligned}$$

where $\widehat{b} \equiv \sum_{k=0}^{K} |\widetilde{b} - b_k|$. Since $\mathbb{E}\eta_1 = 0$, there exists a sufficiently large $z > 0$ such that $\mathbb{E}\eta_1^{[z]} \le \varepsilon$. In view of (15) and (17), as $x \to \infty$,

$$\mathbb{P}\Big(\sup_{n\ge 1}\Big\{\sum_{k=1}^{n} \eta_k^{[z]} - na\Big\} > x\Big) \sim \frac{1}{a - \mathbb{E}\eta_1^{[z]}} G_{B,|b|}(x).$$

Hence,

$$\begin{aligned} &\mathbb{P}\Big(\sup_{n}\Big\{\sum_{k=1}^{n} b_{n-k}\eta_k - na\Big\} > x\Big) \\ &\qquad \le \mathbb{P}\Big(\sup_{n}\Big\{\sum_{k=1}^{n} \eta_k^{[z]} - n(a-\varepsilon)\Big\} > x - \widehat{b}z\Big) \\ &\qquad\qquad \le \frac{1 + o(1)}{a - 2\varepsilon} G_{B,|b|}(x - \widehat{b}z) \sim \frac{1}{a - 2\varepsilon} G_{B,|b|}(x). \end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily, the proof is complete. □

The last lemma implies the following asymptotic upper bound for the tail $\mathbb{P}(\sup_n S_n > x)$.

Theorem 3. *Let* $\overline{C} = \sup\{0, \overline{c}_k, k \in \mathbb{N}\} \geq 0$ *and* $\overline{c} = \inf\{0, \overline{c}_k, k \in \mathbb{N}\} \leq 0$. *If* $G_{\overline{C},|\overline{c}|} \in \mathcal{S}$, *then*

$$\limsup_{x\to\infty} \frac{\mathbb{P}\Big(\sup_n S_n > x\Big)}{G_{\overline{C},|\overline{c}|}(x)} \leq \frac{1}{a} \,. \tag{18}$$

Proof. Put $b_k = \overline{c}_k$, $B = \overline{C}$, and $b = \overline{c}$ in Lemma 8. Condition (16) holds because of (3). □

In the case when the coefficients $\overline{c}_k$ are either all nonnegative or all nonpositive, we obtain the following two immediate consequences of Theorem 3.

Corollary 4. *Assume that* $\overline{c}_k \geq 0$ *for all* $k \in \mathbb{N}$. *Let* $G_+ \in \mathcal{S}$ *and* $\overline{C} = \sup_k \overline{c}_k > 0$. *Then,*

$$\limsup_{x\to\infty} \frac{\mathbb{P}\Big(\sup_n S_n > x\Big)}{\overline{C}G_+(x/\overline{C})} \leq \frac{1}{a}.$$

Corollary 5. *Assume that* $\overline{c}_k \leq 0$ *for all* $k \in \mathbb{N}$. *Let* $G_- \in \mathcal{S}$ *and* $\overline{c} = \inf_k \overline{c}_k < 0$. *Then,*

$$\limsup_{x\to\infty} \frac{\mathbb{P}\Big(\sup_n S_n > x\Big)}{|\overline{c}|G_-(x/|\overline{c}|)} \leq \frac{1}{a}.$$

4 Asymptotics for Regularly Varying Tails

In this section we study the only possible remaining cases not covered by Theorem 1.

The function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called *intermediate regularly varying* if

$$\lim_{\delta\downarrow 0} \lim_{x\to\infty} \frac{f(x(1+\delta))}{f(x)} = 1 \,. \tag{19}$$

By $\mathcal{IR}$ we denote the family of all functions satisfying (19). For example, regularly varying at infinity functions belong to class $\mathcal{IR}$. If a distribution G has an intermediate regularly varying tail, then $G \in \mathcal{S}$.

Theorem 4. *Let $G_{\overline{C},|\overline{c}|} \in \mathcal{S}$ and assume that one of the following conditions hold:*

(i) $\overline{C} > 0$, $\overline{C} > \overline{c}_m$ *for any* m, *and* $\overline{c} = \overline{c}_{m_2} < 0$ *for some* m_2, *and* $G_+ \in \mathcal{IR}$;

(ii) $\overline{C} = \overline{c}_{m_1} > 0$ *for some* m_1, $\overline{c} < 0$ *and* $\overline{c} < \overline{c}_m$ *for any* m, *and* $G_- \in \mathcal{IR}$;

(iii) $\overline{C} > 0$, $\overline{C} > \overline{c}_m$ *for any* m, *and* $\overline{c} < 0$ *and* $\overline{c} < \overline{c}_m$ *for any* m, *and* $G_{\overline{C},|\overline{c}|} \in \mathcal{IR}$.

Then

$$\mathbb{P}\Big(\sup_{n\in\mathbb{N}} S_n > x\Big) \sim a^{-1} G_{\overline{C},|\overline{c}|}(x) \qquad \text{as } x \to \infty\ . \tag{20}$$

Proof. Fix $\varepsilon > 0$ and suppose that condition (i) is fulfilled. Due to (19), there exist $\delta \in (0, \varepsilon]$ and $x_0 > 0$ such that for $x \geq x_0$

$$\frac{G_+(x/(\overline{C}-\delta))}{G_+(x/\overline{C})} \geq 1 - \varepsilon\ . \tag{21}$$

Since $\sup_{k\geq 0} \overline{c}_k = \overline{C}$, there exists k_0 such that $\overline{c}_{k_0} \geq \overline{C} - \delta$. Now it follows from (13) that

$$\liminf_{x\to\infty} \frac{\mathbb{P}\Big(\sup_n S_n > x\Big)}{G_{\overline{c}_{k_0},|\overline{c}_{m_2}|}(x)} \geq \frac{1}{a}\ . \tag{22}$$

Taking into account the equalities

$$\frac{G_{\overline{c}_{k_0},|\overline{c}_{m_2}|}(x)}{G_{\overline{C},|\overline{c}|}(x)} = \frac{G_{\overline{c}_{k_0},|\overline{c}|}(x)}{G_{\overline{C},|\overline{c}|}(x)} = \frac{\overline{c}_{k_0} G_+(x/\overline{c}_{k_0}) + |\overline{c}| G_-(x/|\overline{c}|)}{\overline{C} G_+(x/\overline{C}) + |\overline{c}| G_-(x/|\overline{c}|)}$$

and (21), we obtain for $x \geq x_0$ the inequality

$$\begin{aligned}\frac{G_{\overline{c}_{k_0},|\overline{c}_{m_2}|}(x)}{G_{\overline{C},|\overline{c}|}(x)} &\geq \frac{(\overline{C}-\delta)(1-\varepsilon) G_+(x/\overline{C}) + |\overline{c}| G_-(x/|\overline{c}|)}{\overline{C} G_+(x/\overline{C}) + |\overline{c}| G_-(x/|\overline{c}|)} \\ &\geq (\overline{C}-\delta)(1-\varepsilon)/\overline{C} \geq (\overline{C}-\varepsilon)(1-\varepsilon)/\overline{C}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows from (22) that

$$\liminf_{x\to\infty} \frac{\mathbb{P}\Big(\sup_n S_n > x\Big)}{G_{\overline{C},|\overline{c}|}(x)} \geq \frac{1}{a}\ .$$

Combining this inequality with the upper bound (18), we get (20). The proof for the cases (ii) and (iii) may be carried out in the same way. □

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